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An estimates of a fractional counterpart of the logarithmic derivative of a meromorphic function

Let $h \in L(0,a)$, $a>0$. The Riemann-Liouville fractional integral of order $\alpha>0$ for h is defined as

$$D^{-\alpha}h(x) = \frac{1}{\Gamma(\alpha)} \int\limits_0^x (x-t)^{\alpha-1} h(t)dt, \quad x \in (0,a),$$

where $\Gamma(\alpha)$ is the Gamma function.

Define

$$\mathcal{I}_\alpha[f](r) = \int\limits_0^{2\pi} D^{\alpha-1} \frac{|f'(re^{i\varphi})|}{|f(re^{i\varphi})|} d\varphi, \quad \alpha \in (0,1).$$

Theorem. *Let f be a meromorphic function in $\mathbb{C}$, $f(0) \neq 0, \infty$, $\{c_q\}$ be the sequence of its zeros and poles, $\alpha \in (0,1)$, $\beta \in (1+\infty)$. Then for some $r_0>0$, a constant $C(\alpha,\beta)$, and for all $r \geq r_0$,*

$$\mathcal{I}_\alpha[f](r) = \int\limits_0^{2\pi} D^{\alpha-1} \frac{|f'(re^{i\varphi})|}{|f(re^{i\varphi})|} d\varphi \leq C(\alpha,\beta) \biggl(\frac{T(\beta r, f)}{r^\alpha} + \frac{1}{r^{\alpha-1}} \int\limits_0^{\frac{r}{2}} \frac{n(t,0,\infty,f)}{t^2} dt \biggr),$$

where $n(t,0,\infty,f)$ is the counting function of $\{c_q\}$, $T(r,f)$ is the Nevanlinna characteristic of f.

Moreover, $C(\beta,\alpha) = O(\frac{1}{(\beta-1)(1-\alpha)})$ as $\beta \to 1+$, $\alpha \to 1-$.

In particular, for every $\varepsilon>0$ we have $I_\alpha[f](r) = O(r^{(\rho-\alpha+\varepsilon)^+})$ as $r \to +\infty$, where ρ is the order of f.

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