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Estimates for modulus of continuity of conformal mappings of Jordan domains

1. General case

For a Jordan curve L define its *modulus of oscillation* by $d(L;\delta) := \sup\{d(L;z,t) : z,t \in L,\ |z-t| \le \delta\}$ $(\delta \ge 0)$, where $d(L;z,t)$ denotes diameter of the arc $L(z,t) \subset L$ with endpoints z, t — in the case of unclosed curve L or the least of the diameters of two arcs $L(z,t) \subset L$ with endpoints z, t — in the case of the closed curve L. The definition of the *modulus of rectifiability* $m(L;\delta)$ of the curve L can be obtained by replacing here the diameter $d(L;z,t)$ of the arc $L(z,t)$ by its length $|L(z,t)|$. A rectifiable Jordan curve L is called **Lavrentiev** one if $m(L;\delta) \le c\delta\ \forall\,\delta \ge 0$ for some $c = \mathrm{const} \ge 1$. It is obvious that convex curves are Lavrentiev ones.

Let $g(x)$ be a nondecreasing function for $x \ge 0$, $g(x) \ge x$, $g(0) = g(0+) = 0$.

Define four classes of Jordan curves L (in the sequel $\varepsilon = \mathrm{const} > 0$). We say that $L \in J(g)$ (or that $L \in J(g,\varepsilon)$) if $d(L;\delta) \le g(\delta)\ \forall\,\delta \ge 0$ $(\forall\,\delta \in [0,\varepsilon]$, respectively), and that $L \in J_0(g)$ (or that $L \in J_0(g,\varepsilon)$) if $m(L;\delta) \le g(\delta)\ \forall\,\delta \ge 0$ $(\forall\,\delta \in [0,\varepsilon]$, respectively). Note that the following relatuions hold: $J(g) \subset J(g,\varepsilon)$, $J_0(g) \subset J_0(g,\varepsilon)$ $(\forall\,\varepsilon > 0)$. Note also that every Jordan curve belongs to some class $J(g)$; in the case of recifiability of L this curve belongs also to some class $J_0(g)$.

In what follows $w = \varphi(z)$ is a univalent conformal mapping of a simply connected domain G of the complex plane onto the disk $D := \{z : |z| < 1\}$, ψ is the inverse mapping, $\omega(f,E,\delta)$ is the usial modulus of continuity of f on E. Let $d_g(x) := (\pi^2/4)g^2(\sqrt{x}) + x/2$, $m_g(x) := (g(\sqrt{x}) + \sqrt{x})^2/4$, $d_g^{-1}(y)$ is the function inverse to d_g, and $m_g^{-1}(y)$ is the function inverse to m_g. Let $D_g(X) := \int_1^X dy/d_g^{-1}(y)$, $M_g(X) := \int_1^X dy/m_g^{-1}(y)$, and let $D_g^{-1}(Y)$ and $M_g^{-1}(Y)$ be functions inverse to these ones, respectively.

Theorem 1. $\partial G \in J(g,\varepsilon) \Rightarrow \omega(\varphi,\overline{G},\delta) \le A\sqrt{g(\delta)}$.

Theorem 2. $\partial G \in J(g,\varepsilon) \Rightarrow \omega(\psi,\overline{D},\delta) \le B\, g(\sqrt{D_g^{-1}(\log\delta)}\,)$,

$$\partial G \in J_0(g,\varepsilon) \Rightarrow \omega(\psi,\overline{D},\delta) \le B_0\, g(\sqrt{M_g^{-1}(\log\delta)}\,).$$

Theorem 3. $\partial G \in J_0(g,\varepsilon)\ \&\ g(x) \le cx\ \forall\,x \ge 0 \quad \Rightarrow$

$$\omega(\psi,\overline{D},\delta) \le B_1\,\delta^{\alpha(c)}, \quad \alpha(c) = (2/\pi)\,\mathrm{arcctg}(4B(c)),$$

$B(c) = c^2(\beta(c) - \sin\beta(c))/(2\beta^2(c))$ $(B(1) = 0)$, $\beta(c) \in [0,\pi]$ is the root of the equation $(\beta/2)/\sin(\beta/2) = c$.

Here positive numbers A, B, B_0, and B_1 do not depand on $\delta \ge 0$.

These theorems are proved in [1].

The results of the next sections are obtained jointly with S. V. Kolesnikov.

2. Convex domains

It was proved in [2] that every conformal mapping $w = \varphi(z)$ of an arbitrary convex domain G of the complex plane onto the disk D has the bounded derivative $\varphi'(z)$ on $\overline{G}$ (hence $\varphi \in$Lip 1 on $\overline{G}$) and a criterion of continuity of $\varphi'(z)$ $(z \in \overline{G})$ at points $z_0 \in \partial G$ is established.

3. Domains with smooth boundaries

Let G be a simply connected domain on the complex plane, Γ a smooth open Jordan curve of its boundary ∂G accessible from G by means of a Jordan domain $\Omega(G,\Gamma) \subset G$ (in particular $\Gamma = \partial G$, $\Omega(G,\Gamma) = G$), $D(t,r)$ an open disk or radius

$r > 0$ centered at $t \in \Gamma$. Let a number $R_0 = R_0(t) > 0$ be chosen such that for every $r \in (0, R_0]$ the boundary circle $C(t, r)$ of the disk $D(t, r)$ intersects Γ exactly at two points and has no other points of intersection with the boundary of the domain $\Omega(G, \Gamma)$. The open arc of the circle $C(t, r)$ with ends at these points which lies in the domain $\Omega(G, \Gamma)$ will be denoted by $\Gamma(t, r)$. By $l(t, r)$ we denote the length of the arc $\varphi(\Gamma(t, r))$, $G(t, r) = \Omega(G, \Gamma) \cap D(t, r)$. Let

$$p(t, r) := 1 - \sqrt{\frac{\operatorname{mes} D(t, r)/2}{\operatorname{mes} G(t, r)}},$$

$$F(t, R) := \frac{2}{\pi R^2} \iint\limits_{G(t,R)} |\varphi'(x + iy)| dxdy.$$

Theorem 4. *If $t \in \Gamma$, $z \in \Omega(G, \Gamma)$ and $|z - t| \leq R_0/2$, then*

$$|\varphi(z) - \varphi(t)| \leq 2\pi F(t, R_0) H(t, R_0; |z - t|) \cdot |z - t|,$$

$$H(t, R_0; u) := \exp\left\{2 \int_{2u}^{R_0} \frac{p(t, r)}{r}\, dr\right\} \ (u \in (0,\ R_0/2]).$$

Theorem 5. *Let $t \in \Gamma$, $R \in (0, R_0]$ and $I(t, R) := \int_R^{R_0} (p(t, r)/r)\, dr$. The following is true:*

1) if the $\lim\limits_{R \to 0+} I(t, R)$ exists and is finite, then $\varphi'(z)$ has at the point t a finite angular limit;

2) if $\varphi'(z)$ has at the point t a finite angular limit that does not equal to 0, then there exist $\lim\limits_{R \to 0+} I(t, R) > -\infty$.

Hence, if the integral $I(t, R)$ is bounded for $(0,\ R_0]$, then the function $\varphi'(z)$ has at the point t a finite angular limit if and only if the $\lim\limits_{R \to 0+} I(t, R)$. exists

Theorem 4 implies the following result.

Theorem 6. *If for some $a > 0$ the integral $\int_R^a (p(t, r)/r)\, dr$ is uniformly bounded from above for $t \in \Gamma$ and $R \in (0, a]$, then for every fixed point $t \in \Gamma$ there exist its some one-sided neighbourhood $D(t, R) \cap \Omega(G, \Gamma)$ in which the function $\varphi'(z)$ is bounded.*

Corollary. *If under the condition of Theorem 6 Γ is a smooth rectifiable boundery of a simply connected bounded domain G, then the function $\varphi'(z)$ is bounded on G.*

References

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