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On the Riemann-Hilbert problem for the Beltrami equations

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Let D be a domain in the complex plane $\mathbb{C}$, i.e., a connected open subset of $\mathbb{C}$, and let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. (almost everywhere) in D. A **Beltrami equation** in D is an equation of the form

$$f_{\bar{z}} = \mu(z) f_z \tag{1}$$

where $f_{\bar{z}} = \bar{\partial} f = (f_x + i f_y)/2, f_z = \partial f = (f_x - i f_y)/2, z = x + iy$, and f_x and f_y are partial derivatives of f in x and y, correspondingly. The function μ is called the **complex coefficient** and $K_\mu(z) = \frac{1+|\mu(z)|}{1-|\mu(z)|}$ the **dilatation quotient** of the equation (1). The equation (1) is said to be **degenerate** if esssup $K_\mu(z) = \infty$.

The **Riemann-Hilbert problem** for the Beltrami equation (1) in a Jordan domain D in $\mathbb{C}$, is the problem on the existence of a continuous function $f : \overline{D} \to \mathbb{C}$ having partial derivatives of the first order a.e. in D, satisfying (1) a.e. and such that

$$Re[\overline{\lambda(z)} f(z)] = \gamma(z) \qquad \forall z \in \partial D \tag{2}$$

with continuous functions $\gamma : \partial D \to \mathbb{R}$ and $\lambda : \partial D \to \mathbb{C}$ such that $|\lambda(z)| \equiv 1$ on ∂D. In particular, if $\lambda(z) \equiv 1$, then we have the Dirichlet problem, see, e.g., [1]-[2].

A **regular (pseudoregular) solution** of the Riemann-Hilbert problem, under $\gamma(z) \not\equiv const$, is a continuous in $\mathbb{C}$ ($\overline{\mathbb{C}}$), discrete and open mapping $f : D \to \overline{\mathbb{C}}$ in the class $W^{1,1}_{loc}$ (outside of isolated poles) with the Jacobian $J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2 \neq 0$ a.e. satisfying (1) a.e. and the condition (2).

Further, the integer $\varkappa = \frac{1}{2\pi}\, \Delta_{\partial \mathbb{D}}\ arg\ \lambda \circ \omega$ denotes the **index of the Riemann-Hilbert problem** where $\omega : \mathbb{D} \to D$ is the Riemann conformal mapping from the unit disk $\mathbb{D}$ onto D and $\Delta_{\partial \mathbb{D}}\ arg\ \lambda \circ \omega(t)$ is the increment of argument of the function $\lambda \circ \omega(t)$ as the point t runs through $\partial \mathbb{D}$ one time with $\mathbb{D}$ from the left.

Theorem 1. *Let D be a Jordan domain in $\mathbb{C}$ and $\mu : D \to \mathbb{C}$ be a measurable function such that $|\mu(z)| < 1$ a.e., and*

$$\overline{\lim_{\varepsilon \to 0}} \frac{1}{|B(z_0, \varepsilon)|} \int\limits_{B(z_0,\varepsilon)} K_\mu(z)\, dm(z) < \infty \qquad \forall z_0 \in \overline{D}. \tag{3}$$

If $\varkappa \geq 0$, then the problem (2) for the equation (1) has a regular solution under every continuous function $\gamma : \partial D \to \mathbb{R}, \gamma(z) \not\equiv const$; otherwise, if $\varkappa < 0$, then the problem has a pseudoregular solution with poles at $-\varkappa$ prescribed points in D.

In (3) we assume that K_μ is extended by zero outside of the domain D.

1. Vekua I.N., Generalized analytic functions.-London:Pergamon Press,1962.
2. Kovtonyuk D., Petkov I., Ryazanov V., On the Dirichlet problem for Beltrami equations in finitely connected domains // Ukr.Math.J.-2012.-64,7.-P.1064-1077.