On the angular derivatives of certain class of holomorphic functions in the unit disc

Bülent Nafi ÖRNEK

Department of Mathematics, Gebze Institute of Technology, Kocaeli, Turkey

nornek@gyte.edu.tr

In this paper, a boundary version of the Schwarz lemma is investigated. We take into consideration a function f holomorphic in the unit disc, $f(0)=0$, $f'(0)=1$ and $\left|\frac{f(z)}{\lambda f(z)+(1-\lambda)z}-\alpha\right|<\alpha$ for $|z|<1$, where $\frac{1}{2}<\alpha\leq\frac{1}{1+\lambda}$, $0\leq\lambda<1$. We obtain sharp lower bounds on the angular derivative $f'(b)$ at the point b, where $|b|=1$ and $f(b)=0$. We will obtain more general results at the boundary. In the following theorem, a new inequality of Schwarz inequality at the boundary is obtained and the sharpness of this inequality is proved.

Theorem. *Let* $z+c_{p+1}z^{p+1}+c_{p+2}z^{p+2}+.....$, $p\geq 1$, *be a holomorphic function in the disc* D *and let* $\left|\frac{f(z)}{\lambda f(z)+(1-\lambda)z}-\alpha\right|<\alpha$ *for* $|z|<1$, *where* $\frac{1}{2}<\alpha\leq\frac{1}{1+\lambda}$, $0\leq\lambda<1$.. *Further assume that, for some* $b\in\partial D$, f *has an angular limit* $f(b)$ *at* b, $f(b)=0$. *Let* $a_1,a_2,..,a_n$ *be fixed points of* $f(z)$ *in* D *that are different from zero. Then we have the inequality*

$$|f'(b)|\geq\frac{\alpha(1-\lambda)}{2\alpha-1}\left(p+\sum_{k=1}^{n}\frac{1-|a_k|^2}{|b-a_k|^2}+\frac{(2\alpha-1)\prod_{k=1}^{n}|a_k|-\alpha(1-\lambda)|c_{p+1}|}{(2\alpha-1)\prod_{k=1}^{n}|a_k|+\alpha(1-\lambda)|c_{p+1}|}\right). \tag{1.1}$$

In addition, the equality in (1.1) *occurs for the function*

$$f(z)=\frac{\alpha(1-\lambda)z\left(1-z^p\prod_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}\right)}{\alpha-(1-\alpha)z^p\prod_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}-\alpha\lambda\left(1-z^p\prod_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}\right)},$$

where $a_1,a_2,...,a_n$ *are positive real numbers.*

Similar results for the holomorphic functions with different additional conditions have been obtained by Dubinin in (1) and Osserman in (2).

References

1. Dubinin, V. N., *The Schwarz inequality on the boundary for functions regular in the disc*, Journal of Mathematical Sciences, 2004, 122: 3623-3629.

2. Osserman, R., *A sharp Schwarz inequality on the boundary,* Proc. Math. Soc., 2000,128: 3513-3517 .