The compactness criteria for classes of solutions to the degenerate Beltrami equations

Lomako T.V.
Institute of Applied Mathematics and Mechanics of the NAS of Ukraine
tlomako@yandex.ru

The *Beltrami equation* is equation of the form

$$f_{\overline{z}} = \mu(z) \cdot f_z \tag{1}$$

where $|\mu(z)| < 1$ a.e. in $\mathbb{C}$. The *regular solution* of the Beltrami equation is a homeomorphism f of the class $W^{1,1}_{\mathrm{loc}}$ that satisfies (1) a.e. and $J_f(z) := |f_z|^2 - |f_{\overline{z}}|^2 > 0$ a.e. in $\mathbb{C}$. The Beltrami equation (1) is called *degenerate* if $K_\mu(z) := (1 + |\mu(z)|) \,/\, (1 - |\mu(z)|) \notin L^\infty$. It was recently obtained some new existence theorems for such equations, see e.g. [1]. Below, $dm(z)$ corresponds to the Lebesgue measure in $\mathbb{C}$, $dS(z) = (1 + |z|^2)^{-2} dm(z)$ is the *element of the spherical area* in $\overline{\mathbb{C}}$. In what follows, the *continuity* of the function $\Phi : \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ is understood with respect to the topology of $\overline{\mathbb{R}^+} := [0, \infty]$. The nondecreasing convex function $\Phi : \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ is called *strictly convex* if $\lim_{t\to\infty} \Phi(t)/t = \infty$.

Let $\Phi : \mathrm{I} := [1, \infty] \to \overline{\mathbb{R}^+}$ be an arbitrary function. We denote by $\mathfrak{F}^\Phi_M$, $M \geq 0$, the class of all regular solutions $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of the Beltrami equation (1) with complex coefficients μ such that $\int_{\mathbb{C}} \Phi\left(K_\mu(z)\right) dS(z) \leq M$ and $f(0) = 0$, $f(1) = 1$, $f(\infty) = \infty$.

Theorem 1. *Let $\Phi : \mathrm{I} \to \overline{\mathbb{R}^+}$, $\Phi(\infty) = \infty$, satisfy the condition*

$$\int_\delta^\infty \ln \Phi(\tau)\, \frac{d\tau}{\tau^2} = \infty \tag{2}$$

for some $\delta > \delta_0 := \sup_{\substack{\tau\in \mathrm{I},\\ \Phi(\tau)=0}} \tau$ (if $\Phi(\tau) > 0$ for all $\tau \in \mathrm{I}$, then $\delta_0 = 1$). Then the following assertions are equivalent:

1) the classes $\mathfrak{F}^\Phi_M$ are compact with the topology of the uniform convergence in $\overline{\mathbb{C}}$ with respect to the spherical metric;

2) Φ is strictly convex and left continuous in the sense of the topology of $\overline{\mathbb{R}^+}$ at the point $Q = \sup_{\Phi(t)<\infty} t$.

Remark. The condition (2) is not only sufficient but also necessary for the compactness of $\mathfrak{F}^\Phi_M$, if Φ is continuous, convex and nondecreasing, see [2].

Note that the obtained theorem has an applications in the theory of extremal problems and the theory of the variational method. The matter is that, in the compact classes, it is guaranteed the existence of extremal mappings for every continuous, in particular, nonlinear functionals. Moreover, in the compact classes of mappings with integral constraints the set of complex characteristic is convex, which essentially simplifies the procedure of construction of variations, see e.g. [3].

1. *Gutlyanskii V., Ryazanov V., Srebro U., Yakubov E.* On recent advances in the degenerate Beltrami equations, Ukr. Mat. Visn., 2010, **7**, N. 4, p. 467–515; translation in J. Math. Sci., 2011, **175**, N. 4, p. 413–449.

2. *Ryazanov V.I., Sevostyanov E.A.* Equicontinuity of mean quasiconformal mappings, Sibirsk. Mat. Zh, 2011, **52**, N. 3, p. 665–679; translation in Siberian Math. Journal, 2011, **52**, N. 3, p. 524–536.

3. *Gutlyanskii V.Ya., Lomako T.V., Ryazanov V.I.* To the theory of variational method for Beltrami equations. // Ukr. Mat. Visn., 2011, **8**, N. 4, p. 513–536; translation in J. Math. Sci., 2012, **182**, N. 1, p. 37–54.