Semigroups of conformal mappings with given fixed points

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Let $\mathfrak{P}$ be the set of all functions f holomorphic in $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$ and taking values in $\mathbb{D}$. It is impotent in many areas of analysis and applications that $\mathfrak{P}$ forms a topological semigroup with respect to the operation of composition and the topology uniform convergence in $\mathbb{D}$. In particularly, one-parameter semigroups of analytic functions are often used to describe the dynamics of some processes. Treating $\mathbb{R}^+ = \{t \in \mathbb{R}\colon t \geq 0\}$ as an additive semigroup endowed with the standard topology of the real number line, we understand a one-parameter semigroup in $\mathfrak{P}$ to be a continuous homomorphism $t \mapsto f^t$ acting from $\mathbb{R}^+$ into $\mathfrak{P}$. Synthesis of algebraic and topological properties leads to the infinite differentiability with respect to t of the family of functions $f^t(z)$ of a one-parameter semigroup $t \mapsto f^t$ in $\mathfrak{P}$ (see, for example, [1]). The derivative

$$\frac{\partial}{\partial t} f^t(z)\Big|_{t=0} = \lim_{t\to 0} \frac{f^t(z) - z}{t} = v(z)$$

is an analytic function in $\mathbb{D}$ and characterizes completely the one-parameter semigroup $t \mapsto f^t$. We call this function the infinitesimal generator of the one-parameter semigroup $t \mapsto f^t$. The form of v depends on the set of fixed points of f^t, $t \geq 0$. Note that if $f \in \mathfrak{P}$ is not a fractional linear transformation of the unit disk $\mathbb{D}$ onto itself, then there exists a unique point q, $|q| \leq 1$, such that the sequence of positive-integer iterates f^n, $n = 1, 2, \ldots$, converges to q locally uniformly in $\mathbb{D}$. Furthermore, if $|q| = 1$, then there exist the angular limits

$$\lim_{z\to q} f(z) = f(q) \qquad \lim_{z\to q} f'(z) = f'(q)$$

and $f(q) = q$, $0 < f'(q) \leq 1$. In the literature, q is called the Denjoy–Wolff point of the function f. If $q \in \mathbb{D}$, then $f(q) = q$ and $|f'(q)| \leq 1$. Moreover, if $f(z) \not\equiv z$, then f has no fixed points in the interior of $\mathbb{D}$ other than the Denjoy–Wolff point. On the other hand, it can have fixed points on the boundary $\mathbb{T}$ of the unit disk $\mathbb{D}$ (in the sense of the angular limits).

Let $|q| \leq 1$ and $a_1, \ldots, a_n$ be pairwise distinct points on $\mathbb{T}$. Denote by $\mathfrak{P}[q; a_1, \ldots, a_n]$ the set of functions f in $\mathfrak{P}$ such that q is a Denjoy–Wolff point for f and $a_1, \ldots, a_n$ are fixed points at which the function f has finite angular derivatives. $\mathfrak{P}[q; a_1, \ldots, a_n]$ is a subsemigroup of $\mathfrak{P}$. Note that [1] a function v holomorphic in $\mathbb{D}$ to be infinitesimal generator of a one-parameter semigroup $t \mapsto f^t$ in $\mathfrak{P}[q; a_1, \ldots, a_n]$ if and only if it admits a representation in the form

$$v(z) = \frac{(q - z)(1 - \overline{q}z)}{\displaystyle\sum_{k=1}^{n} \lambda_k \frac{1 + \overline{a}_k z}{1 - \overline{a}_k z} + p(z)},$$

where $\lambda_k > 0$, $k = 1, \ldots, n$, and p is a holomorphic function in $\mathbb{D}$ with non-negative real part.

We study evolution families in semigroup $\mathfrak{P}[q; a_1, \ldots, a_n]$ and obtain a parametric representation of some subsemigroups of $\mathfrak{P}[q; a_1, \ldots, a_n]$ consisting of conformal mappings.

1. Goryainov V. V. Semigroups of analytic functions in analysis and applications, Russian Math. Surveys, 2012, 67:6, 975–1021.