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DENSITY OF SIMPLE PARTIAL FRACTIONS IN FUNCTION SPACES

A simple partial fraction is a rational function of the form

$$r(z) = \sum_{k=1}^{n} \frac{1}{z - a_k},$$

where a_k are complex numbers.

Let K be a compact subset of the complex plane. The set $SPF(\mathbb{C} \setminus K)$ of simple partial fractions with poles outside K is dense in the space $AC(K)$ of functions that are continuous on K and analytic within K iff K has connected complement [1]. Moreover, if the set $SPF(E)$ of simple partial fractions with poles in some compact set $E \subset \mathbb{C} \setminus K$ is dense in $AC(K)$, then the difference $\widehat{E} \setminus K$ is finite [5] (this can also be deduced from the results of [2]). Thus arises the

Problem. *Let K be a compact set with connected complement. Also let E be a compact set, which is disjoint from K and is such that $\widehat{E} \supset K$. Is it true that $SPF(E)$ is dense in $AC(K)$?*

In the case when K lies in a single connected component of the set $\widehat{E} \setminus E$, the positive solvability of this problem is a consequence of the main result of [2]. The problem is also positively solved in case when E contains mutually exterior closed Jordan rectifiable contours $\Gamma_1, \ldots, \Gamma_n$ such that $K \subset \cup_{j=1}^{n} Int\, \Gamma_j$ [5].

Besides that, the density of simple partial fractions has been examined in various spaces of functions defined on unbounded subsets of the complex plane. For example, it was proved [3] that $SPF(\mathbb{C} \setminus \mathbb{R})$ is dense in the space $C_0(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C}, f \in C(\mathbb{R}), f(x) \to 0 \,\text{as}\, x \to \infty\}$, equipped with uniform norm, but is not dense in $L_p(\mathbb{R})$ $(p > 1)$. As for the semiaxis $\mathbb{R}_+$, the set $SPF(\mathbb{C} \setminus \mathbb{R}_+)$ is dense in $L_p(\mathbb{R}_+)$ iff $p \geq 2$ [4].

References

[1] V. I. Danchenko and D.Ya. Danchenko, "Approximation by simplest fractions", Mat. Zametki 70:4 (2001), 553559; English transl. in Math. Notes 70:4 (2001), 502-507.

[2] J.Korevaar, "Asymptotically neutral distribution of electrons and polynomial approximation", Ann. of Math. (2) **80**:2 (1964), 403-410.

[3] P. A. Borodin and O. N. Kosukhin, "Approximation by the simplest fractions on the real axis", Vestn. Moskov. Univ. Ser. 1. Mat. Mekh., 2005, no. 1, 3-8; English transl. in Moscow Univ. Math. Bull. 60:1 (2005), 1-6.

[4] P. A. Borodin, "Approximation by simple partial fractions on the semi-axis", Mat. Sb. 200:8 (2009), 25-44; English transl. in Sb. Math. 200:8 (2009), 1127-1148.

[5] P. A. Borodin, "Approximation by simple partial fractions with constraints on the poles", Mat. Sb. 203:11 (2012), 23-40; English transl. in Sb. Math. 203:11 (2012), 1553-1570.