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THE DIRICHLET PROBLEM FOR THE GENERAL BELTRAMI EQUATIONS

We study the Dirichlet problem for general degenerate Beltrami equations

$$\begin{cases} f_{\overline{z}} = \mu(z)\cdot f_z + \nu(z)\cdot \overline{f_z}, & z\in D,\\ \lim\limits_{z\to\zeta} \operatorname{Re} f(z) = \varphi(\zeta), & \forall\ \zeta\in\partial D, \end{cases} \tag{1}$$

in a Jordan domain D of the complex plane $\mathbb{C}$ with continuous boundary data $\varphi(\zeta)\not\equiv$ const. Here $\mu(z)$ and $\nu(z)$ stand for measurable coefficients satisfying the inequality $|\mu(z)|+|\nu(z)|<1$ a.e. in D. The degeneracy of the ellipticity in (1) is controlled by the dilatation coefficient

$$K_{\mu,\nu}(z)\ :\ =\ \frac{1+|\mu(z)|+|\nu(z)|}{1-|\mu(z)|-|\nu(z)|} \tag{2}$$

Recall that a general Beltrami equation is a complex form of one of the main equations of mathematical physics in anisotropic and inhomogeneous media, see details in the paper [1].

We look for a solution of the Dirichlet problem (1) as a continuous, discrete and open mapping $f: D\to\mathbb{C}$ of the Sobolev class $W^{1,1}_{\rm loc}$ and such that the Jacobian $J_f(z)\neq 0$ a.e. in D. Such a solution we call a **regular solution** of the Dirichlet problem (1) in a domain D. Recall that a mapping $f: D\to\mathbb{C}$ is called **discrete** if the preimage $f^{-1}(y)$ consists of isolated points for every $y\in\mathbb{C}$, and **open** if f maps every open set $U\subseteq D$ onto an open set in $\mathbb{C}$.

In the paper [2], we have established a series of criteria for the existence of regular solutions of the Dirichlet problem (1) and, in particular, the following one.

Theorem 1. *Let D be a Jordan domain in $\mathbb{C}$ and let μ and $\nu: D\to\mathbb{C}$ be measurable functions such that*

$$\int\limits_D \Phi(K_{\mu,\nu}(z))\ dxdy\ <\ \infty \tag{3}$$

where $\Phi:[0,\infty]\to[0,\infty]$ is a non-decreasing convex function with the condition

$$\int\limits_{\delta}^{\infty} \frac{d\tau}{\tau\Phi^{-1}(\tau)}\ =\ \infty \tag{4}$$

for some $\delta>\Phi(+0)$. Then the Dirichlet problem (1) has a regular solution f for each nonconstant continuous function $\varphi:\partial D\to\mathbb{R}$.

Note that the condition (4) is not only sufficient but also necessary to have a regular solution of the Dirichlet problem (1) for all general Beltrami equations with the integral constraint (3).

1. B. Bojarski, V. Gutlyanskii, V. Ryazanov. On existence and representation of solutions for general degenerate Beltrami equations// Complex Variables and Elliptic Equations, 2013, http://dx.doi.org/10.1080/17476933.2013.795955

2. B. Bojarski, V. Gutlyanskii, V. Ryazanov. On the Dirichlet problem for general degenerate Beltrami equations in Jordan domains // Ukr. Math. Visn. – 2012. – V. 9, no. 4. – P. 460-476; transl. in J. Math. Sci. - 2013. - V. 190, no. 4. - P. 525-538.