Some Properties of Two Linearly Independent Meromorphic Solutions of Second Order Linear Differential Equations

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In this paper, we investigate the growth and oscillation of the polynomial of combinaison of two linearly independent meromorphic solutions to the complex differential equation

$$f'' + A(z) f' + B(z) f = 0, \tag{1.1}$$

where $A(z)$ and $B(z)$ are meromorphic functions of finite iterated p-order. We obtain the following three results.

Theorem 1.1 *Let $A(z)$ and $B(z)$ be meromorphic functions with $i(B) = p \geq 1$, $\lambda_p\left(\frac{1}{B}\right) < \rho_p(B)$ such that $\rho_p(B) > \rho_p(A)$ and $\tau_p(B) > \tau_p(A)$ if $\rho_p(B) = \rho_p(A) > 0$. Let $d_j(z)$ $(j = 1, 2)$ be meromorphic functions that are not all vanishing identically such that $\max\{\rho_p(d_1), \rho_p(d_2)\} < \rho_p(B)$. If f_1 and f_2 are two nontrivial linearly independent meromorphic solutions whose poles are of uniformly bounded multiplicities of (1.1), then the polynomial of solutions $w = d_1 f_1 + d_2 f_2$ satisfies $i(w) = p + 1$ and $\rho_p(w) = \infty, \rho_{p+1}(w) = \rho_p(B)$.*

Theorem 1.2 *Under the assumptions of Theorem 1.1, let $\varphi(z) \not\equiv 0$ be a meromorphic function of finite iterated $p-$order such that $\psi(z) = 2\frac{\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h}\varphi^{(3)} + \phi_2\varphi'' + \phi_1\varphi' + \phi_0\varphi \not\equiv 0$, where*

$$\phi_2 = \frac{2\left(d_1 d_2 d_2' - d_2^2 d_1'\right) A - 3 d_1 d_2 d_2'' + 3 d_2^2 d_1''}{h},$$

$$\phi_1 = \frac{1}{h}\left[6 d_2 \left(d_1' d_2'' - d_2' d_1''\right) + 2 d_2 \left(d_1 d_2' - d_2 d_1'\right) B + 2 d_2 \left(d_1 d_2' - d_2 d_1'\right) A' + 3 d_2 \left(d_2 d_1'' - d_1 d_2''\right) A\right],$$

$$\begin{aligned} \phi_0 = \frac{1}{h}[&\left(d_1 d_2' d_2'' - 3 d_2 d_2' d_1'' + 2 d_2 d_1' d_2''\right) A \\ &+ \left(4 d_1 (d_2')^2 + 3 d_2^2 d_1'' - 3 d_1 d_2 d_2'' - 4 d_2 d_1' d_2'\right) B + 2\left(d_2 d_1' d_2' - d_1 (d_2')^2\right) A' \\ &+ 2\left(d_1 d_2 d_2' - d_2^2 d_1'\right) B' + 6 (d_2')^2 d_1'' - 2 d_1 d_2' d_2''' \\ &+ 2 d_2 d_1' d_2''' - 3 d_2 d_1'' d_2'' - 6 d_1' d_2' d_2'' + 3 d_1 (d_2'')^2]. \end{aligned}$$

If f_1 and f_2 are two nontrivial linearly independent meromorphic solutions whose poles are of uniformly bounded multiplicities of (1.1), then the polynomial of solutions $w = d_1 f_1 + d_2 f_2$ satisfies $\overline{\lambda}_p(w - \varphi) = \lambda_p(w - \varphi) = \infty$ and $\overline{\lambda}_{p+1}(w - \varphi) = \lambda_{p+1}(w - \varphi) = \rho_p(B)$.

Theorem 1.3 *Let $A(z)$ and $B(z)$ be meromorphic functions with $i(B) = p \geq 1$, $\lambda_p\left(\frac{1}{B}\right) < \rho_p(B)$ such that $\rho_p(B) > \rho_p(A)$. Let $d_j(z), b_j(z)$ $(j = 1, 2)$ be finite iterated $p-$order meromorphic functions such that $d_2(z) b_1(z) - d_1(z) b_2(z) \not\equiv 0$. If f_1 and f_2 are two nontrivial linearly independent meromorphic solutions whose poles are of uniformly bounded multiplicities of (1.1), then $\rho_p\left(\frac{d_1 f_1 + d_2 f_2}{b_1 f_1 + b_2 f_2}\right) = \infty$ and $\rho_{p+1}\left(\frac{d_1 f_1 + d_2 f_2}{b_1 f_1 + b_2 f_2}\right) = \rho_p(B)$.*

References

1-S. Bank and I. Laine, *On the oscillation theory of* $f'' + A(z)f = 0$. *where* A *is entire*, Trans. Amer. Math. Soc. 273 (1982), no. 1, 351–363.

2-S. Bank and I. Laine, *On the zeros of meromorphic solutions and second-order linear differential equations*, Comment. Math. Helv. 58 (1983), no. 4, 656–677.

3-L. G. Bernal, *On growth* $k-$*order of solutions of a complex homogeneous linear differential equations*, Proc. Amer. Math. Soc. **101** (1987), 317-322.

4- B. Belaïdi, *Oscillation of fixed points of solutions of some linear differential equations*, Acta Math. Univ. Comenian. (N.S.) 77 (2008), no. 2, 263–269.

5- B. Belaïdi, *Complex oscillation of differential polynomials generated by meromorphic solutions of linear differential equations*, Publ. Inst. Math. (Beograd) (N.S.) 90(104) (2011), 125–133.

6-T. B. Cao, J. F. Xu and Z. X. Chen, *On the meromorphic solutions of linear differential equations on the complex plane*, J. Math. Anal. Appl. 364 (2010), no. 1, 130–142.

7- Z. X. Chen, *The fixed points and hyper-order of solutions of second order complex differential equations,* Acta Math. Sci. Ser. A Chin. Ed. 20 (2000), no. 3, 425–432 (in Chinese).

8- W. K. Hayman, *Meromorphic functions*, Clarendon Press, Oxford, 1964.

9- L. Kinnunen, *Linear differential equations with solutions of finite iterated order*, Southeast Asian Bull. Math., **22: 4** (1998), 385-405.

10- I. Laine, *Nevanlinna theory and complex differential equations*, de Gruyter Studies in Mathematics, 15. Walter de Gruyter & Co., Berlin-New York, 1993.

11- Z. Latreuch and B. Belaïdi, *Some properties of solutions of second order linear differential equations*, Journal of Complex Analysis, Volume 2013, Article ID 253168, 5 pages.

12- C. C. Yang and H. X. Yi, *Uniqueness theory of meromorphic functions*, Mathematics and its Applications, 557. Kluwer Academic Publishers Group, Dordrecht, 2003.