On elliptic problems in Hörmander spaces

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We discuss applications of certain Hörmander function spaces of generalized smoothness to elliptic boundary–value problems. These spaces form the extended Sobolev scale $\{H^{\varphi} : \varphi \in \mathrm{RO}\}$, where the smoothness index $\varphi : [1, \infty) \to (0, \infty)$ is a Borel measurable function RO-varying at $+\infty$ in the sense of V. G. Avakumović. The latter property means that the exist numbers $a > 1$ and $c \geq 1$ such that $c^{-1} \leq \varphi(\lambda t)/\varphi(t) \leq c$ for every $t \geq 1$ and $\lambda \in [1, a]$ (a and c may depend on φ).

The extended Sobolev scale over $\mathbb{R}^n$ consists of all Hilbert spaces

$$H^{\varphi}(\mathbb{R}^n) := \left\{ w \in \mathcal{S}'(\mathbb{R}^n) : \|w\|_{\varphi}^2 := \int_{\mathbb{R}^n} \varphi^2(\langle\xi\rangle)\, |(Fw)(\xi)|^2 \, d\xi < \infty \right\}$$

and then is defined in the standard way over Euclidean domains and smooth compact manifolds. Here $\langle\xi\rangle := (1 + |\xi|^2)^{1/2}$, and Fw is the Fourier transform of a tempered distribution w. In the special case where $\varphi(t) \equiv t^s$ we have the inner product Sobolev space $H^{(s)} = H^{\varphi}$ of order $s \in \mathbb{R}$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\Gamma \in C^{\infty}$. We consider a general elliptic boundary–value problem

$$Au = f \ \text{ in } \ \Omega, \quad B_j u = g_j \ \text{ on } \ \Gamma, \ \text{ with } \ j = 1, \ldots, q. \tag{1}$$

Here $A = A(x, D)$, $x \in \overline{\Omega}$, and all $B_j = B_j(x, D)$, $x \in \Gamma$, are linear partial differential expressions. Their coefficients are complex–valued and infinitely smooth; $\operatorname{ord} A = 2q$, with $q \in \mathbb{N}$, and $m_j := \operatorname{ord} B_j \leq 2q - 1$.

We discuss properties of the elliptic problem (1) considered on the extended Sobolev scale. Put $B := (B_1, \ldots, B_q)$ and $\rho(t) := t$ for $t \geq 1$.

Theorem. *Let an increasing function parameter $\varphi \in \mathrm{RO}$ be arbitrary. Then the mapping $u \to (Au, Bu)$, with $u \in C^{\infty}(\overline{\Omega})$, extends uniquely (by continuity) to a bounded operator*

$$(A, B) : H^{\varphi\rho^{2q}}(\Omega) \to H^{\varphi}(\Omega) \oplus \bigoplus_{j=1}^{q} H^{\varphi\rho^{2q-m_j-1/2}}(\Gamma). \tag{2}$$

This operator is Fredholm; its kernel and index do not depend on φ.

Among various applications of this theorem are the following:

- a theorem on isomorphisms generated by (2);
- a priori estimates for solutions to the problem (1);
- a theorem on local increasing in regularity of the solutions;
- new sufficient conditions for the weak solutions to be classical.