Promarz M. Tamrazov: mathematical ideas and results

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TAMRAZOV Promarz Melikovich
(17.06.1933 — 11.02.2012)

Professor Promarz Melikovich Tamrazov is an outstanding and leading specialist in complex analysis, potential theory and related fields of mathematics.

P.M. Tamrazov was born on 17.06.1933 in Kiev. His parents Melik and Shoushan were Assyrians.

His mathematical faculties appeared very early.

According to his own words, he liked being a primary schooler to play under a table, where his elder brother Zhora was solving mathematical school problems under the supervision of parents. And when Zhora could not find an answer to the next problem, little boy Proma loudly gave a correct answer below.

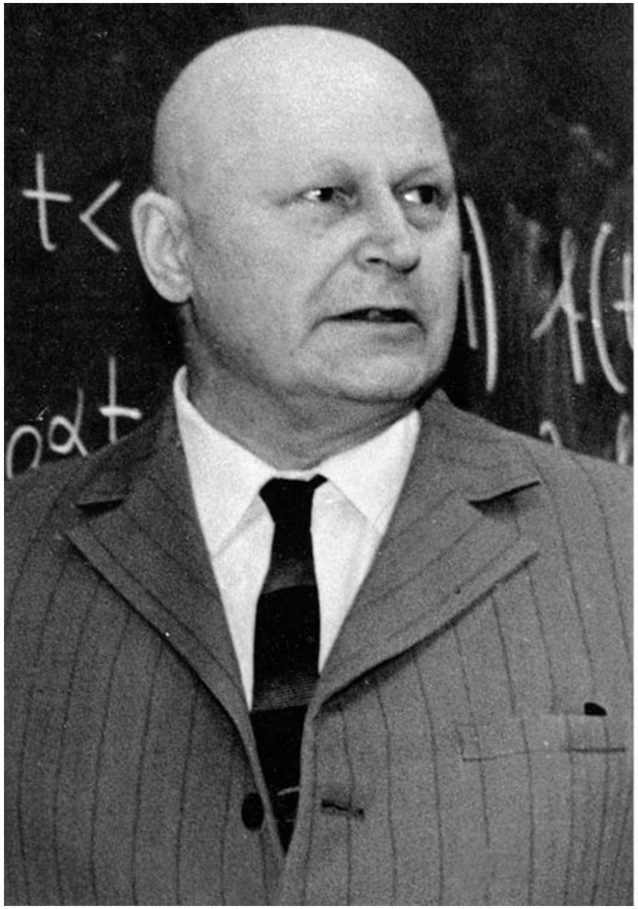
After beginning the Second World War he remained in Kiev together with family. After the occupation of Kiev by the fascist army, the family was transported to Germany in 1942, but they succeeded to escape on the road. In 1944 the family has came back to Kiev, where Proma has renewed the school training.

Being in high school, Promarz wins the Kiev competition in Mathematics among the schoolboys. In 1951 he finished high school with Gold Medal honor.

Proma, Zhora, Alik Tamrazovs with mother Rakhme Shushan

From 1951 he studied at Kiev Polytechnic Institute. Professor Valentin Anatolievich Zmorovich gave enthusiastically lectures of higher mathematics for students of the first year. These lectures have made a deep impression on the young P. Tamrazov. He became a member of a mathematical study group supervised by V.A. Zmorovich. Lessons at this group has strengthened Tamrazov's interest to mathematics.

In 1956 P. Tamrazov graduated with honor in Mechanical and Heat Engineering and received an engineer degree (Soviet equivalent of Master degree). Between 1956 and 1963 he worked in Kiev Polytechnic Institute and institutes of Ukrainian Academy of Sciences as engineer and assistant professor.



V.A. Zmorovich
(1909 – 1994)

In 1958 — 61 P. Tamrazov undertook post-graduate studies in Mathematics under the guidance of Professor V.A. Zmorovich. For his Ph.D./Candidate thesis V.A. Zmorovich has offered a theme closely related to original Grötzsch's ideas in the geometric function theory of the complex variable. Having deeply understood Grötzsch's ideas, P. Tamrazov has developed the theory of extremal length and has solved a series of extremal problems for conformal mappings of multiply and infinitely connected domains. These extremal problems are associated with multipole quadratic differentials.

After proving a Ph.D./Candidate thesis in 1963 P.M. Tamrazov has prepared a brilliant Doctor Sciences thesis that was proved on 29.01.1966 in Institute of Mathematics of Ukrainian Academy of Sciences, Kiev.

P.M. Tamrazov and his wife Janna

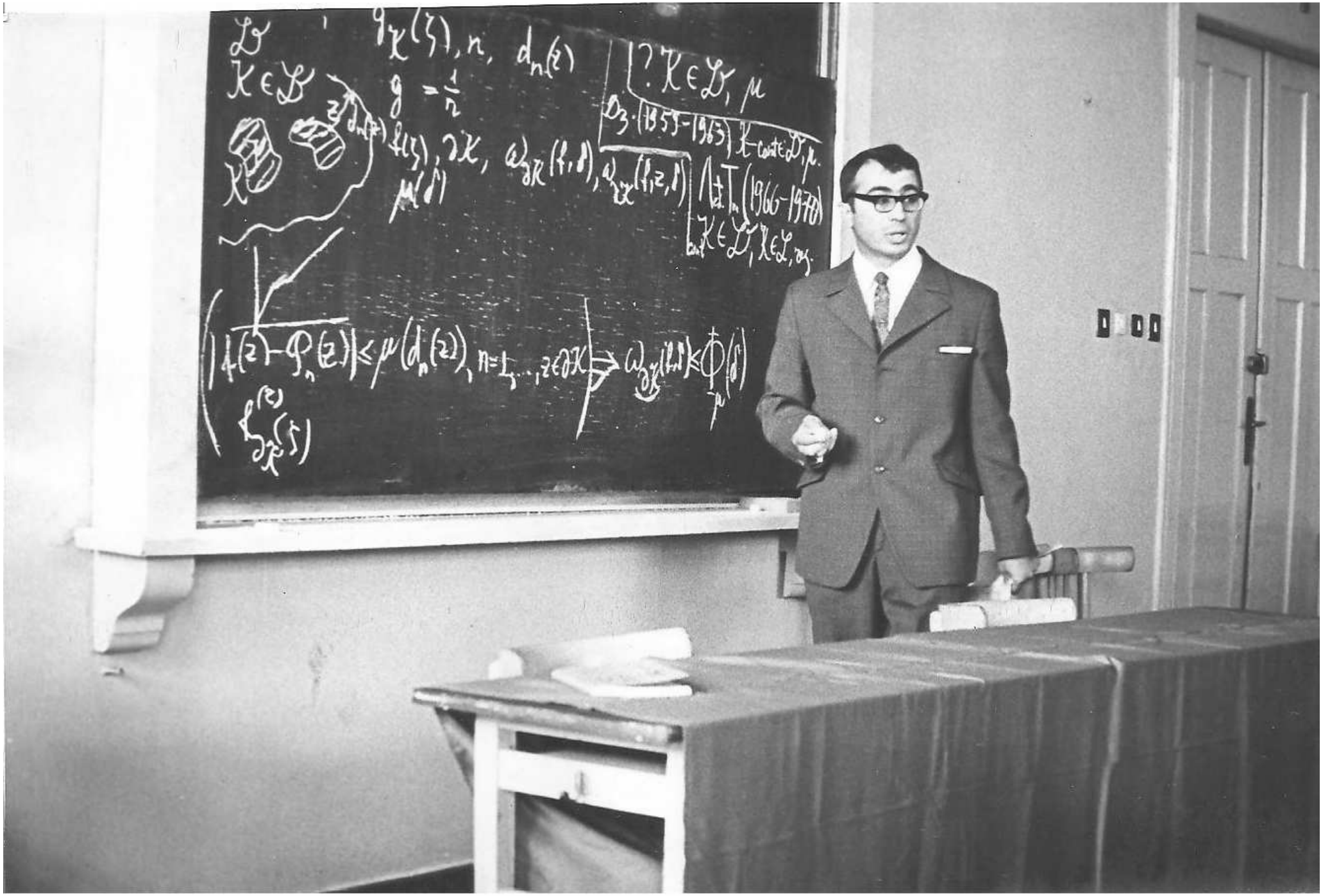
From 1963 P.M. Tamrazov worked in Institute of Mathematics of Ukrainian Academy of Sciences as a research fellow and from 1983 as a head of laboratory. He got the title of Professor in 1982.

In 1989 — 2003 he was the head of Department of Complex Analysis and Potential Theory and from 2003 a leading research fellow of the mentioned department.

In 2006 P.M. Tamrazov was elected to the National Academy of Science of Ukraine its Corresponding member.

He was an active participant on many international congresses and conferences. He was awarded by many grants.

P.M. Tamrazov was a member of ISAAC Board (1998 — 2002) and a member of ISAAC Award Committee (1999).



P.M. Tamrazov is the author of more than 200 research papers and of monograph “Smoothnesses and polynomial approximations”, Naukova Dumka, Kiev, 1975. He was the supervisor of 13 Ph D and 4 Doctor of Sciences theses.

Promarz Tamrazov: Ph.D. – 1963 , Doctor Sciences – 1966

- Volodymyr Gorbaichuk: Ph.D. – 1972
- Aleksandr Bakhtin: Ph.D. – 1975 , Doctor Sciences – 2007
  - A. Targonskyi, V. Viun, I. Vygovska, I. Denega, Ja. Zabolotnyi
- Galina Bakhtina: Ph.D. – 1975
- Elena Karupu: Ph.D. – 1978
- Viacheslav Bardzinskii: Ph.D. – 1978
- Oleg Gerus: Ph.D. – 1980
- Natalia Zorii: Ph.D. – 1981 , Doctor Sciences – 1992
- Varazdat Navoyan: Ph.D. – 1984
- Anatolii Shchekhorskii: Ph.D. – 1984
- Tahir Azeroglu Aliyev: Ph.D. – 1986
- Sergiy Plaksa: Ph.D. – 1989 , Doctor Sciences – 2006
  - S. Gryshchuk, Ju. Kudiavina, V. Shpakivskyi, R. Pukhtairvych
- Vladimir Kudiavin: Doctor Sciences – 1992
  - Anatoliy Golberg: Ph.D. – 1993
- Aleksandr Sarana: Ph.D. – 1995
- Serhii Okhrimenko: Ph.D. – 2003

P.M. Tamrazov solved many open problems which were posed and tackled by other scientists. In particular:

- he developed the theory of complex finite-difference smoothnesses of any order on general sets in the complex plane and solved the difference contour-solid problems for holomorphic functions posed by W.E. Sewell in 1942, and developed a general contour-solid theory for holomorphic and meromorphic and subharmonic functions. The obtained results enabled to solve open problems of approximation theory on complex sets;
- he solved Gonchar's extremal problem on capacities of condensers and for this purpose he developed a method based on mixing signed measures or charges;

- he investigated general properties of extremal lengths and extremal metrics, and solved problems concerning finding extremal metrics and moduli of some nonorientable and twisted Riemannian manifolds, including the problem for Möbius strip that had been tackled by P.M. Pu in 1952 but not solved ;
- he solved extremal problems for conformal mappings associated with multipole quadratic differentials.

His fundamental results gave rise to fruitful investigations of many mathematicians.

Let us remember some P.M. Tamrazov's mathematical ideas and results.

1. Extremal length and extremal metrics

The investigation of geometrical properties of mappings had promoted a search of conformal invariants that could be used for developing effective methods of studying wide classes of mappings. The Ahlfors – Beurling (1950) extremal length and the module of curves family are conformal invariants suitable to this goal.

Let Γ be a family of curves contained in a domain $D \subset \mathbb{R}^n$. The module of Γ is defined by the formula:

$$M(\Gamma) := \inf_{\rho \in \operatorname{adm} \Gamma} \int_D \rho^n(x)\, dm(x)\,, \tag{1}$$

where m is the Lebesgue mesure in $\mathbb{R}^n$, and the infimum is taken over all admissible functions (metrics) $\rho : \mathbb{R}^n \to [0, \infty]$ such that

$$\int_\gamma \rho(x)\, |dx| \geq 1 \qquad \forall\, \gamma \in \Gamma\,. \tag{2}$$

$(M(\Gamma))^{-1}$ is called the extremal length of Γ.

1. Extremal length and extremal metrics

The Ahlfors – Beurling module $M(\Gamma)$ generalized the Faber (1922) "length–area method" and the Grötzsch (1928) stripes method.

H. Renggli (1952) discovered an enough simple proof of uniqueness of the extremal metric ρ. The proof is based on monotony of module that is not fulfilled for the Ahlfors – Beurling module in truth.

1. Extremal length and extremal metrics

J. Hersch (1952) made an attempt to alter the module definition. He gave up a claim of integrability of admissible metrics and suggested to understand the integral

$$M(\Gamma) := \inf_{\rho \in \operatorname{adm} \Gamma} \int_D \rho^n(x)\, dm(x) \tag{1}$$

in the upper Darboux sense and the integral

$$\int_\gamma \rho(x)\, |dx| \geq 1 \qquad \forall\, \gamma \in \Gamma \tag{2}$$

in the lower Darboux sense. But in this case, the uniqueness of extremal metric was lost as well as advantages of the Lebesgue integral.

1. Extremal length and extremal metrics

B. Fuglede (1957) suggested to consider Borelean metrics ρ in

(1) $$M(\Gamma) := \inf_{\rho \in \mathrm{adm}\,\Gamma} \int\limits_D \rho^n(x)\, dm(x)\,,$$

(2) $$\int\limits_\gamma \rho(x)\,|dx| \geq 1 \qquad \forall\, \gamma \in \Gamma$$

and local rectifiable curves γ in (2). In such a case the monotony of module holds and the uniqueness of extremal metric is attainable.

1. Extremal length and extremal metrics

In Ph.D. thesis, P.M. Tamrazov (1963) suggested a more general and universal approach. This approach is based on the definition in which the volume integral

$$(1) \qquad M(\Gamma) := \inf_{\rho \in \operatorname{adm} \Gamma} \int_D \rho^n(x)\, dm(x)$$

is taken in the Lebesgue sense while the linear integral

$$(2) \qquad \int_\gamma \rho(x)\, |dx| \geq 1 \qquad \forall \, \gamma \in \Gamma$$

is taken in the lower Darboux sense. It allowed to take into consideration all metrics L-measurable in space sense and all curves without requirement of local rectifiability. This approach provides many advantages and applications.

1. Extremal length and extremal metrics

P.M. Tamrazov investigated general properties of extremal lengths and extremal metrics:

- he proved a monotony of the module introduced by him and gave a correct proof of uniqueness of the extremal metric;
- he established the local extremal property of extremal metrics;
- he introduced a general limit modulus problem and proved the uniqueness of extremal metric for this problem.

1. Extremal length and extremal metrics

1.2. Extremal metric and modulus problems on nonorientable and twisted Riemannian manifolds

P.M. Tamrazov and V.Kh. Navoyan (1983) solved problems concerning finding extremal metrics and moduli of some nonorientable and twisted Riemannian manifolds.
P.M. Tamrazov (1988) solved also the problem concerning finding extremal metrics for Möbius strip that was tried by P.M. Pu in 1952 but not solved correctly then.
P.M. Tamrazov and S.A. Okhrimenko (1999) obtained estimations for products of moduli of families of curves on a Riemannian Möbius strip.

1. Extremal length and extremal metrics

1.3. Applications to problems of geometric function theory and potential theory

P.M. Tamrazov gave various applications of general properties of extremal lengths and extremal metrics to conformal mappings of multiply (and infinitely) connected domains and problems of the potential theory.
On such a way, he studied a problem (posed by I.P. Mitjuk) on behavior of conformal modulus of multiply connected domains under symmetrization (1971) and a problem (posed by G.D. Suvorov) on boundary behavior of conformal mapping (1974) and other problems.

1. Extremal length and extremal metrics

P.M. Tamrazov wielded masterly the method of extremal lengths and applied skilfully it in the most unexpected situations.

Let us remember a problem on plurisubharmonic extension of functions in complex topological vector spaces.

The classical M. Brelot theorem: Suppose that E is a closed set in an open set $D \subset \mathbb{R}^n$ and $\mathrm{Cap}\, E = 0$. Suppose also that a function u is subharmonic in $D \setminus E$ and bounded above on every compact subset of D. Then u can be uniquely continued to a subharmonic function on D.

The classical M. Brelot theorem: Suppose that E is a closed set in an open set $D \subset \mathbb{R}^n$ and $\operatorname{Cap} E = 0$. Suppose also that a function u is subharmonic in $D \setminus E$ and bounded above on every compact subset of D. Then u can be uniquely continued to a subharmonic function on D.

P. Lelong (1957) replaced the requirement about boundedness above of the function u on every compact subset of D by the condition of existence of a subharmonic in D function v such that for every fixed $\varepsilon > 0$ the function $u(x) + \varepsilon v(x) \to -\infty$ when $D \setminus E \ni x \to y$ for all $y \in E$. Lelong's proof appealed essentially to a subharmonicity of the superior limit of a sequence of subharmonic in D functions bounded above uniformly on every compact subset of D. But the required uniform boundedness above (an essential step on the way to the result) was not proved actually.
P.M. Tamrazov (1988) proved this fact (and more general statements) but the proof was found rather delicate.

1. Extremal length and extremal metrics

A statement similar to Lelong's statement was announced in 1969 for plurisubharmonic functions in topological complex vector spaces but was not proved completely as well as Lelong's statement.

P.M. Tamrazov has understood that his proof suitable for finite-dimensional spaces does not pass in infinitely dimensional spaces. And then P.M. Tamrazov (1989) found another elegant way of the proof using extremal lengths.

V.S. Kudiavin (1992) applied the method of extremal lengths to researching properties of mappings of Sobolev classes in the plane and multidimensional Euclidean space as well.

2. Contour-solid problems

2.1. The Warschawski – Walsh – Sewell contour-solid problem for holomorphic functions

Let $E \subset \mathbb{C}$, $f : E \to \mathbb{C}$ and $z_0 \in E$.

$\omega_E(f, z_0, \delta) := \sup\limits_{z \in E, |z - z_0| \le \delta} |f(z) - f(z_0)|$ is a local centered module of continuity;

$\omega_E(f, \delta) := \sup\limits_{z_1, z_2 \in E, |z_2 - z_1| \le \delta} |f(z_2) - f(z_1)|$ is a global module of continuity.

Let $G \subset \mathbb{C}$ be an open bounded set,
$f(z)$ be a function continuous on $\overline{G}$ and holomorphic in G;
$z_0 \in \partial G$.

2. Contour-solid problems

Problem: G — ? $\mu(\delta)$ — ? *for which:*

1) $\omega_{\partial G}(f,\delta) \le \mu(\delta) \Longrightarrow \omega_{\overline{G}}(f,\delta) \le c\,\mu(\delta), \quad c = \mathrm{const}$.

2) $\omega_{\partial G}(f,z_0,\delta) \le \mu(\delta) \Longrightarrow \omega_{\overline{G}}(f,z_0,\delta) \le c\,\mu(\delta), \quad c = \mathrm{const}$.

3) $\exists f'_{\partial G}(z_0) := \lim\limits_{z\to z_0, z\in\partial G} \frac{f(z)-f(z_0)}{z-z_0} \Longrightarrow \exists f'_{\overline{G}}(z_0) := \lim\limits_{z\to z_0, z\in\overline{G}} \frac{f(z)-f(z_0)}{z-z_0}$.

4) $f'_{\partial G}(z)$ *is continuous on* ∂G *and* $\omega_{\partial G}(f,\delta) \le \mathrm{const}\,\delta \Longrightarrow f'_{\overline{G}}(z)$ *is continuous on* $\overline{G}$.

In the case where G is a Jordan domain:

S.E. Warschawski (1934) proved assertion 2 for $\mu(\delta) = \delta^\alpha, \ \alpha > 0$;

J.L. Walsh and W.E. Sewell (1940) proved the assertion 1 for $\mu(\delta) = \delta^\alpha, \ \alpha > 0,$ with $c = 1$;

W.E. Sewell (1942) proved the assertion 1 for $\mu(\delta) = \delta|\log\delta|$;

J.L. Walsh and W.E. Sewell (1940) proved the assertion 4 and the assertion 3 (under the additional condition $\omega_{\partial G}(f,z_0,\delta) \le \mathrm{const}\,\delta$).

2. Contour-solid problems

In this relation W.E. Sewell (1942) formulated the problems:

A. To extend Warschawski – Walsh – Sewell results to domains more general than Jordan domains.

B. To prove the assertions 1, 2 for majorants $\mu(\delta) \neq \delta^\alpha$ and $\mu(\delta) \neq \delta|\log \delta|$.

C. What is the most general majorant $\mu(\delta)$ for which the assertion 1 is true in the case where G is a Jordan domain?

The problems A, B, C amount to the unified problem on contour-solid properties of holomorphic functions. It is justified to name this problem by the Warschawski – Walsh – Sewell problem. During subsequent 30 years partial results were received by many authors.

2. Contour-solid problems

P.M. Tamrazov (1972) proved the assertions 1 — 4, in particular, for any bounded open set G with connected complement and any majorant $\mu(\delta)$ of the type of module of continuity (i.e. $\mu(\delta) > 0$ for all $\delta > 0$, $\mu(+0) = 0$, $\mu(\delta)$ is a nondecreasing and semiadditive function).

He proved that the concavity of function $\nu(t) := \log \mu(\exp t)$ is necessary and sufficient condition for the truth of assertion

$2)_1 \quad \omega_{\partial G}(f, z_0, \delta) \leq \mu(\delta) \Longrightarrow \omega_{\overline{G}}(f, z_0, \delta) \leq \mu(\delta)$,

and sufficient condition for the truth of assertion

$1)_1 \quad \omega_{\partial G}(f, \delta) \leq \mu(\delta) \Longrightarrow \omega_{\overline{G}}(f, \delta) \leq \mu(\delta)$.

(i.e. with $c = 1$ in solid estimates).

2. Contour-solid problems

The developed methods have allowed essentially to expand statements of problems. In particular, in P.M. Tamrazov's papers:

a) the domain G is considered to be unbounded, multiply-connected (including infinitely connected) and, in general case, G is an open set of very general nature;

b) $\mu(\delta)$ is more general majorant named by normal, i.e. $\mu(\delta)$ is a nondecreasing function for which there exist the constants $\sigma \geq 1$ and $\gamma \geq 0$ such that

$$\mu(t\delta) \leq \sigma\, t^{\gamma} \mu(\delta) \qquad \forall\, \delta > 0 \;\; \forall\, t > 1$$

(generally speaking, $\mu(\delta)$ is discontinuous and is not semiadditive and do not satisfy the condition $\mu(+0) = 0$).

Yu.Yu. Trokhimchuk (2010) proved the equality $\omega_{\overline{G}}(f, \delta) = \omega_{\partial G}(f, \delta)$ in the case where $\overline{G}$ is compact.

2.2. Complex finite-difference smoothnesses

The problem of defining finite-difference smoothnesses of functions in complex domains attacked by many mathematicians during a long time.

P.M. Tamrazov introduced moduli of smoothness of orders $k = 2, 3, \ldots$ which are well defined for any set in the complex plane and enable to solve basic problems of the theory of finite-difference smoothnesses in a general form, under wide assumptions upon sets, majorants and functions. These moduli are free of any approximational features, are uniform with respect to inner and boundary points of sets. They are axiomatically defined on the basis of the notion of localization.

2.2. Complex finite-difference smoothnesses

A localization l is a rule (mapping) under which to each ordered collection (k, E, z, δ) (where k is a nonnegative integer, $E \subset \overline{\mathbb{C}}$, $z \in \mathbb{C}$, $\delta > 0$) there corresponds a unique set $l(k, E, z, \delta) \subset E^{k+1}$ each point $(z_0, z_1, \ldots, z_k)$ of which is simple, i.e. $z_p \neq z_q$ for all $p \neq q$.

Local and global moduli of smoothness corresponding to a given localization l are defined by formulas

$$\omega_{k,E,f,z}(\delta) := \sup_{(z_0,z_1,\ldots,z_k)\in l(k,E,z,\delta)} \left|[z_0, z_1, \ldots, z_k; f, z_0]\right|$$ (local module),

$$\omega_{k,E,f}(\delta) := \sup_{z\in E} \omega_{k,E,f,z}(\delta)$$ (global module),

where $[z_0, z_1, \ldots, z_k; f, z_0] := \prod_{j=1}^{k} (z_0 - z_j) \sum_{q=0}^{k} f(z_q) \prod_{r=0, r\neq q}^{k} (z_q - z_r)^{-1}$ is the finite difference of a function $f(z)$.

2.2. Complex finite-difference smoothnesses

Giving different localizations l, it is possible to introduce different types of moduli of smoothness. In particular, if one consider the localization l for which every set $l(k, E, z, \delta)$ consists of all points $(z_0, z_1, \ldots, z_k)$ satisfying the condition

$$(3) \quad \frac{|z_i - z_j|}{|z_p - z_q|} \le N \quad (\exists\, N \ge 1 \ \forall\, i, j, p, q = 0, 1, , \ldots, k : \ p \ne q),$$

then the corresponding moduli are called uniform. (In particular, classical moduli of smoothness on the real line correspond to the arithmetical localization and are uniform.) The moduli are called free if the constraint (3) is not imposed (or when $N = +\infty$). The introduced moduli of smoothness enabled to extend to finite-difference smoothnesses of orders $k = 2, 3, \ldots$ various results of complex constructive function theory known for $k = 1$.

2.3. Polynomial approximations

In the early seventies in contrast to functions of real variable, for functions given in a closed domain of the complex plane, the results on direct and inverse problems of polynomial approximation had usually relations to moduli of smoothness of the order $k = 1$.

Direct problem: *to study the dependence of rapidity of polynomial approximation on structural properties of functions.*
Inverse problem: *to study the dependence of structural properties of functions on the rapidity of their polynomial approximation.*

J.H. Curtiss (1936), W.E. Sewell (1938, 1942), H.M. Elliott (1951), S.N. Mergelian (1951, 1952), S.Ya. Alper (1955), V.K. Dziadyk (1959), N.A. Lebedev and N.A. Shirokov (1971) obtained results for the direct problem, and J.L. Walsh, W.E. Sewell and H.M. Elliott (1949), V.K. Dziadyk (1959, 1963) obtained results for the contour inverse problem of polynomial approximation under those or other additional restrictions on smoothness.

2.3. Polynomial approximations

V.K. Dziadyk (1959) obtained a result for the solid inverse problem of polynomial approximation of Hölder functions in domains with some good piecewise smooth Jordan boundaries, but such a problem were open for general functional classes defined by means a majorant of the type of module of continuity (the problem was posed by V.K. Dziadyk in the middle sixties).

N.A. Lebedev and P.M. Tamrazov (1970) developed a new method for solving the contour inverse problem of polynomial approximation and obtained results for boundaries of both an arbitrary bounded continuum and a wide class of compacts. P.M. Tamrazov (1971, 1973) extended these results onto the solid inverse problem of polynomial approximation for a wide class of compacts and solved the V.K. Dziadyk problem for general functional classes defined by means normal majorants.

2.3. Polynomial approximations

Using the developed theory of complex finite-difference smoothnesses of any order, P.M. Tamrazov (1975) solved open problems of approximation theory on general sets in the complex plane. He proved direct and inverse theorems of polynomial approximation and obtained constructive characterization of functions on some new classes of sets in terms of the best uniform polynomial approximations. The solution of direct and inverse problems of polynomial approximation on complex sets was obtained mainly due to the introduction of free moduli $\omega_{k,\infty,F,f}(\delta)$ of smoothness of any order k.

2.3. Polynomial approximations

Theorem (P.M. Tamrazov, 1975). *Let* F *be an arbitrary connected compact set with connected complement and let the continuous function* $f(z)$ *be given on* F *such that it is approximated on* ∂F *by polynomials* $p_n(z)$ *of degree* n *in the following rate:*

$$|f(z) - p_n(z)| \leq \mu\left(d\left(\frac{1}{n}, z\right)\right) \qquad \forall\, z \in \partial F \ \ \forall\, n = 1, 2, \dots,$$

where μ *is a normal majorant, and* $d(\frac{1}{n}, z)$ *is the distance between a point* z *and* $\frac{1}{n}$*-th level line of the outer Green function. Then*

$$\omega_{k,\infty,F,f}(\delta) \leq c\,\delta^k \int\limits_0^{ed} \frac{\mu(t)\,dt}{t^k(t+\delta)} \qquad \forall\, \delta \in (0, d]\,,$$

where d *is the diameter of* F*, and the constant* c *does not depend on* δ*.*

2.3. Polynomial approximations

P.M. Tamrazov and V.J. Gorbaichuk (1972) proved some inverse theorems of polynomial approximation on compacts of positive capacity.
P.M. Tamrazov and V.V. Bardzinskii (1976) proved local approximation theorems on complex sets.

2.4. Contour-solid results and some other results

P.M. Tamrazov developed a general contour-solid theory for holomorphic and meromorphic and subharmonic functions.

P.M. Tamrazov and A.J. Shchekhorskii (1977) proved contour-solid theorems for holomorphic functions in $\mathbb{C}^n$.

P.M. Tamrazov and T. Aliyev proved contour-solid theorems for meromorphic (1986) and finely meromorphic (2006) functions. These results take into account zeros and the multivalence of functions.

P.M. Tamrazov and A.A. Sarana (1997) studied contour-solid properties of finely hypoharmonic and finely subharmonic functions.

2.4. Contour-solid results and some other results

P.M. Tamrazov (1977) solved the problem of finite differences and modules of smoothness for superpositions of functions, that was an open problem for a long time. Using a method of solving this problem, E.W. Karupu (1978) obtained some results on finite-difference smoothnesses of conformal mappings.

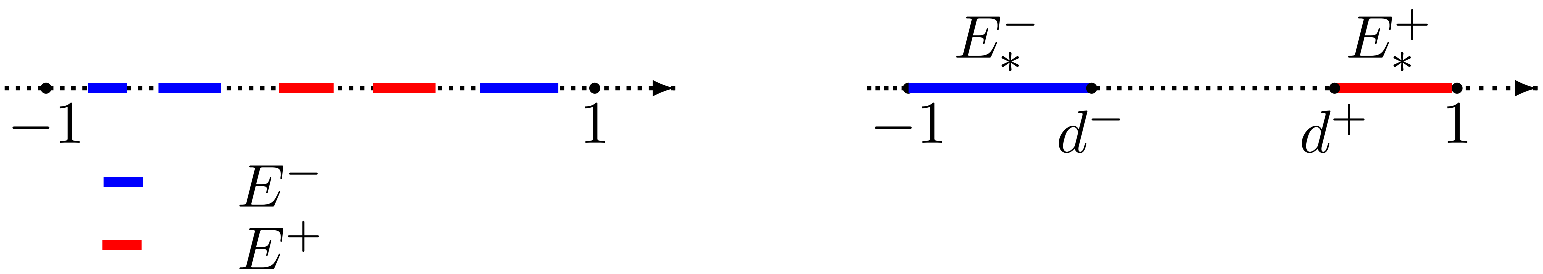
P.M. Tamrazov (1975) and O.F. Gerus (1977, 1998) studied finite-difference smothnesses of Cauchy integral operator and related singular operators. O.F. Gerus (1981) and S.A. Plaksa (1989, 1990) solved some boundary problems for analytic functions in domains with rectifiable Jordan boundaries.

3.1. Gonchar's problem on capacities

An ordered pair $(E^+, E^-) =: E$ of disjoint nonempty closed subsets E^+ and E^- of the extended complex plane $\overline{\mathbb{C}}$ is called a condenser.

Denote by Φ the class of condensers in $[-1, 1]$.
Denote by Φ_* the class of standard condensers (E^+, E^-) such that $E^+ = [d^+, 1]$ and $E^- = [-1, d^-]$, where $-1 \le d^- < d^+ \le 1$.

Let us define a single-valued mapping $s : \Phi \to \Phi_*$ which makes $E \equiv (E^+, E^-) \in \Phi$ correspond to $s(E) := E_* \equiv (E_*^+, E_*^-) \in \Phi_*$ in such a way that $\operatorname{mes} E_*^+ = \operatorname{mes} E^+$ and $\operatorname{mes} E_*^- = \operatorname{mes} E^-$, where mes denotes the linear Lebesgue measure.



$$E \equiv (E^+, E^-) \in \Phi \qquad\qquad s(E) := E_* \equiv (E_*^+, E_*^-) \in \Phi_*$$

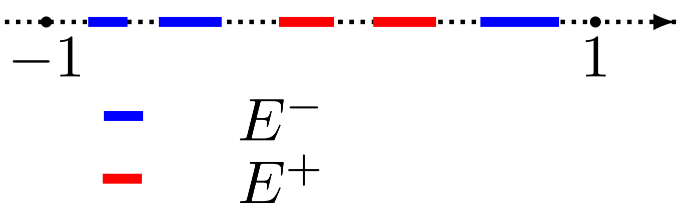
3.1. Gonchar's problem on capacities

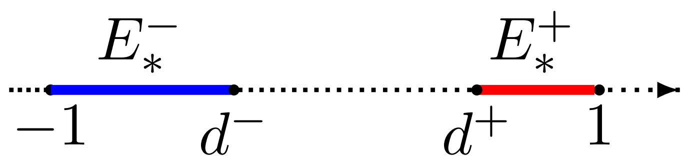
The capacity of $E \equiv (E^+, E^-)$ is denoted by $\operatorname{Cap} E$.

The Gonchar's extremal problem: Show that

$$\operatorname{Cap} E \geq \operatorname{Cap} s(E) \qquad \forall\, E \equiv (E^+, E^-) \in \Phi$$

(at least in the case where $\operatorname{mes} E^+ = \operatorname{mes} E^-$).





$E \equiv (E^+, E^-) \in \Phi$ $\qquad$ $s(E) := E_* \equiv (E_*^+, E_*^-) \in \Phi_*$

3.1. Gonchar's problem on capacities

We call a condenser $D \equiv (D^+, D^-)$ antithetic to the condenser $E \equiv (E^+, E^-) \in \Phi$ if D^+ and D^- are obtained by reflecting E^+ and E^-, respectively, in the imaginary axis.

We say that the condenser $D \equiv (D^+, D^-)$ quasi coincides with $E \equiv (E^+, E^-)$ if the sets $(D^+ \cup E^+) \setminus (D^+ \cap E^+)$ and $(D^- \cup E^-) \setminus (D^- \cap E^-)$ have zero capacity.

Theorem 3.1 (P.M. Tamrazov, 1981). *Let* $E \in \Phi$. *Then* $\operatorname{Cap} E \geq \operatorname{Cap} s(E)$, *and the equality is attained if and only if either* E *quasi coincides with* $s(E)$ *or with the condenser antithetic to* $s(E)$, *or when* $\operatorname{Cap} E = 0$.

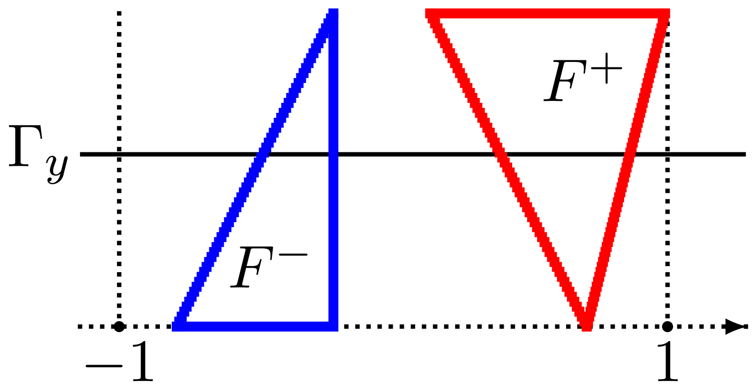
To prove Theorem 3.1, P.M. Tamrazov developed a method based on mixing signed measures (charges).

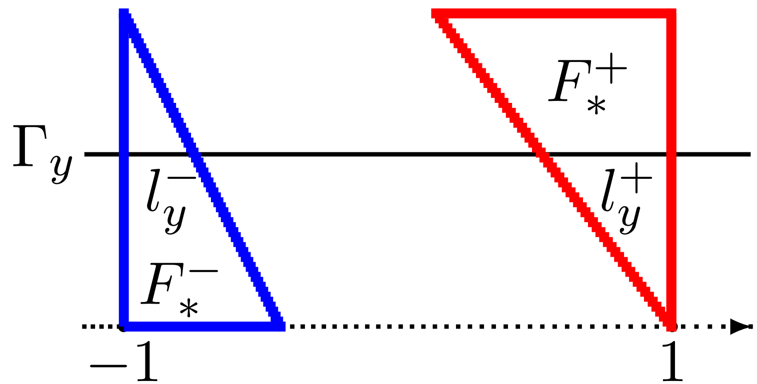
3.1. Gonchar's problem on capacities

Note that a very similar generalized conjecture seems to be intuitively plausible:

$$\operatorname{Cap} F \geq \operatorname{Cap} F_*$$

for the capacity of a condenser $F \equiv (F^+, F^-)$ in the strip $P := \{z \in \mathbb{C} : -1 \leq \operatorname{Re} z \leq 1\}$ as compared to the capacity of the condenser $F_* \equiv (F_*^+, F_*^-)$ related to F in the following way:
on a horizontal line $\Gamma_y := \{z \in \mathbb{C} : \operatorname{Im} z = y\}$ each of the sets
$l_y^+ := F_*^+ \cap \Gamma_y$ and $l_y^- := F_*^- \cap \Gamma_y$ lies in P,
$\operatorname{mes} l_y^+ = \operatorname{mes} F^+ \cap \Gamma_y$ and $\operatorname{mes} l_y^- = \operatorname{mes} F^- \cap \Gamma_y$,
and the right-hand end of l_y^+ and the left-hand end of l_y^- lie on ∂P.



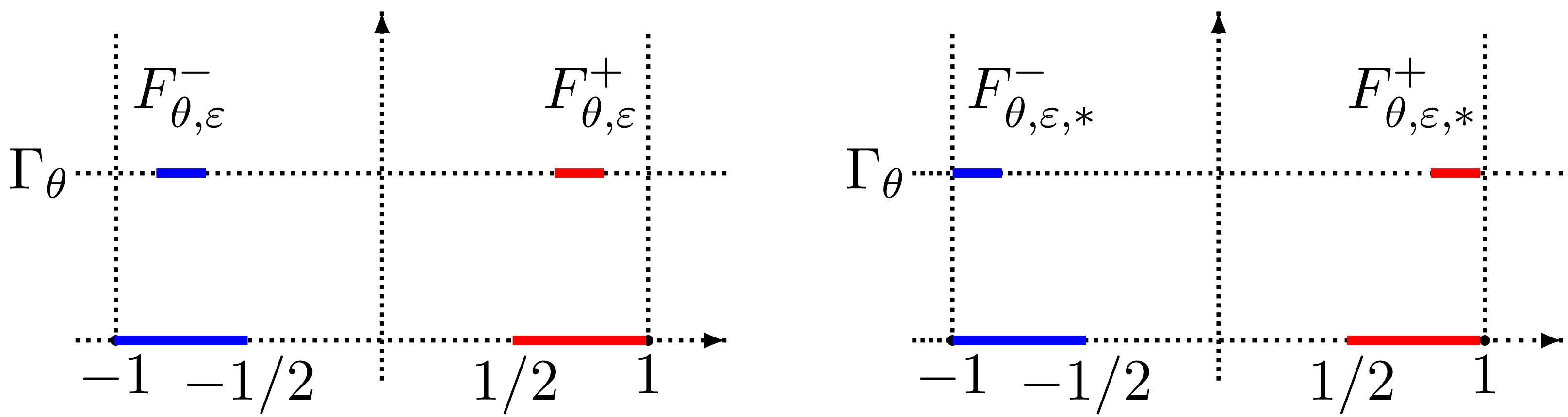


3.1. Goncharâ€™s problem on capacities

But it turns out that this generalized conjecture is false.

In fact, let $F^+_{\theta,\varepsilon}$ consist of the segments $\{z \in \mathbb{C} : 1/2 \le z \le 1\}$ and $\{z \in \mathbb{C} : \operatorname{Im} z = \theta,\ |\operatorname{Re} z - 3/4| \le \varepsilon\}$, and let $F^-_{\theta,\varepsilon}$ be the reflection of $F^+_{\theta,\varepsilon}$ in the imaginary axis.

Let $F^+_{\theta,\varepsilon,*}$ consist of the segments $\{z \in \mathbb{C} : 1/2 \le z \le 1\}$ and $\{z \in \mathbb{C} : \operatorname{Im} z = \theta,\ 1 - 2\varepsilon \le \operatorname{Re} z \le 1\}$, and let $F^-_{\theta,\varepsilon,*}$ be the reflection of $F^+_{\theta,\varepsilon,*}$ in the imaginary axis.

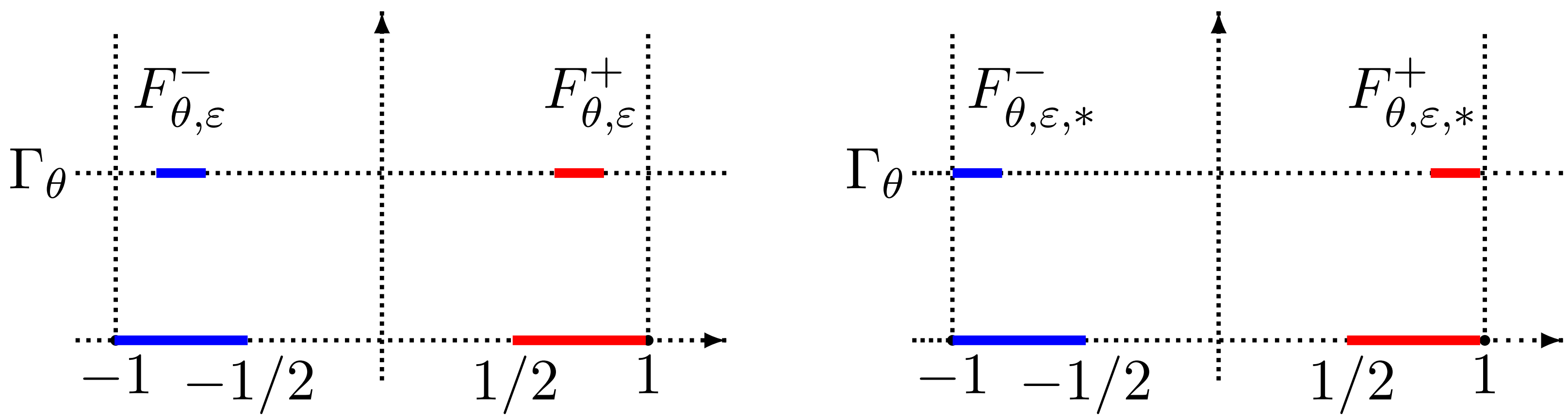


3.1. Gonchar's problem on capacities

Theorem 3.2 (P.M. Tamrazov, 1981). *There is a number* $\theta_0 \in (0, 1/8]$ *such that if* $\theta \in (0, \theta_0]$ *and* $\varepsilon \in [\theta/2, \theta]$*, then the capacities of the condensers* $F_{\theta,\varepsilon} := (F^+_{\theta,\varepsilon}, F^-_{\theta,\varepsilon})$ *and* $F_{\theta,\varepsilon,*} := (F^+_{\theta,\varepsilon,*}, F^-_{\theta,\varepsilon,*})$ *satisfy the inequality*

$$\operatorname{Cap} F_{\theta,\varepsilon} < \operatorname{Cap} F_{\theta,\varepsilon,*} .$$

The proof of Theorem 3.2 is based on the construction of certain families of curves, the effective construction of a conformally invariant metric for them, and the deduction of two-sided estimates for the conformal moduli of the mentioned families.



3.2. Other problems of potential theory

P.M. Tamrazov (1998) solved Eremenko's extremal problem on harmonic functions.

P.M. Tamrazov (1999) solved contour-solid and cluster problems for finely holomorphic and finely subharmonic functions; he solved also some other problems of fine potential theory.

P.M. Tamrazov (2001) solved problems on minimization of energy of charges on batteries of condensers.

N.V. Zorii (1991) solved noncompact essentially problems of the potential theory.

4. Extremal problems for conformal mappings

P.M. Tamrazov (1968) developed methods for solving extremal problems associated with multipole quadratic differentials having free poles.

Let $K_R := \{z \in \mathbb{C} : R < |z| < 1\}$ be a ring, where $R \in (0, 1)$, $C_r := \{z \in \mathbb{C} : |z| = r\}$ be a circle.
Let $\mathcal{F}(R)$ be the class of univalent conformal mappings $f : K_R \to \mathbb{C}$ for which the bounded component of $\mathbb{C} \setminus f(K_R)$ contains the points $0, 1$ and a continuum $f(C_1)$.

Problem: Among all mappings $f \in \mathcal{F}(R)$ to find such f for which the functional $|f'(z_0)|$ is minimal for a fixed $z_0 \in K_R$.

4. Extremal problems for conformal mappings

Let D_R be a doubly-connected domain equivalent conformally to the ring K_R with the boundary consisting of the intervals $[-\infty, -t]$ and $[0, 1]$.
Let $g_R(z)$ be the function with the properties:
$g_R(z) \in \mathcal{F}(R), \;\; g_R(K_R) = D_R, \;\; g_R(1) = 1$.

Theorem 4.1 (P.M. Tamrazov, 1968). *For any function $f \in \mathcal{F}(R)$, the following inequality holds:*

$$|f'(z)| \geq g'_R(-|z|)\,.$$

Moreover, for every fixed point $z = z_0$, the equality is attained only for either

$$f(z) \equiv g_R\left(-z\frac{|z_0|}{z_0}\right) \quad \text{or} \quad f(z) \equiv 1 - g_R\left(-z\frac{|z_0|}{z_0}\right).$$

4. Extremal problems for conformal mappings

Using Theorem 4.1, P.M. Tamrazov (1968) solved some extremal problems for conformal mappings associated with multipole quadratic differentials having 5 free poles.

Using the principal idea to use quadratic differentials with free poles, G.P. Bakhtina (1974) and A.K. Bakhtin (2006) developed the theory of extremal problems for nonoverlapping domains.

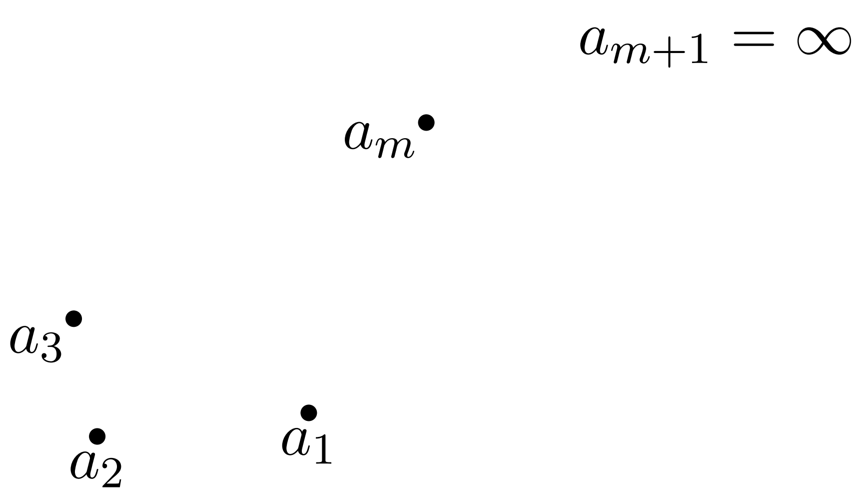
4. Extremal problems for conformal mappings

Parameterization for extremals of Tchebotaröv's problem

Let $K := \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in $\mathbb{C}$.

Let $\{a_j\} := \{a_j\}_{j=1}^{m+1} \subset \overline{\mathbb{C}} \setminus \{0\}$ with $a_{m+1} = \infty$, where $a_1, a_2 \ldots, a_m$ is an unordered collection.

One of equivalent formulations of Tchebotaröv's problem: Among all univalent conformal mappings $f : K \to \overline{\mathbb{C}} \setminus \{a_j\}$ with $f(0) = 0$, to find such f for which the functional $|f'(0)|$ is maximal.



4. Extremal problems for conformal mappings

It is known that the extremal function f of Tchebotaröv's problem satisfies the following functional-differential equation:

$$\left(\frac{zf'(z)}{f(z)}\right)^2 = \frac{p(f(z))}{q(f(z))},$$

where $p(w) := \prod_{j:a_j \neq \infty} (a_j - w)$, and q is a polynomial of the degree $m-1$ with $q(0) = p(0)$ (which is uniquely determined by the collection of points $\{a_j\}$). The quadratic differential of Tchebotaröv's problem

$$Q(w)dw^2 = -\frac{q(w)}{w^2 p(w)}\, dw^2 .$$

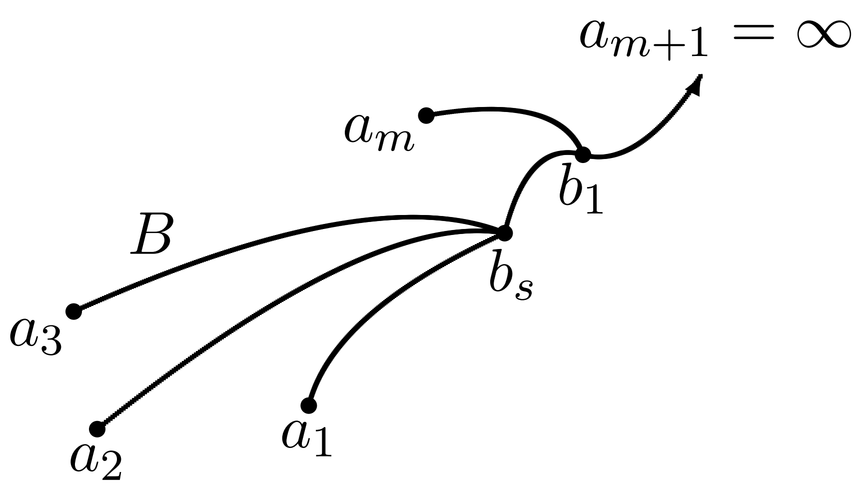
4. Extremal problems for conformal mappings

Let $B := \overline{\mathbb{C}} \setminus f(K)$. One may consider B as an undirected, connected, simple, acyclic, plane graph (it is a tree) on $\overline{\mathbb{C}}$ consisting of:

1) nodes of order one at all points a_j and only at them,

2) nodes of orders $\nu_s + 2$ at all zeros b_s of the order $\nu_s \geq 1$ of $Q(w)dw^2$ and only at them,

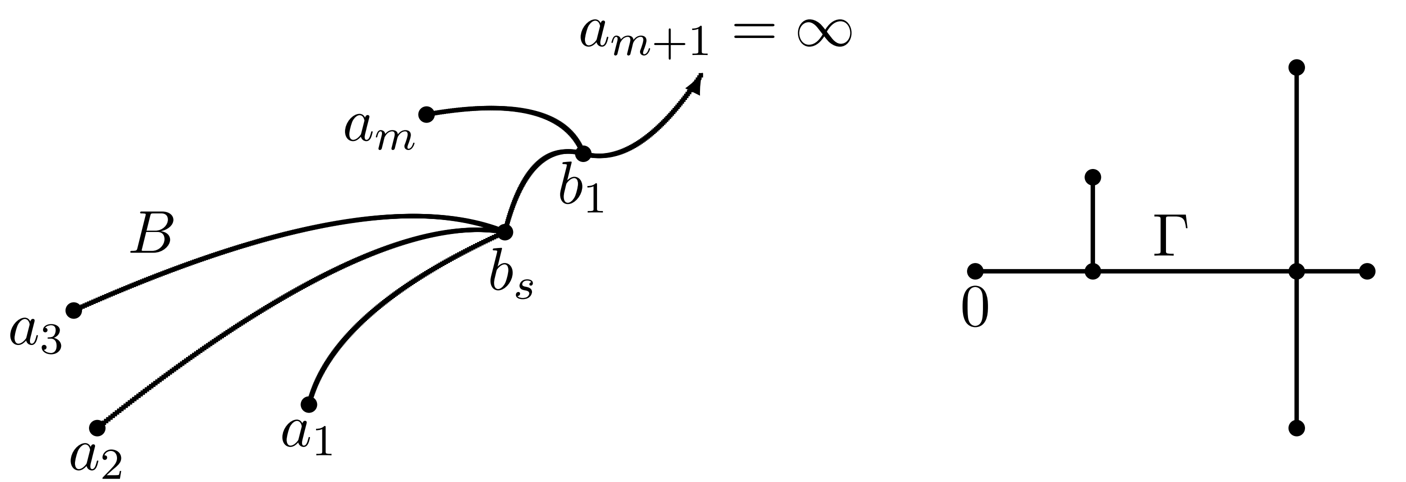
3) all critical analytic trajectories of $Q(w)dw^2$ (ending at zeros or simple poles of $Q(w)dw^2$) as edges of the graph.



4. Extremal problems for conformal mappings

P.M. Tamrazov proved that B is isomorphic (in a certain sense) to a rectilinear graph (tree) Γ for which the sum of lengths of all segments (composing Γ) equals π, with the correspondence of the nodes

$\Gamma \ni 0 \longleftrightarrow \infty \in B$.



4. Extremal problems for conformal mappings

Theorem 4.2 (P.M. Tamrazov, 2005). *For every* Γ *under consideration, there is a function* ϕ_T *with the following properties:*

1) ϕ_T *is holomorhic and univalent in* K*, and continuous on* $\overline{K} \setminus \{1\}$*;*

2) $\phi_T(\overline{K}) = \overline{\mathbb{C}}$*,* $\phi_T(0) = 0$*,* $\phi'_T(0) = 1$*,* $\phi_T(1) = \infty$*;*

3) ϕ_T *is continuous (with respect to topology of* $\overline{\mathbb{C}}$ *in the image) on* $\overline{K}$*;*

4) ϕ_T *is extremal in Tchebotaröv's problem for the collection of points* $\{a_j\}$ *which are completely defined by* Γ *and* ϕ_T*.*

The extremal function $f(z)$ *in this problem for the mentioned collection of points* $\{a_j\}$ *is unique up to rotation of the disc* K *in the* z*-plane around the origin.*

4. Extremal problems for conformal mappings

P.M. Tamrazov introduced a certain relation of equivalence in the set G of all considered rectilinear graphs.

Let $\widetilde{G}$ denote the factor-set of G with respect to the equivalence.

For a graph $\Gamma \in G$, let $\widetilde{\Gamma}$ denote the class of all graphs equivalent to Γ (with the "same configuration").

Theorem 4.3 (P.M. Tamrazov, 2005). *The class of all extremals of Tchebotaröv's problem is parametrized by elements of the set* $\widetilde{G} \times \partial K$*, and this parametrization is one-to-one correspondence:*

1) to every element $\widetilde{\Gamma} \in \widetilde{G}$ *and any* $t \in \partial K$ *there corresponds one (and only one) point collection* $\{a_j\}$ *for which the function* $f(z) := t\phi_T(z)$ *(with* ϕ_T *mentioned in Theorem 4.2 and any graph* $\Gamma \in \widetilde{\Gamma}$*) is extremal in Tchebotaröv's problem, and conversely,*

2) for every collection of points $\{a_j\}$ *there exists one and only one class* $\widetilde{\Gamma} \in \widetilde{G}$ *and the single* $t \in \partial K$ *such that the function* $f(z) := t\phi_T(z)$ *is extremal in Tchebotaröv's problem for* $\{a_j\}$*.*

4. Extremal problems for conformal mappings

P.M. Tamrazov (2010) established a parameterization for extremals of the problem formulated by H. Grötzsch in 1930 as the hyperbolic analog of Tchebotaröv's problem. He established also a parameterization for extremals of some generalization of Tchebotaröv's problem.

P.M. Tamrazov has died on 11.02.2012 after a prolonged illness.

Promarz Melikovich Tamrazov was a remarkable person, being very kind, responsive and exceptionally attentive to the people. Mathematics was his true love, a sense and happiness in all his life. In our contacts and mathematical discussions with him, we were convinced repeatedly by his great mathematical talent. Sometimes the impression was created that there are practically no mathematical difficulties for him. He had tremendous mathematical intuition and shared generously his ideas with disciples.

We have a good luck to meet this eminent person on our creative way. He will remain in our memory as an intellectually gifted and outstanding person for ever.