Using Engel series
for construction of continuous functions
with complicated local properties

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We construct and study the infinite-parameter family of continuous functions with complicated local properties:

\[
f(\Delta^E_{g_1(x)g_2(x)\ldots g_n(x)}\ldots) = r_{g_1(x)} + \sum_{k=2}^{\infty} \left( r_{g_k(x)} \prod_{i=1}^{k-1} u_{g_i(x)} \right). \tag{1}
\]

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O. Baranovskyi and M. Pratsiovytyi, One class of continuous functions with complicated local properties related to Engel series, Manuscript.

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These slides are available at
https://www.imath.kiev.ua/~baranovskyi/talks/20210604msta5.pdf
Engel series, $E$-representation

For any $x \in (0, 1]$, there exists a unique sequence $(g_n)$, $g_n \in \mathbb{Z}_0 = \{0, 1, 2, \ldots\}$, such that

\[
    x = \sum_{n=1}^{\infty} \frac{1}{(2 + g_1)(2 + g_1 + g_2) \cdots (2 + g_1 + g_2 + \cdots + g_n)} \equiv \Delta^E_{g_1g_2\ldots g_n\ldots}.
\]

The series (2) is called *Engel series*, symbolic notation (3) is called *$E$-representation* of the number $x$, and $g_n = g_n(x)$ is $n$th symbol (digit) of this representation.
Let \((u_n)_{n=0}^\infty\) be an infinite sequence, \(u_n \in \mathbb{R}\):

\[
\sum_{n=0}^{\infty} u_n = u_0 + u_1 + \ldots + u_n + r_n = S_n + r_n = 1; \quad (4)
\]

\(|u_n| < 1 \quad \text{for any } n \in \mathbb{Z}_0; \quad (5)
\]

\[0 < r_n \equiv \sum_{i=n+1}^{\infty} u_i < 1 \quad \text{for any } n \in \mathbb{Z}_0. \quad (6)\]
Definition of the function $I$

Let $x \in (0, 1]$,

$$x = \sum_{n=1}^{\infty} \frac{1}{(2 + g_1)(2 + g_1 + g_2) \cdots (2 + g_1 + g_2 + \cdots + g_n)} \quad (7)$$

$$\equiv \Delta^E_{g_1 g_2 \cdots g_n \cdots}, \quad (8)$$

and let $(u_n)_{n=0}^{\infty}$ be a sequence with the given properties. Then

$$f(x) = r_{g_1} + \sum_{k=2}^{\infty} \left( r_{g_k} \prod_{i=1}^{k-1} u_{g_i} \right) \quad (9)$$

$$\equiv \Delta_{g_1 g_2 \cdots g_n \cdots}, \quad (10)$$

where $g_n = g_n(x)$ is $n$th symbol of $E$-representation of the number $x \in (0, 1]$. 
Definition of the function II

1. The function $f$ is well defined.

2. The function $f$ is continuous at any point of interval $(0, 1)$, and it is right-continuous at the point $x = 0$, left-continuous at the point $x = 1$. (Put $f(0) = 0$.)

3. The set of values of the function $f$ is a closed interval $[0, 1]$.

4. The function $f$ is a unique solution of the system of functional equations

$$f(x) = r_i + u_i f(\omega(x)), \quad i \in \mathbb{Z}_0,$$  \hspace{1cm} (11)

in the class of bounded functions defined at every point of $(0, 1]$, where $\omega(\Delta^E_{g_1(x)g_2(x)...g_n(x)...}) = \Delta^E_{g_2(x)g_3(x)...g_n(x)...}$ is a shift operator on symbols of $E$-representation of a number.
Most of continuous on the unit interval functions have complicated local properties.

In particular, singular functions (their derivative is equal to zero almost everywhere with respect to Lebesgue measure), nowhere monotonic functions (they do not have any arbitrary small monotonicity interval), and nowhere differentiable functions (they do not have derivative in any point) are among them.

There exist some problems in development of general as well as individual theory of such functions. The reason is an absence of effective means of their definition (description) and tools for their study.
Infinite series, infinite products, continued fractions, systems of functional equations, iterated function systems, automata with finite memory, and other tools and methods are often used to model and study such functions.

We use the so-called $E$-representation of real numbers to construct and study the infinite-parameter family of continuous functions with complicated local properties.
Motivation III

Classic strictly increasing singular function (Salem function):

Let $x \in [0, 1]$,

$$x = \frac{\alpha_1}{2} + \frac{\alpha_2}{2^2} + \ldots + \frac{\alpha_n}{2^n} + \ldots \equiv \Delta^2_{\alpha_1 \alpha_2 \ldots \alpha_n}, \quad (12)$$

$$S(x) = \beta_{\alpha_1} + \sum_{k=2}^{\infty} \left( \beta_{\alpha_k} \prod_{i=1}^{k-1} p_{\alpha_i} \right) \equiv \Delta^{Q_2}_{\alpha_1 \alpha_2 \ldots \alpha_n}, \quad (13)$$

where $0 < p_0, p_1 < 1$, $p_0 + p_1 = 1$, $\beta_0 = 0$, $\beta_1 = p_0$.

Equivalent definition:

$$\begin{cases} 
S \left( \frac{0+x}{2} \right) = \beta_0 + p_0 S(x), \\
S \left( \frac{1+x}{2} \right) = \beta_1 + p_1 S(x).
\end{cases}$$
A cylinder of rank $m$ with base $c_1c_2\ldots c_m$ is the set $\Delta^E_{c_1c_2\ldots c_m}$ of all numbers $x \in (0,1]$ having $E$-representation with first $m$ symbols $c_1$, $c_2$, $\ldots$, $c_m$ respectively, i.e.,

$$\Delta^E_{c_1c_2\ldots c_m} = \left\{ x : x = \Delta^E_{g_1g_2\ldots g_mg_{m+1}\ldots g_{m+k}\ldots}, g_i = g_i(x) = c_i, \ i = 1, m \right\}.$$

**Lemma**

If $u_p = 0$, then function $f$ is constant on every cylinder $\Delta^E_{c_1c_2\ldots c_m}$.

**Corollary**

If $(c_1, c_2, \ldots, c_m)$, $c_i \in \mathbb{Z}_0$, such that

$$u_{c_1} u_{c_2} \ldots u_{c_m} = 0,$$

then function $f$ is constant on cylinder $\Delta^E_{c_1c_2\ldots c_m}$.
Probability distribution function \((u_n \geq 0)\)

**Theorem**

If \(u_n \geq 0\) for any \(n \in \mathbb{Z}_0\), then \(f\) is

1. a probability distribution function on \([0, 1]\), moreover, it is a distribution function of random variable

\[
\xi = \sum_{k=1}^{\infty} \frac{1}{(2 + \eta_1)(2 + \eta_1 + \eta_2) \ldots (2 + \eta_1 + \eta_2 + \ldots + \eta_k)}
\]

\[= \Delta_{\eta_1 \eta_2 \ldots \eta_k}^E \]

such that its \(E\)-symbols \(\eta_k\) are i.i.d. random variables having the distribution \(P\{\eta_k = n\} = u_n\);

2. a strictly increasing function if \(u_n > 0\) for any \(n \in \mathbb{Z}_0\);

3. a pure absolutely continuous or pure singularly continuous function.
Nowhere monotonic function \((u_p < 0)\)

**Theorem**

*If sequence \((u_n)\) does not contain zeroes but contains negative terms, then function \(f\) is nowhere monotonic on \([0, 1]\) (i.e., it does not have any arbitrary small monotonicity interval).*
Level sets

Level set $y_0$ of function $f$ is a set

$$f^{-1}(y_0) = \{x : f(x) = y_0\}.$$ 

**Theorem**

*If there exist negative terms in the sequence $(u_n)$ and $E$-representation of number $x = \Delta^E_{g_1(x)g_2(x)...g_n(x)...}$ has the following property:*

$$u_{g_i(x)}u_{g_{i+1}(x)} < 0$$  \hspace{1cm} (14)

*for infinite set of values $i \in \mathbb{N}$, then level set $f^{-1}(y_0)$, where $y_0 = f(x)$, is a countable set.*
"Symmetries" of the graph

Theorem

Graph $\Gamma_f$ of the function $f$ is a scale-invariant set, namely:

$$\Gamma_f = \Delta^E_{(0)} \cup \bigcup_{i=0}^{\infty} \Gamma_i,$$

where $\Gamma_i = \varphi_i(\Gamma_f)$,

and

$$\varphi_i: \begin{cases} x' = \delta_i(x) \equiv \Delta^E_{ig_1(x)g_2(x)\ldots g_n(x)}, \\ y' = r_i + u_i f(x). \end{cases}$$
Theorem

For Lebesgue integral,

\[ \int_{0}^{1} f(x) \, dx \leq \left( 1 - \sum_{n=0}^{\infty} \frac{u_n}{2+n} \right)^{-1} \sum_{n=0}^{\infty} \frac{r_n}{2+n}. \]  

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There exist some problems in studying continuous functions with complicated local properties (singular, nowhere monotonic, and nowhere differentiable functions). Effective means and tools for their analytical definition and study does not exist.

We construct and study the infinite-parameter family of continuous functions with complicated local properties using the so-called $E$-representation of a real number, i.e., its encoding by infinite alphabet in the form of Engel series (positive series such that their terms are reciprocal to cumulative products of natural numbers).