Using Engel series for construction of continuous functions with complicated local properties

Oleksandr Baranovskyi[2] Mykola Pratsiovytyi[1,2]

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[1]Faculty of Physics and Mathematics, National Pedagogical Dragomanov University
[2]Institute of Mathematics, National Academy of Sciences of Ukraine

Outline

We construct and study the infinite-parameter family of continuous functions with complicated local properties:

$$f(\Delta^E_{g_1(x)g_2(x)\ldots g_n(x)\ldots}) = r_{g_1(x)} + \sum_{k=2}^{\infty}\left(r_{g_k(x)} \prod_{i=1}^{k-1} u_{g_i(x)}\right). \qquad (1)$$

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These slides are available at
`https://www.imath.kiev.ua/~baranovskyi/talks/20210604msta5.pdf`

Engel series, E-representation

For any $x \in (0,1]$, there exists a unique sequence (g_n), $g_n \in \mathbb{Z}_0 = \{0, 1, 2, \dots\}$, such that

$$x = \sum_{n=1}^{\infty} \frac{1}{(2+g_1)(2+g_1+g_2)\dots(2+g_1+g_2+\dots+g_n)} \tag{2}$$

$$\equiv \Delta^E_{g_1 g_2 \dots g_n \dots}. \tag{3}$$

The series (2) is called *Engel series*, symbolic notation (3) is called *E-representation* of the number x, and $g_n = g_n(x)$ is nth symbol (digit) of this representation.

Generating sequence

Let $(u_n)_{n=0}^{\infty}$ be an infinite sequence, $u_n \in \mathbb{R}$:

$$\sum_{n=0}^{\infty} u_n = u_0 + u_1 + \ldots + u_n + r_n = S_n + r_n = 1; \tag{4}$$

$$|u_n| < 1 \quad \text{for any } n \in \mathbb{Z}_0; \tag{5}$$

$$0 < r_n \equiv \sum_{i=n+1}^{\infty} u_i < 1 \quad \text{for any } n \in \mathbb{Z}_0. \tag{6}$$

Definition of the function I

Let $x \in (0, 1]$,

$$x = \sum_{n=1}^{\infty} \frac{1}{(2+g_1)(2+g_1+g_2)\ldots(2+g_1+g_2+\ldots+g_n)} \tag{7}$$
$$\equiv \Delta^{E}_{g_1 g_2 \ldots g_n \ldots}, \tag{8}$$

and let $(u_n)_{n=0}^{\infty}$ be a sequence with the given properties. Then

$$f(x) = r_{g_1} + \sum_{k=2}^{\infty} \left(r_{g_k} \prod_{i=1}^{k-1} u_{g_i} \right) \tag{9}$$
$$\equiv \Delta_{g_1 g_2 \ldots g_n \ldots}, \tag{10}$$

where $g_n = g_n(x)$ is nth symbol of E-representation of the number $x \in (0, 1]$.

Definition of the function II

1. The function f is well defined.
2. The function f is continuous at any point of interval $(0, 1)$, and it is right-continuous at the point $x = 0$, left-continuous at the point $x = 1$. (Put $f(0) = 0$.)
3. The set of values of the function f is a closed interval $[0, 1]$.
4. The function f is a unique solution of the system of functional equations

   $$f(x) = r_i + u_i f(\omega(x)), \quad i \in \mathbb{Z}_0, \tag{11}$$

   in the class of bounded functions defined at every point of $(0, 1]$, where $\omega(\Delta^E_{g_1(x)g_2(x)\ldots g_n(x)\ldots}) = \Delta^E_{g_2(x)g_3(x)\ldots g_n(x)\ldots}$ is a shift operator on symbols of E-representation of a number.

Motivation I

Most of continuous on the unit interval functions have complicated local properties.

In particular, singular functions (their derivative is equal to zero almost everywhere with respect to Lebesgue measure), nowhere monotonic functions (they do not have any arbitrary small monotonicity interval), and nowhere differentiable functions (they do not have derivative in any point) are among them.

There exist some problems in development of general as well as individual theory of such functions. The reason is an absence of effective means of their definition (description) and tools for their study.

Motivation II

Infinite series, infinite products, continued fractions, systems of functional equations, iterated function systems, automata with finite memory, and other tools and methods are often used to model and study such functions.

We use the so-called E-representation of real numbers to construct and study the infinite-parameter family of continuous functions with complicated local properties.

Motivation III

Classic strictly increasing singular function (Salem function):

Let $x \in [0,1]$,

$$x = \frac{\alpha_1}{2} + \frac{\alpha_2}{2^2} + \ldots + \frac{\alpha_n}{2^n} + \ldots \equiv \Delta^2_{\alpha_1\alpha_2\ldots\alpha_n\ldots}, \tag{12}$$

$$S(x) = \beta_{\alpha_1} + \sum_{k=2}^{\infty}\left(\beta_{\alpha_k}\prod_{i=1}^{k-1} p_{\alpha_i}\right) \equiv \Delta^{Q_2}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}, \tag{13}$$

where $0 < p_0, p_1 < 1$, $p_0 + p_1 = 1$, $\beta_0 = 0$, $\beta_1 = p_0$.

Equivalent definition: $\begin{cases} S\left(\dfrac{0+x}{2}\right) = \beta_0 + p_0 S(x), \\ S\left(\dfrac{1+x}{2}\right) = \beta_1 + p_1 S(x). \end{cases}$

Monotonicity intervals ($u_p = 0$)

A *cylinder of rank m with base $c_1c_2\dots c_m$* is the set $\Delta^E_{c_1c_2\dots c_m}$ of all numbers $x \in (0,1]$ having E-representation with first m symbols c_1, c_2, ..., c_m respectively, i.e.,

$$\Delta^E_{c_1c_2\dots c_m} = \left\{x \colon x = \Delta^E_{g_1g_2\dots g_m g_{m+1}\dots g_{m+k}\dots},\ g_i = g_i(x) = c_i,\ i = \overline{1,m}\right\}.$$

Lemma

If $u_p = 0$, then function f is constant on every cylinder $\Delta^E_{c_1c_2\dots c_m p}$.

Corollary

If $(c_1, c_2, \dots, c_m)$, $c_i \in \mathbb{Z}_0$, such that

$$u_{c_1}u_{c_2}\dots u_{c_m} = 0,$$

then function f is constant on cylinder $\Delta^E_{c_1c_2\dots c_m}$.

Probability distribution function ($u_n \geq 0$)

Theorem

If $u_n \geq 0$ for any $n \in \mathbb{Z}_0$, then f is

1. *a probability distribution function on $[0,1]$, moreover, it is a distribution function of random variable*

$$\xi = \sum_{k=1}^{\infty} \frac{1}{(2+\eta_1)(2+\eta_1+\eta_2)\ldots(2+\eta_1+\eta_2+\ldots+\eta_k)}$$
$$= \Delta^E_{\eta_1\eta_2\ldots\eta_k\ldots}$$

   such that its E-symbols η_k are i.i.d. random variables having the distribution $\mathrm{P}\{\eta_k = n\} = u_n$;

2. *a strictly increasing function if $u_n > 0$ for any $n \in \mathbb{Z}_0$;*
3. *a pure absolutely continuous or pure singularly continuous function.*

Nowhere monotonic function ($u_p < 0$)

Theorem

If sequence (u_n) *does not contain zeroes but contains negative terms, then function* f *is nowhere monotonic on* $[0, 1]$ (*i.e., it does not have any arbitrary small monotonicity interval*).

Level sets

Level set y_0 of function f is a set

$$f^{-1}(y_0) = \{x\colon f(x) = y_0\}.$$

Theorem

If there exist negative terms in the sequence (u_n) and E-representation of number $x = \Delta^E_{g_1(x)g_2(x)\ldots g_n(x)\ldots}$ has the following property:

$$u_{g_i(x)} u_{g_{i+1}(x)} < 0 \tag{14}$$

for infinite set of values $i \in \mathbb{N}$, then level set $f^{-1}(y_0)$, where $y_0 = f(x)$, is a countable set.

“Symmetries” of the graph

Theorem

Graph Γ_f of the function f is a scale-invariant set, namely:

$$\Gamma_f = \Delta^E_{(0)} \cup \bigcup_{i=0}^{\infty} \Gamma_i, \quad \textit{where} \quad \Gamma_i = \varphi_i(\Gamma_f),$$

and

$$\varphi_i \colon \begin{cases} x' = \delta_i(x) \equiv \Delta^E_{ig_1(x)g_2(x)\ldots g_n(x)\ldots}, \\ y' = r_i + u_i f(x). \end{cases}$$

Integral

Theorem

For Lebesgue integral,

$$\int_0^1 f(x)\,dx \leq \left(1 - \sum_{n=0}^{\infty} \frac{u_n}{2+n}\right)^{-1} \sum_{n=0}^{\infty} \frac{r_n}{2+n}. \tag{15}$$

References I

W. Sierpiński,
***O kilku algorytmach dla rozwijania liczb rzeczywistych na szeregi*,**
Sprawozdania z posiedzeń Towarzystwa Naukowego Warszawskiego, Wydział III **4** (1911), 56–77;
French translation: *Sur quelques algorithmes pour développer les nombres réels en séries*, Oeuvres choisies, tm. I, PWN, Warszawa, 1974, pp. 236–254.

F. Engel,
***Entwicklung der Zahlen nach Stammbrüchen*,**
Verh. d. 52. Versamml. dtsch. Philologen u. Schulmänner Marburg 1913, Teubner, Leipzig, 1914, pp. 190–191.

References II

P. Erdős, A. Rényi, and P. Szüsz,
***On Engel's and Sylvester's series*,**
Ann. Univ. Sci. Budapest. Sect. Math. **1** (1958), 7–32.

A. Rényi,
***A new approach to the theory of Engel's series*,**
Ann. Univ. Sci. Budapest. Sect. Math. **5** (1962), 25–32.

M. V. Pratsiovytyi and B. I. Hetman,
***Engel series and their applications*,**
Nauk. Chasop. Nats. Pedagog. Univ. Mykhaila Drahomanova. Ser. 1. Fiz.-Mat. Nauky (2006), no. 7, 105–116 (in Ukrainian).

- M. V. Pratsiovytyi and O. M. Baranovskyi,
***Properties of distributions of random variables with independent differences of consecutive elements of the Ostrogradskiĭ series*,**
Teor. Ĭmovīr. Mat. Stat. (2004), no. 70, 131–143
(in Ukrainian); translation in Theory Probab. Math. Statist. (2005) no. 70, 147–160.

- M. Pratsiovytyi and Yu. Khvorostina,
***Topological and metric properties of distributions of random variables represented by the alternating Lüroth series with independent elements*,**
Random Oper. Stoch. Equ. **21** (2013), no. 4, 385–401.

References IV

- S. Albeverio, M. Pratsiovytyi, and G. Torbin,
  Singular probability distributions and fractal properties of sets of real numbers defined by the asymptotic frequencies of their s-adic digits,
  Ukr. Math. Zhurn. **57** (2005), no. 9, 1163–1170; also in Ukrainian Math. J. **57** (2005), no. 9, 1361–1370.

- S. Albeverio, Ya. Gontcharenko, M. Pratsiovytyi, and G. Torbin,
  Jessen–Wintner type random variables and fractal properties of their distributions,
  Math. Nachr. **279** (2006), no. 15, 1619–1633.

References V

M. V. Pratsiovytyi,
***Continuous Cantor projectors*,**
Methods of investigation of algebraic and topological structures, Kyiv State Pedagog. Inst., Kyiv, 1989, pp. 95–105 (in Russian).

M. V. Pratsiovytyi,
***Fractal properties of one continuous nowhere differentiable function*,**
Nauk. Zap. Nats. Pedagog. Univ. Mykhaila Drahomanova. Fiz.-Mat. Nauky (2002), no. 3, 351–362 (in Ukrainian).

K. A. Bush,
***Continuous functions without derivatives*,**
Amer. Math. Monthly **59** (1952), no. 4, 222–225.

W. Wunderlich,
***Eine überall stetige und nirgends differenzierbare Funktion*,**
Elem. Math. **7** (1952), no. 4, 73–79.

O. B. Panasenko,
***Fractal properties of one class of one-parameter continuous nowhere differentiable functions*,**
Nauk. Chasop. Nats. Pedagog. Univ. Mykhaila Drahomanova. Ser. 1. Fiz.-Mat. Nauky (2006), no. 7, 160–167 (in Ukrainian).

- O. B. Panasenko,
  ***Fractal dimension of graphs of continuous Cantor projectors*,**
  Nauk. Chasop. Nats. Pedagog. Univ. Mykhaila Drahomanova. Ser. 1. Fiz.-Mat. Nauky (2008), no. 9, 104–111 (in Ukrainian).
- O. B. Panasenko,
  ***Hausdorff–Besicovitch dimension of the graph of one continuous nowhere-differentiable function*,**
  Ukr. Math. Zhurn. **61** (2009), no. 9, 1225–1239 (in Ukrainian); translation in Ukrainian Math. J. **61** (2009), no. 9, 1448–1466.

References VIII

M. V. Pratsiovytyi and N. A. Vasylenko,
***Fractal properties of functions defined in terms of Q-representation*,**
Int. J. Math. Anal. (Ruse) **7** (2013), no. 64, 3155–3167.

M. V. Pratsiovytyi and N. A. Vasylenko,
***One family of continuous nowhere monotonic functions with fractal properties*,**
Nauk. Chasop. Nats. Pedagog. Univ. Mykhaila Drahomanova. Ser. 1. Fiz.-Mat. Nauky (2013), no. 14, 176–188 (in Ukrainian).

References IX

- M. V. Pratsiovytyi and A. V. Kalashnikov, ***On one class of continuous functions with complicated local structure such that most of them are singular or non-differentiable functions*****,** Tr. Inst. Prikl. Mat. Mekh. **23** (2011), 180–191 (in Ukrainian).
- M. V. Pratsiovytyi and A. V. Kalashnikov, ***Self-affine singular and nowhere monotone functions related to the*** Q***-representation of real numbers*****,** Ukr. Math. Zhurn. **65** (2013), no. 3, 405–417 (in Ukrainian); translation in Ukrainian Math. J. **65** (2013), no. 3, 448–462.

References X

S. Albeverio, O. Baranovskyi, Yu. Kondratiev, and M. Pratsiovytyi,
***On one class of functions related to Ostrogradsky series and containing singular and nowhere monotonic functions*,**
Nauk. Chasop. Nats. Pedagog. Univ. Mykhaila Drahomanova. Ser. 1. Fiz.-Mat. Nauky (2013), no. 15, 24–41.

Summary

There exist some problems in studying continuous functions with complicated local properties (singular, nowhere monotonic, and nowhere differentiable functions). Effective means and tools for their analytical definition and study does not exist.

We construct and study the infinite-parameter family of continuous functions with complicated local properties using the so-called E-representation of a real number, i.e., its encoding by infinite alphabet in the form of Engel series (positive series such that their terms are reciprocal to cumulative products of natural numbers).