Fractal Analysis of Singular, Nowhere Differentiable, and Nowhere Monotonic Functions

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Simple Examples

Example 1: Cantor Set

$$C = \left\{ x \in [0,1] \colon x = \underbrace{\frac{\alpha_1}{3^1} + \frac{\alpha_2}{3^2} + \ldots + \frac{\alpha_n}{3^n} + \ldots}_{\text{ternary expansion}}, \quad \alpha_n = \alpha_n(x) \in \{0,1,2\} \right\}$$

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Informal Explanation of Fractals

“Fractal dimension” = a “generalization” of a “usual” dimension that can take fractional (noninteger) values.

Fractal set = a set whose fractal dimension is a noninteger number.

Self-similar Set and Self-similar Dimension

Definition

E is called a *self-similar set* if

1. $E = E_1 \bigcup E_2 \bigcup \ldots \bigcup E_n$, $n > 1$,
2. $E_i \overset{k_i}{\sim} E$, $i = \overline{1, n}$,
3. $E_i \bigcap E_j$ is "small" with respect to E for $i \neq j$.

Definition

Self-similar dimension $\alpha_s(E)$ of a set E is a solution of

$$k_1^x + k_2^x + \ldots + k_n^x = 1.$$

For Cantor set, $C = C_1 \cup C_2$, $C_i \overset{\frac{1}{3}}{\sim} C$.

$\alpha_s(C) = x$, $\left(\frac{1}{3}\right)^x + \left(\frac{1}{3}\right)^x = 1 \quad \Rightarrow \quad \alpha_s(C) = \log_3 2 \approx 0.6309.$

Example 2: Tribin Function

$$x = \frac{\alpha_1}{3} + \frac{\alpha_2}{3^2} + \ldots + \frac{\alpha_n}{3^n} + \ldots \equiv \Delta^3_{\alpha_1\alpha_2\ldots\alpha_n\ldots},$$

$$\alpha_n = \alpha_n(x) \in A_3 = \{0, 1, 2\},$$

$$y = f(x) = \frac{\beta_1}{2} + \frac{\beta_2}{2^2} + \ldots + \frac{\beta_n}{2^n} + \ldots \equiv \Delta^2_{\beta_1\beta_2\ldots\beta_n\ldots},$$

$$\beta_n = \beta_n(y) \in A_2 = \{0, 1\},$$

where

$$\beta_1 = \begin{cases} 0 & \text{if } \alpha_1 = 0 \\ 1 & \text{if } \alpha_1 \neq 0, \end{cases} \quad \beta_n = \begin{cases} \beta_{n-1} & \text{if } \alpha_n = \alpha_{n-1} \\ 1 - \beta_{n-1} & \text{if } \alpha_n \neq \alpha_{n-1}, \quad n > 1. \end{cases}$$

Example 2: Tribin Function

1. Continuous,
2. nowhere differentiable function (i.e., it does not have derivative at any point).

Example 3: Minkowski Function

$x \in [0, 1]$

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}} = [a_1, a_2, a_3, \ldots], \quad a_n \in \mathbb{N},$$

$$G(x) = \frac{1}{2^{a_1 - 1}} - \frac{1}{2^{a_1 + a_2 - 1}} + \ldots + \frac{(-1)^{n-1}}{2^{a_1 + a_2 + \ldots + a_n - 1}} + \ldots$$

is called the *Minkowski function*.

Example 3: Minkowski Function

1. Continuous,
2. strictly increasing,
3. singular function (i.e., its derivative is equal to 0 almost everywhere with respect to Lebesgue measure).

Problems and Methods I

We study

- Fractal sets:
    - sets on the real line (Cantor-type sets, Besicovitch–Eggleston sets, etc.),
    - curves on the plane (graphs of functions, Koch snowflake, etc).

  Their analytical definition (by formulae, equations, etc).
  Fractal properties (self-similarity, self-affinity, Hausdorff–Besicovitch dimension).

Problems and Methods II

- Continuous functions (singular, nowhere monotonic, nowhere differentiable).
  They have fractal properties (i.e., their level sets, graphs, spectra are fractal sets).
- Singular probability measures.
  They are supported on the sets of zero Lebesgue measure, which can be fractal sets.

Problems and Methods III

We need to study various **systems of representation for real numbers** and create new systems with finite and infinite alphabet, with constant and variable alphabet, with standard and redundant alphabet, etc.

Calculation of Hausdorff–Besicovitch dimension (fractal dimension) is a complicated problem usually.

We need to develop and use different techniques for this problem: for example, faithful systems of covering (restricted systems of covering).

Some Systems of Representation for Real Numbers

Let $A_s = \{0, 1, 2, \ldots, s-1\}$ be an alphabet, $s \geq 2$, let $Q_s = (q_0, q_1, \ldots, q_{s-1})$ be a stochastic vector, where $q_i \in (0, 1)$, $q_0 + q_1 + \ldots + q_{s-1} = 1$.

Theorem

For any $x \in [0, 1]$, there exists a sequence (α_n), $\alpha_n = \alpha_n(x) \in A_s$ such that

$$x = \beta_{\alpha_1} + \sum_{n=2}^{\infty} \left(\beta_{\alpha_n} \prod_{j=1}^{n-1} q_{\alpha_j} \right) \qquad Q_s\text{-expansion} \qquad (1)$$

$$\equiv \Delta^{Q_s}_{\alpha_1 \alpha_2 \ldots \alpha_n \ldots}, \qquad Q_s\text{-representation} \qquad (2)$$

where $\beta_{\alpha_n} = \sum_{i=0}^{\alpha_n - 1} q_i$.

Geometry

Definition

Let $(c_1, c_2, \ldots, c_m)$ be a fixed m-tuple, $c_i \in A_s$. *Cylinder of rank m with the base $c_1 c_2 \ldots c_m$* is

$$\Delta^{Q_s}_{c_1 c_2 \ldots c_m} = \left\{ x \in [0,1] \colon x = \Delta^{Q_s}_{c_1 c_2 \ldots c_m \alpha_{m+1} \alpha_{m+2} \ldots \alpha_{m+i} \ldots}, \ \alpha_{m+i} \in A_s \right\}.$$

1. Cylinder is a closed interval.
2. Its length is
$$|\Delta^{Q_s}_{c_1 c_2 \ldots c_m}| = q_{c_1} q_{c_2} \ldots q_{c_m},$$
3. Cylinders of the same rank do not overlap.

Basic Metric Relation

Lemma

For any $c \in A_s$ *and any m-tuple* $c_1, c_2, \ldots, c_m \in A_s$,

$$\frac{|\Delta^{Q_s}_{c_1 c_2 \ldots c_m c}|}{|\Delta^{Q_s}_{c_1 c_2 \ldots c_m}|} = q_c.$$

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Other Representations with Finite Alphabet

Q_s^*-representation:

Let $A_s = \{0, 1, 2, \dots, s-1\}$ be an alphabet, let

$$Q_s^* = \begin{pmatrix} q_{01} & q_{02} & \cdots & q_{0k} & \cdots \\ q_{11} & q_{12} & \cdots & q_{1k} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ q_{(s-1)1} & q_{(s-1)2} & \cdots & q_{(s-1)k} & \cdots \end{pmatrix}$$

be an infinite stochastic matrix, where $q_{ik} \in (0, 1)$, $q_{0k} + q_{1k} + \ldots + q_{(s-1)k} = 1$ for any $k \in \mathbb{N}$.

Representations with redundant set of digits, etc.

Particular Cases

$q_0 = q_1 = \ldots = q_{s-1} = \frac{1}{s} \quad \Rightarrow \quad$ classic s-adic expansion (representation)

$s = 2$, $Q_2 = (q_0, q_1) \quad \Rightarrow \quad$ important particular case

$q_0 = q_1 = \frac{1}{2} \quad \Rightarrow \quad$ classic binary expansion (representation)

First Ostrogradsky Series

Theorem

Any $x \in (0,1)$ *can be represented in the form of the first Ostrogradsky series*

$$x = \frac{1}{q_1} - \frac{1}{q_1 q_2} + \ldots + \frac{(-1)^{n-1}}{q_1 q_2 \ldots q_n} + \ldots \tag{3}$$

$$= \mathrm{O}^1(q_1, q_2, \ldots, q_n, \ldots), \tag{4}$$

where $q_n \in A = \mathbb{N} = \{1, 2, 3, \ldots\}$ *and* $q_{n+1} > q_n$ *for any* $n \in \mathbb{N}$*. If* x *is irrational then the expression* (3) *is unique and it has an infinite number of terms. If* x *is rational then it can be represented in the form* (3) *in the following different ways:*

$$x = \mathrm{O}^1(q_1, q_2, \ldots, q_{n-1}, q_n, q_n{+}1) = \mathrm{O}^1(q_1, q_2, \ldots, q_{n-1}, q_n{+}1).$$

$\bar{\text{O}}^1$-representation

Let $g_1 = q_1$ and $g_{n+1} = q_{n+1} - q_n$ for any $n \in \mathbb{N}$.

Then one can rewrite series (3) in the form

$$\frac{1}{g_1} - \frac{1}{g_1(g_1+g_2)} + \ldots + \frac{(-1)^{n-1}}{g_1(g_1+g_2)\ldots(g_1+g_2+\ldots+g_n)} + \ldots, \tag{5}$$

where $g_n \in \mathbb{N}$.

For any $x \in (0,1)$ there exists a sequence (g_n), $g_n = g_n(x) \in \mathbb{N}$, such that

$$x = \sum_n \frac{(-1)^{n-1}}{g_1(g_1+g_2)\ldots(g_1+g_2+\ldots+g_n)} \tag{6}$$

$$= \bar{\text{O}}^1(g_1, g_2, \ldots, g_n, \ldots). \tag{7}$$

Cylindrical Set (Cylinder)

Definition

A set $\bar{\mathrm{O}}^1_{[c_1 c_2 \ldots c_m]}$ of all $x \in (0, 1)$, which can be represented by the $\bar{\mathrm{O}}^1$-representation such that first m $\bar{\mathrm{O}}^1$-symbols are equal to $c_1, c_2, \ldots, c_m$ respectively, is said to be *the cylindrical set* (*cylinder*) *of rank m with the base* $(c_1, c_2, \ldots, c_m)$.

Properties of Cylindrical Sets

1. $\bar{\mathrm{O}}^1_{[c_1 c_2 \ldots c_m]} = [a, b]$ (i.e., closed interval).
2. $\bar{\mathrm{O}}^1_{[c_1 c_2 \ldots c_m]} = \bigcup_{c=1}^{\infty} \bar{\mathrm{O}}^1_{[c_1 c_2 \ldots c_m c]} \bigcup \bar{\mathrm{O}}^1(c_1, c_2, \ldots, c_m)$,

$$\begin{aligned} \sup \bar{\mathrm{O}}^1_{[c_1 c_2 \ldots c_m c]} &= \inf \bar{\mathrm{O}}^1_{[c_1 c_2 \ldots c_m (c+1)]}, \quad \text{if } m \text{ is odd,} \\ \inf \bar{\mathrm{O}}^1_{[c_1 c_2 \ldots c_m c]} &= \sup \bar{\mathrm{O}}^1_{[c_1 c_2 \ldots c_m (c+1)]}, \quad \text{if } m \text{ is even,} \end{aligned}$$

$$\bar{\mathrm{O}}^1_{[c_1 c_2 \ldots c_m c]} \cap \bar{\mathrm{O}}^1_{[c_1 c_2 \ldots c_m (c+1)]} = \bar{\mathrm{O}}^1(c_1, c_2, \ldots, c_m, c+1).$$

3. Length $|\bar{\mathrm{O}}^1_{[c_1 c_2 \ldots c_m]}| = \dfrac{1}{\sigma_1 \sigma_2 \ldots \sigma_m (\sigma_m + 1)}, \quad \sigma_k = \sum_{i=1}^{k} c_i$.

Basic Metric Relation

Lemma

$$\frac{|\bar{\mathrm{O}}^1_{[c_1 c_2 \ldots c_m c]}|}{|\bar{\mathrm{O}}^1_{[c_1 c_2 \ldots c_m]}|} = \frac{a}{(a+c-1)(a+c)} = f_c(a), \quad a = 1 + \sum_{i=1}^{m} c_i. \quad (8)$$

$$f_c(a) \leq \frac{1}{2 \cdot (2c-1)}.$$

$$\frac{|\bar{\mathrm{O}}^1_{[c_1 c_2 \ldots c_m c]}|}{|\bar{\mathrm{O}}^1_{[c_1 c_2 \ldots c_m]}|} \leq \frac{m+1}{(m+c)(m+c+1)} \quad \textit{for } m \geq c-1.$$

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Other Representations with Infinite Alphabet

Second Ostrogradsky series

(Positive and alternating) Lüroth series

Engel series

Sylvester series

Q_∞-representation

A_2-continued Fractions

Let $A_2 = \{\alpha_1, \alpha_2\}$ be an alphabet, $0 < \alpha_1 < \alpha_2$.

Definition (A_2-continued fraction)

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}} = [a_1, a_2, \dots, a_n, \dots],\ a_n \in A_2$$

A_2-continued Fractions

$$L_{A_2} = \{x\colon x = [a_1, a_2, \ldots, a_n, \ldots],\ a_n \in A_2,\ n = 1, 2, \ldots\},$$

$$\min L_{A_2} = \inf L_{A_2} = \beta_1,\ \max L_{A_2} = \sup L_{A_2} = \beta_2,\ L_{A_2} \subseteq [\beta_1, \beta_2],$$

$$\beta_1 = [(\alpha_2, \alpha_1)] = \frac{\sqrt{\alpha_1^2\alpha_2^2 + 4\alpha_1\alpha_2} - \alpha_1\alpha_2}{2\alpha_2},$$

$$\beta_2 = [(\alpha_1, \alpha_2)] = \frac{\sqrt{\alpha_1^2\alpha_2^2 + 4\alpha_1\alpha_2} - \alpha_1\alpha_2}{2\alpha_1}.$$

Cylinder Sets I

Cylinder of rank m with the base $c_1c_2\ldots c_m$

$$\Delta'_{c_1c_2\ldots c_m} = \{x\colon x = [c_1, c_2, \ldots, c_m, a_{m+1}, a_{m+2}, \ldots], \\ a_{m+i} \in A_2 \text{ for all } i \in \mathbb{N}\},$$

cylindrical closed interval of rank m with the base $c_1c_2\ldots c_m$

$$\Delta_{c_1c_2\ldots c_m} = [\min \Delta'_{c_1c_2\ldots c_m}, \max \Delta'_{c_1c_2\ldots c_m}].$$

1. $\Delta'_{c_1\ldots c_m c} \subset \Delta'_{c_1\ldots c_m}$, $\Delta'_{c_1\ldots c_m} = \Delta'_{c_1\ldots c_m\alpha_1} \cup \Delta'_{c_1\ldots c_m\alpha_2}$.
2. $\Delta_{c_1\ldots c_m c} \subset \Delta_{c_1\ldots c_m}$, but, in general, $\Delta_{c_1\ldots c_m} \neq \Delta_{c_1\ldots c_m\alpha_1} \cup \Delta_{c_1\ldots c_m\alpha_2}$.

Cylinder Sets II

3. $\inf \Delta_{c_1\ldots c_m\alpha_1} < \inf \Delta_{c_1\ldots c_m\alpha_2}$, if m is odd,
   $\inf \Delta_{c_1\ldots c_m\alpha_1} > \inf \Delta_{c_1\ldots c_m\alpha_2}$, if m even.
4. If $\alpha_2 - \alpha_1 = \beta_2 - \beta_1$, then

$$\Delta_{c_1\ldots c_m\alpha_1} \cap \Delta_{c_1\ldots c_m\alpha_2} = [c_1, \ldots, c_m, \alpha_1 + \beta_2] = [c_1, \ldots, c_m, \alpha_2 + \beta_1].$$

5. If $\alpha_2 - \alpha_1 < \beta_2 - \beta_1$, then $\Delta_{c_1\ldots c_m\alpha_1} \cap \Delta_{c_1\ldots c_m\alpha_2} = [a, b]$, where

$$a = \begin{cases} [c_1, \ldots, c_m, \alpha_1 + \beta_2] & \text{for even } m, \\ [c_1, \ldots, c_m, \alpha_2 + \beta_1] & \text{for odd } m; \end{cases}$$

$$b = \begin{cases} [c_1, \ldots, c_m, \alpha_2 + \beta_1] & \text{for even } m, \\ [c_1, \ldots, c_m, \alpha_1 + \beta_2] & \text{for odd } m. \end{cases}$$

Cylinder Sets III

6. If $\alpha_2 - \alpha_1 \leq \beta_2 - \beta_1$, then

$$\Delta_{c_1 \ldots c_m} = \Delta_{c_1 \ldots c_m \alpha_1} \cup \Delta_{c_1 \ldots c_m \alpha_2}.$$

7. If $\alpha_2 - \alpha_1 > \beta_2 - \beta_1$, then

$$\Delta_{c_1 \ldots c_m \alpha_1} \cap \Delta_{c_1 \ldots c_m \alpha_2} = \varnothing.$$

Basic Metric Relation I

Length of cylindrical closed interval

$$|\Delta_{c_1\ldots c_n}| = \frac{\beta_2 - \beta_1}{(q_n + \beta_1 q_{n-1})(q_n + \beta_2 q_{n-1})},$$

where q_n is a denominator of convergent of rank n.

Basic metric relation:

$$\frac{|\Delta_{c_1\ldots c_n c}|}{|\Delta_{c_1\ldots c_n}|} = \frac{\left(1 + \beta_1 \frac{q_{n-1}}{q_n}\right)\left(1 + \beta_2 \frac{q_{n-1}}{q_n}\right)}{\left(c + \beta_1 + \frac{q_{n-1}}{q_n}\right)\left(c + \beta_2 + \frac{q_{n-1}}{q_n}\right)}.$$

Basic Metric Relation II

If $\alpha_2 - \alpha_1 = \beta_2 - \beta_1$ (that is $\alpha_1\alpha_2 = \frac{1}{2}$, $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$), then

$$\frac{|\Delta_{c_1\ldots c_n c}|}{|\Delta_{c_1\ldots c_n}|} = \frac{\left(1 + c\frac{q_{n-1}}{q_n}\right)}{\left(2c^2 + 1 + 2c\frac{q_{n-1}}{q_n}\right)},$$

$$\frac{|\Delta_{c_1\ldots c_n \alpha_1}|}{|\Delta_{c_1\ldots c_n \alpha_2}|} = \frac{\left(1 + \alpha_1\frac{q_{n-1}}{q_n}\right)\left(2\alpha_2^2 + 1 + 2\alpha_2\frac{q_{n-1}}{q_n}\right)}{\left(1 + \alpha_2\frac{q_{n-1}}{q_n}\right)\left(2\alpha_1^2 + 1 + 2\alpha_1\frac{q_{n-1}}{q_n}\right)}.$$

Basic Metric Relation III

If $\alpha_1 = \frac{1}{2}$, $\alpha_2 = 1$, then $\beta_1 = \frac{1}{2}$, $\beta_2 = 1$ and

$$\frac{|\Delta_{c_1\dots c_n\frac{1}{2}}|}{|\Delta_{c_1\dots c_n}|} = \frac{2+\frac{q_{n-1}}{q_n}}{3+2\frac{q_{n-1}}{q_n}},$$

$$\frac{|\Delta_{c_1\dots c_n 1}|}{|\Delta_{c_1\dots c_n}|} = \frac{1+\frac{q_{n-1}}{q_n}}{3+2\frac{q_{n-1}}{q_n}},$$

$$\frac{|\Delta_{c_1\dots c_n\frac{1}{2}}|}{|\Delta_{c_1\dots c_n 1}|} = \frac{2+\frac{q_{n-1}}{q_n}}{1+\frac{q_{n-1}}{q_n}} = 1+\frac{1}{1+\frac{q_{n-1}}{q_n}}.$$

Properties of L_{A_2}

Theorem

If $\alpha_1\alpha_2 \leq \frac{1}{2}$, *then* $L_{A_2} = [\beta_1, \beta_2]$.

Corollary

If $\alpha_1\alpha_2 \leq \frac{1}{2}$, *then* $\Delta'_{c_1\ldots c_m} = \Delta_{c_1\ldots c_m}$.

Theorem

If $\alpha_1\alpha_2 = \frac{1}{2}$, *then only countable set of points* $x \in [\beta_1, \beta_2]$ *have two representations in the form of* A_2*-continued fraction. Other points have a unique representation.*

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Research Topics at the Department

Let $\{V_n\}$ be a fixed sequence of nonempty subsets of alphabet A.

$$C[f, \{V_n\}] = \{x \in [0,1] \colon x = \Delta^f_{\alpha_1\alpha_2\ldots\alpha_n\ldots}, \\ \alpha_n = \alpha_n(x) \in V_n,\ n \in \mathbb{N}\},$$

where f is one of the above-mentioned representations.

This set can be a spectrum of probability distribution, level set of a function, etc.

We study topological, metric, and fractal properties of $C[f, \{V_n\}]$.

$x = \Delta^{Q_s}_{\alpha_1\alpha_2\ldots\alpha_n\ldots} \in [0,1]$.

Definition

Let $N_i(x,k) = \#\{j\colon \alpha_j(x) = i,\ j \le k\}$.

Frequency of digit "i" in Q_s-representation of x is

$$\nu_i(x) = \nu_i^{Q_s}(x) = \lim_{k\to\infty} \frac{N_i(x,k)}{k}.$$

Besicovitch–Eggleston Sets

Let $(p_0, p_1, \dots, p_{s-1})$ be a stochastic vector.

The *Besicovitch–Eggleston set* is

$$E \equiv E[Q_s, (p_0, p_1, \dots, p_{s-1})] = \big\{x \in [0,1]\colon \nu_i(x) = p_i, \\ i \in A_s = \{0, 1, \dots, s-1\}\big\}.$$

Theorem

E is

- *everywhere dense set in* $[0,1]$,
- *set of zero Lebesgue measure if* $p_i \neq q_i$ *for some* $i \in A_s$,
- *set with Hausdorff–Besicovitch dimension*

$$\alpha_0(E) = \frac{\ln p_0^{p_0} p_1^{p_1} \dots p_{s-1}^{p_{s-1}}}{\ln q_0^{p_0} q_1^{p_1} \dots q_{s-1}^{p_{s-1}}}$$

Random A_2-continued Fraction with Independent Elements

Consider random variable

$$\xi = \cfrac{1}{\eta_1 + \cfrac{1}{\eta_2 + \dots}} \equiv [\eta_1, \eta_2, \dots],$$

where η_k are independent random variables with distribution $\mathrm{P}\{\eta_k = \alpha_1\} = p_{\alpha_1 k} \geq 0$, $\mathrm{P}\{\eta_k = \alpha_2\} = p_{\alpha_2 k} \geq 0$, $p_{\alpha_1 k} + p_{\alpha_2 k} = 1$, $0 < \alpha_1 < \alpha_2$, $\alpha_1 \alpha_2 \geq \frac{1}{2}$.

Lebesgue Structure of Distribution

Theorem (Lebesgue theorem)

Let $F(x)$ be a probability distribution function. Then

$$F(x) = \alpha_1 F_d(x) + \alpha_2 F_{ac}(x) + \alpha_3 F_s(x), \qquad (9)$$

where $\alpha_i \geq 0$, $\alpha_1 + \alpha_2 + \alpha_3 = 1$, and

1. *discrete function F_d increases only by jumps at atoms of distribution;*
2. *absolutely continuous F_{ac} is an improper integral of its derivative*

$$F_{ac}(x) = \int_{-\infty}^{x} F'_{ac}(t)dt;$$

3. *singular F_s is a continuous function such that its derivative is equal to 0 almost everywhere w.r.t. Lebesgue measure.*

Lebesgue Structure of Distribution

Eq. (9) is called *Lebesgue structure of distribution* (of probability distribution function F). To establish Lebesgue structure of distribution = to find α_1, α_2, α_3 and F_d, F_{ac}, F_s.

If one of the α_1, α_2, α_3 = 1, then distribution has a pure Lebesgue type (pure discrete, pure absolutely continuous or pure singularly continuous).

Otherwise it is a mixture of two or three distributions of pure Lebesgue types.

Structure of Singular Distribution

Theorem (Pratsiovytyi theorem)

Let $F(x)$ be a singular probability distribution function. Then

$$F(x) = \gamma_1 F_S(x) + \gamma_2 F_C(x) + \gamma_3 F_K(x), \tag{10}$$

where $\gamma_i \geq 0$, $\gamma_1 + \gamma_2 + \gamma_3 = 1$, and

1. *F_S is of Salem type,*
2. *F_C is of Cantor type,*
3. *F_K is of quasi-Cantor type.*

Structure of Singular Distribution

A singular probability distribution function $F(x)$ is

- of Salem type if $S_\xi = \cup_i [a_i, b_i]$,
- of Cantor type if Lebesgue measure $\lambda(S_\xi) = 0$,
- of quasi-Cantor type if S_ξ is a nowhere dense set and $\lambda(S_\xi) > 0$.

Spectrum S_ξ of random variable ξ (or of its probability distribution function F_ξ) is

$$S_\xi = \{x \in [0,1] \colon F_\xi(x+\varepsilon) - F_\xi(x-\varepsilon) > 0 \text{ for any } \varepsilon > 0\}.$$

Structure of Singular Distribution

Eq. (10) is called the *structure of singular distribution* (of singular probability distribution function F). To establish structure of singular distribution = to find γ_1, γ_2, γ_3 and F_S, F_S, F_K.

If one of the γ_1, γ_2, γ_3 = 1, then distribution has a pure singular type.

Random A_2-continued Fraction with Independent Elements

Theorem

Distribution of ξ *is discrete* $\Leftrightarrow$ $M \equiv \prod_{k=1}^{\infty} \max\{p_{\alpha_1 k}, p_{\alpha_2 k}\} > 0.$

Distribution of ξ *is continuous* $\Leftrightarrow$ $M = 0.$

Random A_2-continued Fraction with Independent Elements

Theorem

If $\alpha_1\alpha_2 > \frac{1}{2}$ and distribution of random variable ξ is continuous, then ξ has a singular distribution of Cantor type.

Theorem

For $\alpha_1\alpha_2 = \frac{1}{2}$, distribution of ξ has a pure Lebesgue type.

Random A_2-continued Fraction with Independent Elements

Let ξ has a continuous distribution, i.e., $M = 0$, and $\alpha_1\alpha_2 = \frac{1}{2}$.

Theorem

If matrix $\|p_{ik}\|$ contains a finitely many zeroes, then ξ has a singular distribution of Salem type.

Theorem

Random variable ξ has a singular distribution of Cantor type if and only if matrix $\|p_{ik}\|$ contains an infinitely many zeroes.

References

M. Pratsiovytyi and D. Kyurchev,
Properties of the distribution of the random variable defined by A_2-continued fraction with independent elements,
Random Oper. Stoch. Equ. **17** (2009), no. 1, 91–101.

Tribin Function

$$x = \frac{\alpha_1}{3} + \frac{\alpha_2}{3^2} + \ldots + \frac{\alpha_n}{3^n} + \ldots \equiv \Delta^3_{\alpha_1\alpha_2\ldots\alpha_n\ldots},$$

$$\alpha_n = \alpha_n(x) \in A_3 = \{0, 1, 2\},$$

$$y = f(x) = \frac{\beta_1}{2} + \frac{\beta_2}{2^2} + \ldots + \frac{\beta_n}{2^n} + \ldots \equiv \Delta^2_{\beta_1\beta_2\ldots\beta_n\ldots},$$

$$\beta_n = \beta_n(y) \in A_2 = \{0, 1\},$$

where

$$\beta_1 = \begin{cases} 0 & \text{if } \alpha_1 = 0 \\ 1 & \text{if } \alpha_1 \neq 0, \end{cases} \quad \beta_n = \begin{cases} \beta_{n-1} & \text{if } \alpha_n = \alpha_{n-1} \\ 1 - \beta_{n-1} & \text{if } \alpha_n \neq \alpha_{n-1}, \end{cases} \quad n > 1.$$

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W. Wunderlich,
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Elem. Math. **7** (1952), no. 4, 73–79.

Level Sets of Function f

For any $y_0 \in [0,1]$, the level set of the function f is $f^{-1}(y_0) = \{x \in [0,1] \colon f(x) = y_0\}$.

Theorem

1. *If y_0 is a binary rational number, then set $f^{-1}(y_0)$ is finite and Hausdorff–Besicovitch dimension $\alpha_0\left(f^{-1}(y_0)\right) = 0$.*
2. *For binary irrational number y_0, $\alpha_0\left(f^{-1}(y_0)\right) = B\log_3 2$, where $B = \varliminf_{k\to\infty} \frac{d_k}{k}$, d_k is an amount of pairs of consecutive digits of y_0 (to kth place) such that their components are different.*

Graph of Function f

Theorem

Box-counting dimension of graph Γ_f *is* $2 - \log_3 2 \approx 1.36907$.

Theorem

Hausdorff–Besicovitch dimension of Γ_f *is*

$$\alpha_0(\Gamma_f) = \log_2\left(1 + 2^{\log_3 2}\right) \approx 1.34968.$$

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O. B. Panasenko,
***Fractal properties of one class of one-parameter continuous nowhere differentiable functions*,**
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O. B. Panasenko,
***Fractal dimension of graphs of continuous Cantor projectors*,**
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***Hausdorff–Besicovitch dimension of the graph of one continuous nowhere-differentiable function*,**
Ukr. Math. Zhurn. **61** (2009), no. 9, 1225–1239 (in Ukrainian); translation in Ukrainian Math. J. **61** (2009), no. 9, 1448–1466.

M. V. Pratsiovytyi and N. A. Vasylenko,
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Nauk. Chasop. Nats. Pedagog. Univ. Mykhaila Drahomanova. Ser. 1. Fiz.-Mat. Nauky (2013), no. 14, 176–188 (in Ukrainian).

Shukla Function

Using nontrivial arithmetic construction, Shukla proposed a first example of singular function which is nonmonotonic.

U. K. Shukla,
On points of non-symmetrical differentiability of a continuous function. III,
Ganita **8** (1957), no. 2, 81–104.

Class of Functions 1

Let (a_k) be a given infinitesimal sequence of positive real numbers, $0 < a_k < \frac{1}{2}$, $A_3 = \{0, 1, 2\}$,
$g_{0k} = g_{2k} = \frac{1}{2} + a_k$, $g_{1k} = -2a_k$,
$\gamma_{0k} = 0$, $\gamma_{1k} = g_{0k}$, $\gamma_{2k} = g_{0k} + g_{1k} = \frac{1}{2} - a_k$.

$$f(x) = \gamma_{\alpha_1(x)1} + \sum_{k=2}^{\infty} \left(\gamma_{\alpha_k(x)k} \prod_{j=1}^{k-1} g_{\alpha_j(x)j} \right), \tag{11}$$

where

$$x = \frac{\alpha_1(x)}{3} + \frac{\alpha_2(x)}{3^2} + \ldots + \frac{\alpha_k(x)}{3^k} + \ldots \equiv \Delta^3_{\alpha_1(x)\alpha_2(x)\ldots\alpha_k(x)\ldots}.$$

Class of Functions 1

Lemma

Function $f(x)$ is well defined and has the following properties:

1. $0 \le f(x) \le 1$ *and* $f(0) = 0$, $f(1) = 1$,
2. *it is continuous,*
3. *it is nowhere monotonic.*

Theorem

If $g_{0k} = g_{2k} = \frac{1}{2} + \frac{1}{6^k}$, $g_{1k} = -\frac{2}{6^k}$, *then* $f(x)$ *is a nowhere monotonic singular function.*

References

M. V. Pratsiovytyi,
***Nowhere monotonic singular functions*,**
Nauk. Chasop. Nats. Pedagog. Univ. Mykhaila Drahomanova. Ser. 1. Fiz.-Mat. Nauky (2011), no. 12, 24–36 (in Ukrainian).

Class of Functions 2

Let (ε_n) be a sequence of positive real numbers from $[0, 1]$,
$\overline{g_n} = (g_{0n}, g_{1n}, g_{2n}, g_{3n}, g_{4n})$,
$g_{0n} = g_{4n} = \frac{2+\varepsilon_n}{4}$, $g_{1n} = g_{3n} = \frac{-\varepsilon_n}{4}$, $g_{2n} = 0$,
$\delta_{0n} = 0$, $\delta_{1n} = \frac{2+\varepsilon_n}{4}$, $\delta_{2n} = \frac{2}{4} = \delta_{3n}$, $\delta_{4n} = \frac{2-\varepsilon_n}{4}$,
i.e., $\delta_{[i+1]n} = \delta_{in} + g_{in} = \sum\limits_{j=0}^{i} g_{jn}$, $n \in \mathbb{N}$.

$$f(x) = \delta_{\alpha_1(x)1} + \sum_{k=2}^{\infty}\left(\delta_{\alpha_k(x)k}\prod_{j=1}^{k-1} g_{\alpha_j(x)j}\right) \equiv \Delta^G_{\alpha_1\alpha_2\ldots\alpha_k\ldots}. \qquad (12)$$

Class of Functions 2

Theorem

Function $f(x)$ is

1. *well defined and continuous on* $[0,1]$*;*
2. *constant on every cylinder* $\Delta^5_{c_1c_2...c_m2}$ *as well as on cylinder* $\Delta^5_{c_1c_2...c_{n-1}1}$ *and* $\Delta^5_{c_1c_2...c_{n-1}3}$ *if* $\varepsilon_n = 0$*;*
3. *monotonic* (*nondecreasing*) *if and only if* $\varepsilon_n = 0$*,* $n \in \mathbb{N}$*;*
4. *singular function of Cantor type, its set of nonconstancy is Cantor-type set* $C[5, A_5] = \{x \in [0,1] \colon \alpha_n(x) \in A_5\}$*,* $A_5 = \{0,1,3,4\}$*, with Hausdorff–Besicovitch dimension* $\log_5 4$*. It takes all values from closed interval* $[0,1]$*, does not have intervals of monotonicity, except for intervals of constancy, if inequality* $\varepsilon_n \neq 0$ *holds for infinite set of n, and its graph is symmetric with respect to point* $C\left(\frac{1}{2}, \frac{1}{2}\right)$*.*

α-dimensional Hausdorff Measure

Let E be any bounded set of $\mathbb{R}^n$ and $\alpha > 0$.

Definition (α-dimensional Hausdorff measure)

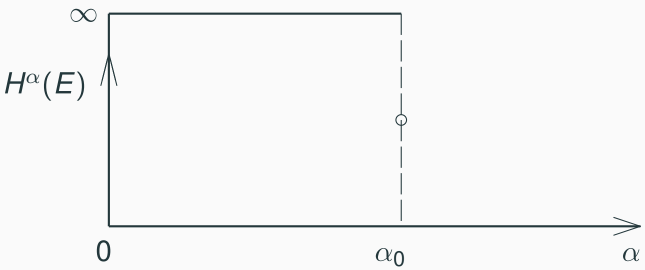
$$H^\alpha(E) = \lim_{\varepsilon \to 0} m_\varepsilon^\alpha(E) = \sup_{\varepsilon > 0} m_\varepsilon^\alpha(E),$$

where $m_\varepsilon^\alpha(E) = \inf_{d(E_j) \leq \varepsilon} \left\{ \sum_j d^\alpha(E_j) \right\}$, $d(E_j)$ is a diameter of the set E_j and the infimum is taken over all at most countable ε-coverings $\{E_j\}$ of the set E by sets $E_j \in \mathbb{R}^n$.

Properties of the Hausdorff Measure

1. $H^\alpha\left(\bigcup_i E_i\right) \leq \sum_i H^\alpha(E_i)$;
2. If $\alpha_1 < \alpha_2$, then $H^{\alpha_1}(E) \geq H^{\alpha_2}(E)$;
3. If $H^{\alpha_1}(E) = 0$, then $H^{\alpha_2}(E) = 0$ for $\alpha_1 < \alpha_2$;
4. If $H^{\alpha_2}(E) = \infty$, then $H^{\alpha_1}(E) = \infty$ for $0 < \alpha_1 < \alpha_2$.

Hausdorff–Besicovitch Dimension



Definition (Hausdorff–Besicovitch dimension)

$$\alpha_0(E) = \inf\{\alpha \colon H^\alpha(E) = 0\} = \sup\{\alpha \colon H^\alpha(E) \neq 0\}$$

Properties of the Hausdorff–Besicovitch Dimension

1. $\alpha_0(E) = 0$ for any at most countable set E;
2. $\alpha_0(E_1) \leq \alpha_0(E_2)$ if $E_1 \subset E_2$;
3. $\alpha_0(\bigcup\limits_n E_n) = \sup\limits_n \alpha_0(E_n)$;
4. If E_1 and E_2 are geometrically similar, then $\alpha_0(E_1) = \alpha_0(E_2)$.

Billingsley Dimension

Let E be any bounded subset of $[0,1]$ and $\alpha > 0$.

Let ν be a continuous probability measure on $[0,1]$.

Definition

$$H^{\alpha}(E,\nu) = \lim_{\varepsilon\to 0} m_{\varepsilon}^{\alpha}(E,\nu) = \sup_{\varepsilon>0} m_{\varepsilon}^{\alpha}(E,\nu),$$

where $m_{\varepsilon}^{\alpha}(E,\nu) = \inf_{\nu(E_j)\le\varepsilon} \left\{ \sum_j \nu^{\alpha}(E_j) \right\}$, and the infimum is taken over all at most countable ε-coverings $\{E_j\}$ of the set E by sets $E_j \in [0,1]$.

Definition (Billingsley dimension or Hausdorff–Besicovitch dimension with respect to measure ν)

$$\alpha_{\nu}(E) = \inf\{\alpha\colon H^{\alpha}(E,\nu) = 0\} = \sup\{\alpha\colon H^{\alpha}(E,\nu) \neq 0\}$$

Why We Study Such Objects?

Theorem (Banach–Mazurkiewicz)

The set of all nowhere differentiable functions in the space $C[0,1]$ of continuous on $[0,1]$ functions with uniform distance is a set of second category.

- S. Banach,
  ***Über die Baire'sche Kategorie gewisser Funktionenmengen*,**
  Studia Math. **3** (1931), no. 1, 174–179.
- S. Mazurkiewicz,
  ***Sur les fonctions non dérivables*,**
  Studia Math. **3** (1931), no. 1, 92–94.

Domination

Theorem (T. Zamfirescu)
The set of all singular functions in the space of all continuous monotonic functions with supremum-distance is a set of second category.

T. Zamfirescu,
***Most monotone functions are singular*,**
Amer. Math. Monthly **88** (1981), no. 1, 47–49.

Summary

Summary

We study various systems of encoding (representation) for real numbers with finite and infinite alphabet.

We use these systems for analytical definition and studying some mathematical objects with complicated local structure: fractal sets, singular probability distribution functions, nowhere differentiable functions, and nowhere monotonic functions.

We study topological, metric, and fractal properties of the sets, Lebesgue structure of singular probability distributions, properties of level sets and graphs of nowhere differentiable functions, and fractal properties of such objects.

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