Sharkovsky ordering

and

combinatorial dynamics

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The **Sharkovsky ordering** describes the coexistence of cycles with different periods for discrete-time dynamical systems given by maps $f : I \to I$ where I is an interval in the real line $\mathbf{R}$ and, possibly, $I = \mathbf{R}$. One can also say that it provides the forcing relation $\prec$ for the existence of cycles of certain periods due to the presence of a cycle of another period.

This ordering is the following ordering of natural numbers

$$1 \prec 2 \prec 2^2 \prec 2^3 \prec \cdots \prec 2^n \prec \ldots$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\cdots \prec 7 \cdot 2^n \prec 5 \cdot 2^n \prec 3 \cdot 2^n \prec \ldots$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\cdots \prec 7 \cdot 2 \prec 5 \cdot 2 \prec 3 \cdot 2 \prec \ldots$$

$$\cdots \prec 11 \prec 9 \prec 7 \prec 5 \prec 3$$

or, if use the relation $\succ$,

$$3 \succ 5 \succ 7 \succ \cdots \succ 2{\cdot}3 \succ 2{\cdot}5 \succ \cdots \succ 2^2{\cdot}3 \succ 2^2{\cdot}5 \succ \cdots \succ 2^2 \succ 2 \succ 1$$

Let f^n, $n \geq 1$, denote the n-th iteration of f, i.e., $f^n = f(f^{n-1})$, where f^0 is the identity map. The point $x \in I$ is a periodic point of period m $(m \geq 1)$ for f, if $f^m(x) = x$ and $f^n(x) \neq x$ for any $1 \leq n < m$. In this case, the points $x, f(x), \ldots, f^{m-1}(x)$ form a periodic orbit or a cycle of period m.

Theorem (Sharkovsky, 1964) *If a continuous map of an interval into itself has a cycle of period m, then it has a cycle of any period $\widetilde{m} \prec m$. Moreover, for any m there exists a continuous map that has a cycle of period m but does not have cycles of periods $\overline{m}$, $m \prec \overline{m}$.*

The lecture will deal with some properties of the ordering, its possible generalizations on various classes of maps, spaces, and the history of this ordering birth.

This theorem also shows how cycles of different periods can be arranged on I. If B is a cycle, let $S(B)$ be the interval $[\min\{x \in B\}, \max\{x \in B\}]$, referred to as the support of the cycle. If m is the period of the cycle B and $\widetilde{m}$ is any number such that $\widetilde{m} \prec m$, then the map f also has a cycle $\widetilde{B}$ of period $\widetilde{m}$ such that $S(\widetilde{B}) \subset S(B)$. Indeed, instead of the map f, one can consider a continuous map f_B that coincides with f on $S(B)$ and equals *const* outside the interval $S(B)$. The theorem remains true for f_B, in particular, f_B has cycles of period $\widetilde{m}$, but all cycles of f_B are in $S(B)$.

The ordering $(*)$ can be interpreted in terms of stratification (Block, Coppel 1992). Let $C(I, I)$ denote the set of all continuous maps of I into itself and $\mathbb{P}_n$ be the subset of $C(I, I)$ consisting of maps which have cycles of period n. According to $(*)$, if $m \prec \overline{m}$ then $\mathbb{P}_m \supset \mathbb{P}_{\overline{m}}$. Hence, $\mathbb{P}_1 \supset \mathbb{P}_2 \supset \mathbb{P}_4 \supset ... \supset \mathbb{P}_5 \supset \mathbb{P}_3$.

The ordering $(*)$ has a property of C^0-stability (Block 1981): if f has a cycle of period m, then there exists $\varepsilon = \varepsilon(f, m) > 0$ such that whatever $\widetilde{m} \prec m$, any map $\widetilde{f} : \sup_{x \in I} |\widetilde{f}(x) - f(x)| < \varepsilon$ has a cycle of period $\widetilde{m}$.

The following important corollary of the theorem relates to bifurcation theory: if the map f depends on a parameter, the ordering $(*)$ also gives a universal ordering for the birth of cycles of new periods when this parameter varies. For example, the bifurcation diagram for the logistic family of maps

$$x \mapsto \lambda x(1 - x),$$

shown in Fig. 1, displays the birth of attracting cycles of new periods according to $(*)$, when λ increases from 2.9 up to 4. At first, there is an attracting cycle of period 1 (fixed point), then there arises an attracting cycle of period 2, then of period 2^2, then of period 2^3; the cycle of period 3 appears for $\lambda = 1 + 2\sqrt{2} \approx 3.83$.

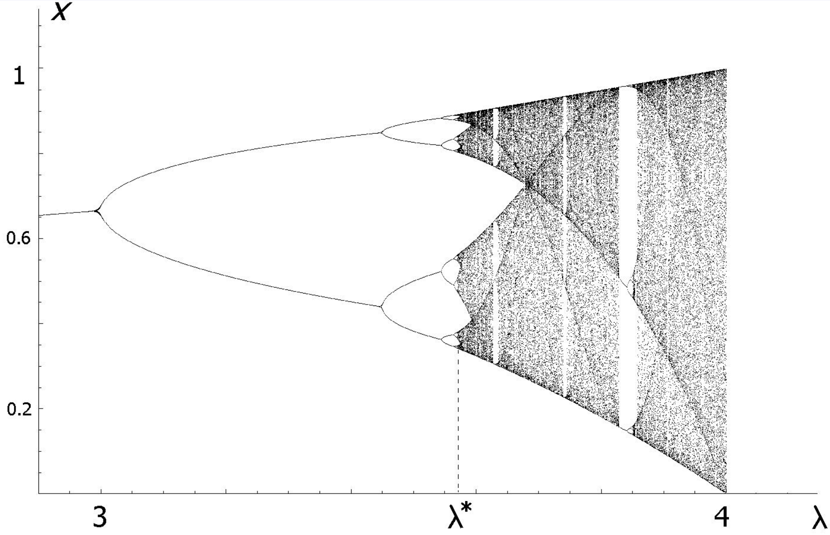
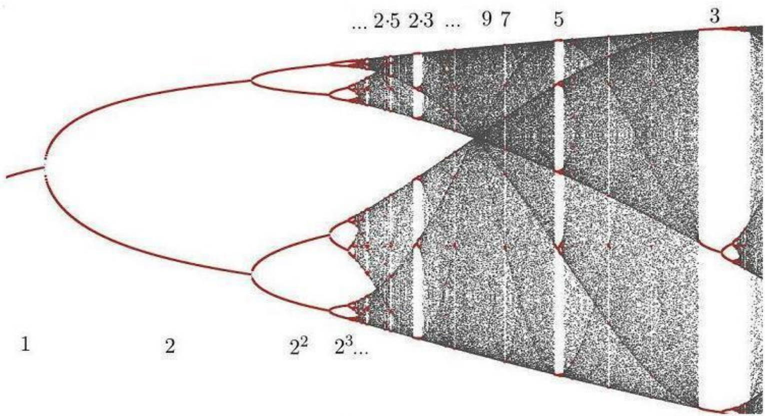


Рис.: The bifurcation diagram for the logistic family of maps
$x \mapsto \lambda x(1-x),\ x \in [0,1]$.

If λ_n denotes the parameter value corredponding to the birth of the first cycle of period 2^n, then, as noticed by Feigenbaum, Coullet, and Tresser,

$$\delta_n = (\lambda_n - \lambda_{n-1})/(\lambda_{n+1} - \lambda_n) \to \delta = 4.66920... \qquad \text{as} \quad n \to \infty,$$

that is, the rate of appearance of cycles of double periods is characterized by the number δ which is often called the *Feigenbaum constant*. It turns out that not only the sequence of bifurcations, defined by $(*)$, but also the rate of bifurcations, defined by the constant δ, are "universal" in the sense that they are valid for the whole class of differentiable maps (and not only the logistic family).



$$1 \prec 2 \prec 2^2 \prec 2^3 \prec 2^4 \prec \dots$$

$$\dots \prec 2^2\cdot 7 \prec 2^2\cdot 5 \prec 2^2\cdot 3 \prec \cdots \prec 2\cdot 7 \prec 2\cdot 5 \prec 2\cdot 3 \prec \dots$$

$$\cdots \prec 9 \prec 7 \prec 5 \prec 3$$

The proof of the theorem is based on the intermediate value theorem and actually uses only the fact that if f is a continuous map and J is an interval such that $f(J) \supset J$ then on J there exists a fixed point of the map f.
Since the end of 1970th, there have been published many papers with various proofs of the theorem or its parts, as well as proofs of the theorem for special classes of maps.

K. Burns, and B. Hasselblatt, *The Sharkovsky theorem: a natural direct proof,* Amer. Math. Monthly **118**(2011), 229-244.

D.-S. Du *A simple proof of theorem,* Amer. Math. Monthly **111**(2004), 595-599.

B.-S. Du *A collection of simple proofs of Sharkovsky's theorem,* arXiv:math/0703592

Сосуществование циклов непрерывного преобразования прямой в себя

А. Н. Шарковский

Всякая непрерывная функция действительного переменного $f(x)$, $-\infty < x < +\infty$, порождает непрерывное преобразование T прямой в себя: $x \to f(x)$. Свойства преобразования T определяются в основном структурой множества неподвижных точек преобразования T.

Напомним, что точку α называют неподвижной точкой порядка k преобразования T, если $T^k\alpha = \alpha$, $T^j\alpha \neq \alpha$, $1 \leqslant j < k$. Точки $T\alpha, T^2\alpha, \ldots, T^{k-1}\alpha$ также являются неподвижными порядка k и вместе с точкой α составляют цикл порядка k.

В этой работе исследуется вопрос о зависимости между существованием циклов различных порядков.

Основной результат настоящей работы может быть сформулирован в следующей форме. Рассмотрим множество натуральных чисел, в котором введено отношение: n_1 предшествует n_2 ($n_1 \preceq n_2$), если для всякого непрерывного преобразования прямой в себя существование цикла порядка n_1 влечет за собой существование цикла порядка n_2. Такое отношение, очевидно, обладает свойствами рефлексивности и транзитивности, и, следовательно, множество натуральных чисел с этим отношением есть квазиупорядоченное множество*. Ниже доказывается

Т е о р е м а. *Введенное отношение превращает множество натуральных чисел в упорядоченное множество и притом упорядоченное следующим образом*

$$3 \prec 5 \prec 7 \prec 9 \prec 11 \prec \ldots \prec 3\cdot 2 \prec 5\cdot 2 \prec \ldots \prec 3\cdot 2^2 \prec 5\cdot 2^2 \prec \ldots \prec$$
$$\prec 2^3 \prec 2^2 \prec 2 \prec 1. \qquad (*)$$

Терминологией упорядоченных множеств в дальнейшем мы пользоваться не будем. Доказательства теорем по существу опираются только на теорему Больцано—Коши о промежуточном значении.

Из непрерывности преобразования T сразу вытекает, что *если у преобразования T существует цикл порядка $k > 1$, то преобразование T имеет и неподвижную точку первого порядка.*

Т е о р е м а 1. *Если преобразование T имеет цикл порядка $k > 2$, то оно имеет и цикл второго порядка.*

Пусть $\alpha_1, \alpha_2, \ldots, \alpha_k$ — точки цикла, причем $T\alpha_i = \alpha_{i+1}$, $i = 1, 2, \ldots, k-1$, $T\alpha_k = \alpha_1$. Пусть $\alpha_1 < \alpha_i\ (i \neq 1)$, $\alpha_r > \alpha_i\ (i \neq r)$. Рассмотрим интервал (α_1, α_{r-1}) (считаем, что $r > 2$; если $r = 2$, следует взять интер-

* Г. Биркгоф, Теория структур, Гостехиздат, М., 1952, стр. 16—21.

Теорема 7. *Между любыми двумя точками цикла порядка $k > 1$ лежит хотя бы одна точка цикла порядка $l < k$.*

Пусть $\alpha > \beta$ — точки цикла порядка k; n_α, n_β — количество точек этого цикла, меньших соответственно точек α и β. Очевидно, $k > n_\alpha > n_\beta \geqslant 0$. Существует n_α различных целых положительных чисел s_i, $i = 1, 2, \ldots, n_\alpha$, меньших k и таких, что $T^{s_i}\alpha < \alpha$. Так как $n_\alpha > n_\beta$, найдется s_{i_0}, $1 \leqslant i_0 \leqslant n_\alpha$, такое, что $T^{s_{i_0}}\alpha < \alpha$, $T^{s_{i_0}}\beta > \beta$. А это означает, что существует точка $\gamma \in (\beta, \alpha)$, для которой $T^{s_{i_0}}\gamma = \gamma$; γ есть точка цикла порядка $l \leqslant s_{i_0} < k$.

В заключение отметим еще, что все результаты можно перевести на язык периодических решений функционального уравнения $y(x+1) = f(y(x))$ (x пробегает дискретную последовательность значений). Например, если преобразование прямой в себя $y \to f(y)$ непрерывно, то 1) если функциональное уравнение имеет периодическое решение с периодом k, то у него есть и периодические решения с любым периодом, следующим в (*) за k, 2) если уравнение не имеет периодического решения с периодом k, то у него нет периодических решений ни с каким периодом, предшествующим k в (*).

Автор приносит благодарность Ю. М. Березанскому и Ю. А. Митропольскому, ознакомившимся с рукописью работы и давшим ряд полезных советов.

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Поступила 22.III 1962 г.
Киев

Co-existence of the cycles of a continuous mapping of the line into itself

A. N. Sharkovsky

Summary

The basic result of this investigation may be formulated as follows. Consider a set of natural numbers in which the following relationship is introduced: n_1 precedes n_2 ($n_1 \preceq n_2$), if for any continuous mappings of the real line into itself the existence of a cycle of order n_2 follows from the existence of a cycle of order n_1. The following theorem holds.

Theorem. The introduced relationship transforms the set of natural numbers into an ordered set, ordered in the following way: $3 \prec 5 \prec 7 \prec 9 \prec 11 \prec \ldots \prec 3\cdot 2 \prec 5\cdot 2 \prec \ldots \prec 3\cdot 2^2 \prec 5\cdot 2^2 \prec \ldots \prec 2^3 \prec 2^2 \prec 2 \prec 1$.

The progress towards the ultimate goal — the publication of the article "Coexistence of the cycles of a continuous map of the line into itself" in the Ukrainian Mathematical Journal [UMZh, 1964, 16, No. 1, 61-71] — was more or less traditional and lasted about two years. According to dates in the journal publications, it began in May 1960 from the statement $\forall (k > 2) \ \succ 2$ appeared in the article "Necessary and sufficient conditions for the convergence of one-dimensional iterative processes" [UMZh, 1960, 12, No. 4], continued with the statement $\forall k \neq 2^i \succ ... 2^m \succ 2^{m-1} \succ ... \succ 1$ in the article "On the reducibility of a continuous function..." [Reports of Acad.Sci.USSR, 1961, 130, No.5], and completed in March 1962 with a submission and acceptance of the aforementioned final article to the journal, where it was printed already in the 1964. True, in order to pay attention by mathematicians to it, it took another 13 years or more.

Probably the first time the words "Sharkovsky ordering" as a mathematical term were used by Peter Kloeden in his article "On Sharkovsky's cycle coexistence ordering" [Bull. Austral. Math. Soc., 1979, vol. 20, 171-177].

In that article P.Kloeden showed that the ordering is also true for the multidimensional maps $x \mapsto F(x)$, $F = (f_1, ..., f_n)$ of the kind

$$f_i(x_1, ..., x_n) \;=\; f_i(x_1, ..., x_i), \quad i = 1, ..., n, \quad n > 1;$$

these maps are now called *triangular*. For them, the i-th coordinate depends only on the first i coordinates, i.e. a triangular map is a skew product of one-dimensional maps.

I was asked many times why it came to my mind to investigate such a topic, not very popular at that time. As I already wrote, I became acquainted with the iterations of functions during the first and second years of study at Taras Shevchenko Kiev State University, where I was involved in mathematical circles and discovered some interesting facts such as, for example, that the iterated sine $\sin_n(x) = \sin(\sin_{n-1}(x))$, $n = 1, 2, \ldots,$ converges to 0 as $\sqrt{3/n}$. And the decision to study one-dimensional iterations came in 1958, at the last 5th year of study, when it was time to write a graduate work.

The graduation work on the iterations was written and successfully defended, but in the course of its implementation, new questions arose that seemed interesting to the author. So, in November 1958, when I became a graduate student at the Institute of Mathematics of the Academy of Sciences of Ukraine, the problem of choosing the subject of research was not stand in front of me. (Even though there was a problem with a supervisor of the graduate student: all potential supervisors insisted that the young man had to deal with the topics that they do. As a result, at the suggestion by Yuri Alekseevich Mitropolsky, who was the Director of the Institute, it was approved that the official supervisor will be Nikolai Nikolaevich Bogolyubov, who already had moved to Moscow (Dubna)).

After the first year of graduate school, which was mainly devoted to the preparation of so-called candidate exams, I was able to actively engage in research. As a result, at June 1961, I presented my thesis that was adopted to defense. The thesis was entitled “On some problems of the theory of one-dimensional iterative processes” and it was based on four articles by the author, three of which were published in 1960-61 in UMZh, and the fourth one in the above-mentioned Reports of the Academy of Sciences of the USSR. Thus, the Ph.D. thesis already contained a part of the o r d e r i ng $\forall k \neq 2^i \succ ... 2^m \succ 2^{m-1} \succ ... \succ 1$, and since Reports of the Academy of Sciences of the USSR was translated (by AMS) into English already at that time, this statement became available also to English-speaking readers.

Yuri Makarovich Berezansky was an official opponent to my thesis. He worked at our institute, and I met him many times at that time discussing my dissertation and other problems. Since my head was busy "clarifying the details" on the coexistence of cycles, I talked to him about my progress in this direction, and Yu. M. expressed his doubts that specific "details" (or parts) of the coexistence ordering for cycles actually occur, because it sounded very unusual. However, soon after the defense that took place on October 28th, 1961, the proof crystallized out in the whole within two days, and then, as well as I remember, it took (as many as!) 11 days to put everything on paper. The title arose "by itself": at that time, the political term "peaceful coexistence of two systems, capitalism and socialism" was used very often in mass media, and it seemed that the word c o e x i s t e n c e had to be highly appropriate for the situation that loomed with periods of cycles (although, perhaps, the word "forcing" would reflect the essence of the matter more accurately).

It took about 3 months more to finalize the draft version, to print the handwritten text on a typewriter in several copies as required by the journal staff and then write the formulas in all printed copies by hand. The article also included more than ten drawings that had to be made on separate sheets. Finally, in March of 1962, the article was sent to the Ukrainian Mathematical Journal, and the editorial sent it for review. As Y. M. told me later, at his suggestion the article was sent for review to a well-known topologist (it seems to be Albert Solomonovich Schwartz) who could understand the proposed proof and dispel any doubts that arise. About a year later, a positive review was received and it contained one recommendation to use the term Λ-scheme instead of Λ-construction. The recommendation was accepted by the author, and the replacement was made. I myself had doubts about the Lemmas 1-3, which are rather trivial: is it worth to include them? The reviewer dispelled my doubts by writing that, of course, it is worth having these lemmas in the text for completeness.

After reviewing and editing the text, the manuscript was sent to the printing house, from where I was soon asked how to handle the $\succ$ badge that was not available in the typography (at that time, for each letter or icon, it was necessary to have a cast made from lead, and all the text was typed by a typesetter from such casts by hand). To the question from the printing house, I answered that the easiest way is probably to lay the letter Y on its side, which was done as a result (though they put it on the wrong side).

In 1967, I had my first travel abroad to Prague, where I participated in a conference on nonlinear oscillations. My report was devoted to one-dimensional difference equations and included, in particular, the theorem on the coexistence of periodic solutions with different periods. The organizers published texts of almost all reports in the Proceedings of the conference, but my report was presented by an abstract only [“Proc. 4th Conf. on Nonlinear Oscillations”, Academia Publ. House, Prague, 1968, p. 249]: According to the organizing committee, a strange ordering of natural numbers, and moreover written for the simplest difference equation, can hardly be related to a very serious theory of nonlinear oscillations.
At that time, there was nothing surprising in the similar attitude of many mathematicians to one-dimensional dynamical systems. For example, very known mathematician Yakov Sinai wrote in his book “Modern Problems of Ergodic Theory” (Fizmatlit, Moscow, 1995) : “About twenty years ago I had the general feeling that the structure of one-dimensional dynamical systems is relatively simple and can be fully understood, and at the same time, the results valid for the one-dimensional case do not have natural multidimensional analogs.

The years after this have shown that both of these sensations are wrong. First, new surprising and unexpected patterns were discovered here, and, second, some of them are naturally transferred to the case of any dimension." [Lecture 11 "Sharkovsky order and Feigenbaum universality"].

Finally, in 1975, the appearance of the article by T.Li and J.Yorke "Period three implies chaos" attracted the attention of mathematicians to one-dimensional dynamic systems (and, of course, to the notion of *chaos*) and the article by P.Stefan "A theorem of Sarkovskii on the existence of periodic orbits of continuous endomorphisms of the real line"[Comm. Math. Phys., 1977, 54, 237-248] literally pulled Sharkovsky the ordering out of non-existence and showed that very interesting facts in one-dimensional systems had already been found. We can say that from this point the ordering began its own life, eventful and independent of the author.

Since the original proof was far from optimal, many people were tempted to suggest their own proofs that would be more or less "normal", and as a result, in the late 1970s – early 1980s, several proofs were built by the efforts of several mathematicians or groups of mathematicians (.....). Since the study of one-dimensional systems seemed very promising, it attracted quite a lot of mathematicians. Soon it was already possible to say about the emergence of a new direction in dynamical systems — a new section called *"Combinatorial dynamics"*. Some summaries and prospects of these studies were considered at a special conference "Thirty years after Sharkovskii's theorem: new perspectives" (Murcia, Spain, 1994) [Proceed. Conf.(eds Alseda L.,Balibrea F., Llibre J., Misiurewicz M.), Intern. J. Bifurcation and Chaos **5**(5), 1995, and World Sci. Ser. Nonlinear Sci. B, vol. 8, 1996].

* * * * *

Let us consider several statements (examples, facts) of combinatorial dynamics.

1) the simplest, or minimal, cycles ;

2) rotation theory by A. Blokh and M.Misiurewicz ;

3) coexistence of homoclinic trajectories and

stratification of the space $C^0(I, I)$

The Sh-theorem started a new field in the dynamical systems theory that can be appropriately called *combinatorial dynamics.*

While the Sh-theorem is stated in the language of a specific ordering among the periods of cycles of an interval map, in reality it solves the problem of fully describing all possible sets of periods of an interval map. Thus, a cycle is labeled by its period viewed as the type of the cycle, and we describe all possible sets of types of cycles of interval maps. Therefore, one direction of the one-dimensional combinatorial dynamics is to describe possible sets of types of periodic orbits of one-dimensional maps.
This can be done by describing special ordering among types of cycles (so-called *forcing* relation) and then using it in the same way as the Sh-Theorem is used for the full characterization of all possible sets of periods of interval maps.

We will first define *simplest cycles*. In what follows when considering cycles of a map f we will denote the first point of a cycle by p_0 and the like while setting $f^i(p_0) = p_i$, etc.

Definition

A periodic orbit of an odd period n is called a *simplest cycle* if $n \geq 3$ and the orbit can be described as

$$p_{n-1} < p_{n-3} < \cdots < p_4 < p_2 < p_0 = p_n < p_1 < p_3 < \cdots <$$
$$\cdots < p_{n-4} < p_{n-2}$$

or

$$p_{n-2} < p_{n-4} < \cdots < p_3 < p_1 < p_0 = p_n < p_2 < p_4 < \cdots <$$
$$\cdots < p_{n-3} < p_{n-1}$$

Evidently, any cycle of period three is a simplest cycle.

If the period is five this is no longer the case. Indeed, using the same notation as in definition we see that if $p_4 < p_2 < p_0 = p_5 < p_1 < p_3$ or $p_3 < p_1 < p_0 = p_5 < p_2 < p_4$, then the cycle is a simplest cycle while it is easy to see that there are other cycles of period five too (such as, e.g., $p_0 = p_5 < p_1 < p_2 < p_3 < p_4$).

Theorem (L.Block, 1979). If $f \in C^0(I, I)$ has a cycle of period n, then f has also a simplest cycle of period n.

The Sh-theorem uses a specific order among the periods of cycles of an interval map. One can think of periods having different strengths, so that stronger periods *force* weaker periods (to be among periods of cycles of an interval map). Simply put, periods force periods. As a result, one describes all possible periods of cycles of continuous interval maps. However, there are finer than periods but still numerical ways to describe interval cycles (i.e., a cycle is characterized by a fixed number of integers - say, two - however long the cycle is). The concepts of rotation pair/number (A.M.Blokh, M.Misiurewicz,1995,1997) fit into this description. Here it turns out to be very useful concept of the number of rotations per period.

The movement on $\mathbb{R}$ is to the right or left. Let $A = \{a_1 < a_2 < \cdots < a_k\}$ be points of a cycle and let a map f act on these points according to a cyclic permutation $i_1 = 1 \to i_2 \to i_3 \to \ldots \to i_k \to 1$: first $a_{i_1} = a_1$ moves to the right to a_{i_2}, then a_{i_2} maps in some direction to a_{i_3}, etc. Each time the direction of the point's movement changes, it can visualized as the turn (rotation) of the point by 180 degrees *in the positive direction*. Taking two 180-degree rotation as one (full) rotation, after k steps the point comes back to itself and the direction in which it moves is the same. The cumulative rotation p is just the **number of rotations per period** corresponding to $2p$ changes of the direction of the movement of a point.

Define the *rotation pair* of a cycle as (p, q), where q is the period of the cycle and p is the number of rotations per period of this cycle. The number p/q is called the *rotation number* of the cycle. Let us introduce the following partial ordering among all pairs of integers (p, q).

We will write $(p, q) \gtrdot (r, s) \quad$ if $\quad p/q < r/s$ or $p/q = r/s = m/n$ with m and n coprime and $p/m \succ r/m$ (notice that $p/m, r/m \in \mathbb{N}$).

Theorem

(A.M.Blokh, M.Misiurewicz, 1995,1997) If $f : [0,1] \to [0,1]$ *is continuous and has a cycle of rotation pair* (p,q) *then* f *has cycles of any rotation pair* (r,s) *such that* $(p,q) \gtrdot (r,s)$*.*

This theorem can be understood in the sense of *forcing* among rotation pairs of interval cycles: the fact that (p,q) is the rotation pair of a cycle of a map f *forces* the presence of other cycles of f with every rotationt pair (r,s) such that $(p,q) \gtrdot (r,s)$. Evidently, this theorem is modeled after the Sh-theorem. Moreover, the theorem implies a full description of the sets of rotation pairs for continuous maps; as in the Sh-theorem, all theoretically possible sets really occur.

One example: suppose that we know that an interval map f has a cycle of period, say, 11; then according to the Sh-theorem we can only guarantee that it has cycles of periods 13, 15 etc. However we cannot say in what cases the existence of a cycle of period 11 forces the existence of cycles of periods, say, 3, or 5, or 7, or 9. Are there any cycles of period 11 that in fact force the existence of cycles of odd periods of less than 11?

Assume now, that there exists an f-cycle of rotation pair, say, $(2, 11)$. Then not only can we guarantee that f has cycles of periods 9, 7, 5 and 3 but also that some of these cycles have rotation pairs $(2, 9), (3, 9), (4, 9), (2, 7), (3, 7)$, $(1, 5), (2, 5)$ and $(1, 3)$. Thus, since we are now using more informative input we are getting a slightly richer output. Also, Theorem 1 and the definition of the order $\gtrdot$ are easy to follow as both are basically related to the order of rotation numbers (all of whom must be less than or equal to $1/2$) with respect to their distance to $1/2$; this order is rather transparent and easy to grasp.

Coexistence of periodic and homoclinic trajectories

Along with periodic trajectories, homoclinic trajectories play an important role in the dynamics. Their presence in a system indicates also the presence of trajectories with very complex behavior. In particular, homoclinic trajectories are thoroughly studied for multidimensional dynamical systems.

Usually, the trajectory of a dynamical system, different from periodic, call homoclinic if its α-limit and ω-limit sets coincide and are the same cycle. If $f \in C^0(I, I)$, then the map f^{-1} is multivalued, therefore the definition needs to be adjusted.

Definition H *Call a trajectory* $x_0, x_1, x_2, \ldots$ *(and a point* x_0*) homoclinic (to a cycle* $(\beta_1, \ldots, \beta_m)$*), if* x_0 *is not periodic, its* ω*-limit set is a cycle* $(\beta_1, \ldots, \beta_m)$*, and there exists a sequence of points* $x_{-1}, x_{-2}, \ldots$ *with* $f(x_{-i}) = x_{-i+1},\ i = 1, 2, \ldots$ *such that* $x_{-jm+k} \to \beta_k,\ k = 1, \ldots, m,\ j = 0, 1, 2, \ldots,$ *when* $j \to \infty$*; here the sequence* $x_{-1}, x_{-2}, \ldots$ *is simply a branch of the backward trajectory of* $x = x_0$*.*

The term *homoclinic trajectory* appeared in one-dimensional dynamics for the first time probably back in 1969 where it was remarked that a homoclinic trajectory exists in a system then and only then when there exists a cycle of period $\neq 2^m,\ m > 0$, and later this statement was proved. Namely, in 1978 Louis Block proved the followig theorem.

Theorem *Let $f \in C^0(I, I)$. Then the following are equivalent:*
(i) f has a periodic point whose period is not a power of 2.
(Here, $1 = 2^0$ is included as a power of 2.)
(ii) f has a homoclinic point.
(iii) There are disjoint closed intervals J and K in I, and a positive integer n, such that $f^n(J) \supseteq J \cup K$ and $f^n(K) \supseteq J \cup K$.

The statement (iii) means that f^n has on $J \cup K$ so-called the Smale' horseshoe or the Λ–scheme.

It is natural to expect that the coexistence of homoclinic trajectories is closely related to the coexistence of cycles that are limits of these homoclinic trajectories. To describe the coexistence of homoclinic trajectories for one-dimensional dynamical systems, the following classification of homoclinic trajectories was proposed.

Definition $H_{1,2}$
We say that a homoclinic trajectory to some cycle is one-sided, if there exists its backward trajectory to this cycle, which tends to each point of the cycle from one side only. If such a backward trajectory do not exists, we will call this trajectory as two-sided homoclinic trajectory.

Definition H_m
We shall call a homoclinic trajectory as m-homoclinic one if it is an one-sided homoclinic trajectory to a cycle of period m or a two-sided homoclinic trajectory to a cycle of period $m/2$.

Theorem Homoclinic trajectories induce in the space $C^0(I,I)$ the following order among integers

$$1 \triangleright 3 \triangleright 5 \triangleright 7 \triangleright 9 \triangleright \ldots \triangleright 2{\cdot}1 \triangleright 2{\cdot}3 \triangleright 2{\cdot}5 \triangleright \ldots \triangleright 2^2{\cdot}1 \triangleright 2^2{\cdot}3 \triangleright 2^2{\cdot}5 \triangleright \ldots : \qquad (**)$$

if $f \in C^0(I,I)$ has a m-homoclinic trajectory, then f has also a k-homoclinic trajectory for every k, $m \triangleright k$.

The difference between $(**)$ and Sh-ordering

$$3 \succ 5 \succ 7 \succ \cdots \succ 2{\cdot}3 \succ 2{\cdot}5 \succ \cdots \succ 2^2{\cdot}3 \succ 2^2{\cdot}5 \succ \cdots \succ 2^2 \succ 2 \succ 1$$

is that all powers of two "migrate" in $(**)$ in the respective blocks generated by (with the participation of) odd numbers, where they become the "strongest". Specifically, the "strongest" of all numbers is 1.

Let $F(m)$ be the set of all continuous functions of interval which has a cycle of period m and
let $H(m)$ be the set of all continuous functions of interval which has a one-sided homoclinic trajectory to a cycle of period m or a two-sided homoclinic trajectory to a cycle of period $m/2$.

Then in these notations

$$H(1) \subset H(3) \subset H(5) \subset H(7) \subset \dots$$

$$\dots \subset H(2) \subset H(2 \cdot 3) \subset H(2 \cdot 5) \subset \dots$$

$$\dots \subset H(2^2) \subset H(2^2 \cdot 3) \subset H(2^2 \cdot 5) \subset \dots$$

and the following theorem is true

Theorem $H(n) = F(n)$ for any $n = m2^k$, where $m > 1$ is odd and $k \geq 0$.

As the consequence of all statements above, we obtain the following stratification of space $C^0(I, I)$:

Theorem For $f \in C^0(I, I)$, the following is true

$$F(1) \supset F(2) \supset F(2^2) \supset \dots \supset F(2^\infty) \supset$$

$$. \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; .$$

$$\dots \supset F(5 \cdot 2^k) = H(5 \cdot 2^k) \supset F(3 \cdot 2^k) = H(3 \cdot 2^k) \supset H(2^k) \supset$$

$$. \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; .$$

$$\dots \supset F(5 \cdot 2^2) = H(5 \cdot 2^2) \supset F(3 \cdot 2^2) = H(3 \cdot 2^2) \supset H(2^2) \supset$$

$$\dots \supset F(5 \cdot 2) = H(5 \cdot 2) \supset F(3 \cdot 2) = H(3 \cdot 2) \supset H(2) \supset$$

$$\dots \supset F(5) = H(5) \supset F(3) = H(3) \supset H(1)$$