ON THE REDUCIBILITY OF A CONTINUOUS FUNCTION OF A REAL VARIABLE AND THE STRUCTURE OF THE STATIONARY POINTS OF THE CORRESPONDING ITERATION PROCESS

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Let $f(x)$ be a real function defined on the whole real axis R. We say that a set $M \subseteq R$ reduces $f(x)$ if $f(M) \subseteq M$ and $f(R \setminus M) \cap M = 0$.

The reducibility of a function is naturally closely connected with the structure of the iteration process generated by the function, i.e., with the regions of attraction of its fixed points and their boundaries. As we shall see below, the latter are, in the case of a function of one real variable, completely determined by the fixed points and the points that are carried into them, i.e., the zeros of the functions $f_{k+m}(x) - f_m(x)$, $k = 1, 2, \cdots$; $m = 0, 1, 2, \cdots$, where $f_i(x) = f_{i-1}(f(x))$, $i = k + m$, m.

As usual, a point α is called a fixed point of order k if $f_k(\alpha) = \alpha$ and $f_j(\alpha) \neq \alpha$ for $1 \leq j < k$. The fixed points of order k of the iteration process generated by a continuous function form a closed set which, like every closed set, can be represented as the union of a perfect set and an at most countable set. The points of the second kind of the perfect set ordinarily do not play an important role in the structure of the iteration process, and in what follows we shall, for the sake of simplicity, suppose that the set of fixed points has no perfect component. Then the set of fixed points of all orders is at most countable. It is known, in addition, that there are attractive, repulsive, and so-called indifferent fixed points. Among the latter are limit points of isolated fixed points and fixed points that are points of the second kind of perfect sets * which, as agreed above, we are not considering. Finally, we note that the character of a fixed point may be different on opposite sides, for example it may be an attractive point on one side and a repulsive point on the other.

The following three theorems are concerned with the structure of sets of fixed points. Broadly speaking, they consider relations between the existence of fixed points of different orders, and between the existence of attractive and repulsive fixed points.

It was shown in [2] that if the iteration process generated by a continuous function has a fixed point of order $k > 2$ then it also has a fixed point of the second order. Hence there follows at once:

Theorem 1. *If the iteration process generated by the continuous function $f(x)$ has no fixed point of order 2^m, then it can have only fixed points of order 2^i, $i = 0, 1, \cdots, m-1$. But if it has a fixed point of order different from 2^j, $j = 0, 1, 2, \cdots$, then it has fixed points of arbitrarily high order.*

In fact, if there is no fixed point of order 2^m, then there is none of order $s \cdot 2^{m-1}$, $s = 3, 4, \cdots$, and the zeros of the functions $f_{s \cdot 2^{m-1}}(x) - x$, $s = 2, 3, \cdots$, coincide with the zeros of $f_{2^{m-1}}(x) - x$. If the iteration process has a fixed point of order $l \neq 2^i$, $i = 0, 1, \cdots, m-1$, then the function $f_l(x) - x$, and therefore also the function $f_{l \cdot 2^{m-1}}(x) - x$, has zeros different from those of $f_{2^{m-1}}(x) - x$, which is impossible. If, finally, the iteration process has a fixed point of order different from a power of two, e.g., one of order 3, then it also has fixed points of orders 2^j, $j = 0, 1, 2, \cdots$, i.e., points of arbitrarily high order.

* In [1] a fixed point α of order k is called indifferent if $df_k(x)/dx|_{x=\alpha} = 1$.

Theorem 2. *If $f(x)$ is continuous on R, there is always a repulsive fixed point between any two attractive fixed points.*

If $f(x)$ is differentiable on R, then between any two repulsive fixed points there is at least either an attractive fixed point or a countable number of repulsive fixed points including some of arbitrarily high order.

This theorem follows directly from the continuity of the function and the fact that if a fixed point of order k is attractive, then $f_k(x) > x$ to the left of it and $f_k(x) < x$ to the right; if a fixed point is repulsive, then because of the differentiability $f_{2k}(x) < x$ to the left and $f_{2k}(x) > x$ to the right of the fixed point.

The conclusion holds for fixed points that have the required character only on one side.

The requirement of differentiability is essential. For example, the iteration process generated by the function

$$f(x) = \begin{cases} ax - x^2, & x \leq a - 1, \\ ax - (a-1)^2, & x \geq a - 1, \end{cases} \quad a > 2,$$

has only two fixed points and both are repulsive.

It remains to clarify the structure of the set of repulsive fixed points.

Theorem 3. *The closure of the set of repulsive fixed points can be any closed set.*

In other words, the closure of the set of repulsive fixed points can contain a perfect set even though the set itself may contain only a countable number of points. In fact, in the first place, an iteration process can have isolated repulsive fixed points which may have a limit point. An example is the iteration process generated by $f(x) = x + x^2 \sin(1/x)$. The indifferent fixed point $x = 0$ is a limit point of repulsive (and of attractive) isolated fixed points.

Furthermore, the closure of the set of repulsive fixed points may contain a nowhere-dense perfect set. For example, when an iteration process has fixed points of arbitrarily high order, these form a perfect set which is usually nowhere dense. This can easily be shown, but to prove the theorem it is enough to exhibit one example.

Let M be a closed interval, $f(M) \supset M$, $f(R \setminus M) \cap M = 0$, and let the set M^1 for which $f(M^1) = M$, $f(M \setminus M^1) \cap M = 0$, consist of at least two closed intervals, on each of which $f(x)$ is monotonic. If there is no set $\widetilde{M} \subset M$ such that $f(\widetilde{M}) \subseteq \widetilde{M}$ and $\operatorname{mes} \widetilde{M} > 0$, then M contains a nowhere dense perfect set M^0 such that $f(M^0) = M^0$, and M^0 is a closed set of repulsive fixed points.

For example, these conditions are satisfied by the function $ax - bx^2$ with $M = [0, a/b]$, if $a > 4$, $b > 0$.

Take sets M^i such that $f_i(M^i) = M$ and $f_i(M \setminus \bigcup_{j=1}^{i} M^j) \cap M = 0$, $i = 2, 3, \cdots$. Since M^1 is a perfect set and contains at least two closed intervals, the sets M^i and $M^0 = \lim_{i\to\infty} M^i$ are also perfect. We show that M^0 is the required set. Clearly $f(M^0) = M^0$ and M^0 is nowhere dense. Furthermore, every closed interval $M_p^i \subset M^i$, $i, p = 1, 2, \cdots$, contains at least one fixed point (repulsive, since we know that there are no attractive points in M, and M contains M^0), since $f_i(M_p^i) = M$.

Finally we show that the closure of the repulsive fixed points can also be a closed interval. Again let M be a closed interval $f(M) = M$, and let there exist no set $\widetilde{M}$ such that $f(\widetilde{M}) \subseteq \widetilde{M}$ and $\operatorname{mes} \widetilde{M} > 0$. Then fixed points fill out M densely. In fact, if M' is any closed interval contained in M, then $\bigcup_{i=1}^{\infty} f_i(M') = M$, which means that there is a q such that $\bigcup_{i=1}^{q} f_i(M') = M$. The set M clearly contains at least one repulsive fixed point, which we suppose belongs to the set $f_r(M')$, $r \leq q$. We can always find a closed set $M'' \subseteq f_r(M')$ containing this fixed point and such that $f(M'') \supset M''$. In a similar way there is a p such that $\bigcup_{j=1}^{p} f_j(M'') = M$ and consequently $f_p(M'') = M$. Thus $f_{rp}(M') = M$, i.e., M' contains a repulsive fixed point of order at most rp.

An illustrative example is $f(x) = 4x - bx^2$, $b > 0$, on $[0, 4/b]$. Moreover, this case occurs for every $f(x)$ such that $f(\phi(x)) = \phi(\theta(x))$ where $\phi(x)$ is continuous and periodic, and $\theta(x)$ is a continuous function such that for every closed interval $\hat{M}$ there is an m such that the closed interval $\theta_m(\hat{M})$ ($\theta_m(x)$ is the mth iterate of $\theta(x)$) contains at least one complete period of $\phi(x)$. For example, we may take $\theta(x)$ to be the linear function $cx + d$ with $|c| > 1$. Taking $\theta(x) = nx$, $\phi(x) = \cos x$, we obtain the Čebyšev polynomial $\cos(n \arccos x)$ $(M = [-1, 1])$. For the function $(x^2 - 1)/2x = \operatorname{ctg}(2 \operatorname{arc\,ctg} x)$ the fixed points are dense on the whole real axis.

We now state a theorem on the reducibility of a continuous function of a real variable.

Theorem 4. *Let $f(x)$ be defined and continuous on R. Then R falls into sets M_i, $i = 0, 1, 2, \cdots$, such that $f(M_i) \subseteq M_i$. The set M_0 is the closure of the set consisting of the repulsive fixed points of the iteration process generated by $f(x)$, and the points that are carried into them; each of the sets $M_1, M_2, \cdots$ is the region of attraction of k attractive fixed points of order k which are carried into each other, and its boundary consists of fixed points of order not exceeding $2k$ and of points that are carried into these.*

We note that the sets $M_1, M_2, \cdots$ do not intersect. Each of them is open and has no points in common with M_0, if it contains a two-sided attractive point; and can be represented as the union of an open set and a set of points belonging to M_0, if it contains a one-sided attractive point. All the sets $M_0, M_1, M_2, \cdots$ reduce $f(x)$.

The question of the distribution of the points of R into the sets M_i is sufficiently clear, and we omit the proof of the theorem. It was shown in [2] that the boundary of a domain of attraction of points of order k belongs to a repulsive fixed point of order at most $2k$.

The reducibility of a function of a real variable can be used for solving various equations in which the argument of the unknown function contains not only x but also a known function $f(x)$, for example for solving functional equations of the form $\Phi(x, \phi(x), \phi(f(x))) = 0$, where $\phi(x)$ is the unknown function.

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Received 28/MAR/61

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