Симетрія та інтегровність рівнянь математичної фізики

Національна академія наук України
Інститут математики

Збірник праць
Інституту математики НАН України
Том 16, № 1

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Київ — 2019

УДК 517.95:517.958:512.816(06)

Симетрія та інтегровність рівнянь математичної фізики / Відп. ред. А.Г. Нікітін // Зб. праць Інституту математики НАН України. Т. 16, № 1. — Київ: Ін-т математики НАН України, 2019. — 204 с.

ISSN 1815–2910

У збірнику представлено статті з сучасних проблем групового аналізу диференціальних рівнянь, інтегровних систем та алгебраїчних методів математичної фізики.

Розраховано на наукових працівників, аспірантів, які цікавляться застосуваннями груп і алгебр Лі в теорії диференціальних рівнянь та математичній фізиці.

Papers on modern problems of group analysis of differential equations, integrable systems, and on algebraic methods in mathematical physics are presented in the volume.

The volume is intended for scientists and post-graduate students interested in applications of Lie groups and Lie algebras to the theory of differential equations and to mathematical physics.

Затверджено до друку Вченою радою
Інституту математики НАН України
Протокол № 9 від 03.09.2019 р.

Свідоцтво про державну реєстрацію – серія КВ № 8459,
видане 19 лютого 2004

Передмова

21–24 грудня 2018 року в Інституті математики НАН України проходив міжнародний семінар “Симетрія та інтегровність рівнянь математичної фізики” з нагоди 40-ї річниці створення відділу прикладних досліджень (з 2016 року — відділ математичної фізики), див. https://www.imath.kiev.ua/~appmath/conf2018. Семінар традиційно проводиться у грудні в пам’ять про засновника відділу, видатного українського вченого В.І. Фущича (18.12.1936–07.04.1997).

Мета зустрічі — обмін думками вчених, що працюють у галузі групового аналізу диференціальних рівнянь, інтегровності та математичного моделювання. Головними темами семінару були застосування групових методів до дослідження моделей, що описують процеси реального світу, теорія інтегровності, сучасна теорія алгебр Лі включно з контракціями та інваріантами таких алгебр. У семінарі взяли участь 33 учасника з України, Польщі, Кіпру, Австрії, Італії, Канади та Німеччини.

Цей збірник містить статті учасників семінару. До нього увійшли 16 статей, що будуть корисними аспірантам та науковим співробітникам, які цікавляться груповим аналізом диференціальних рівнянь та теорією інтегровності.

Редактори

Збірник праць Інституту математики НАН України 2019, т. 16, № 1, 6–15

УДК 517.912:512.816

Точні розв'язки нелінійного рівняння теплопровідності $u_t = (F(u)u_x)_x + H(u)$

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Запропоновано метод побудови точних розв'язків нелінійного рівняння теплопровідності $u_t = (F(u)u_x)_x + H(u)$, який ґрунтується на використанні підстановки $p(x) = w_1(t)\varphi(u)$, де функція $p(x)$ є розв'язком одного з рівнянь $(p')^2 = Ap^2 + B$, $(p')^2 = Ap^4 + Bp^2 + C$, а функції $w_1(t)$ і $\varphi(u)$ знаходяться з умови, що ця підстановка редукує рівняння до звичайного диференціального рівняння з невідомою функцією $w_1(t)$.

A method for construction of exact solutions to nonlinear heat equation $u_t = (F(u)u_x)_x + H(u)$ which is based on ansätz $p(x) = w_1(t)\varphi(u)$ is proposed. Here the function $p(x)$ is a solution to one of the equations $(p')^2 = Ap^2 + B$, $(p')^2 = Ap^4 + Bp^2 + C$, and the functions $w_1(t)$ and $\varphi(u)$ can be found from the condition that this ansatz reduces the equation to an ordinary differential equation with unknown function $w_1(t)$.

1. Вступ. Робота присвячена побудові точних розв'язків нелінійного рівняння теплопровідності

$$u_t = (F(u)u_x)_x + H(u), \tag{1}$$

яке описує нестаціонарну теплопровідність в нерухомому середовищі, якщо коефіцієнт теплопровідності і швидкість реакції є довільними функціями температури. Групова класифікація рівнянь цього виду, а також точні розв'язки для різних функцій $F(u)$ і $H(u)$ описано в роботах (див. [1, 2, 3] і цитовану там літературу).

У цій статті ми використовуємо метод побудови точних розв'язків рівняння (1), який ґрунтується на класичному методі відокремлення змінних та його узагальненні, а також методі редукції, що лежить в основі симетрійного методу С. Лі. Для побудови точних розв'язків рівняння (1) застосовується підстановка

$$p(x) = w_1(t)\varphi(u), \tag{2}$$

яка містить дві невідомі функції $w_1(t)$ і $\varphi(u)$, а також функцію $p(x)$, яка задається апріорно. Детально розглядаються випадки, коли $p(x)$ є розв'язком одного з таких рівнянь:

$$(p')^2 = Ap^2 + B,$$
$$(p')^2 = Ap^4 + Bp^2 + C,$$

де A, B, C — сталі. При такому виборі функції $p(x)$ невідомі функції $w_1(t)$ і $\varphi(u)$ визначаються з умови, що підстановка (2) редукує рівняння (1) до звичайного диференціального рівняння з невідомою функцією $w_1(t)$.

Відмітимо, що такий підхід був використаний для побудови точних розв'язків рівняння типу Кортевега–де Фріза в [4, 5] і нелінійного рівняння

$$u_{tt} = F(u)u_{xx} + F'(u)u_x^2.$$

2. Розв'язки рівняння (1), що виражаються через тригонометричні функції. Введемо означення

Означення 1. Будемо говорити, що рівняння (1) допускає підстановку (2), якщо вона редукує рівняння (1) до звичайного диференціального рівняння на функцію $\omega_1(t)$.

Для побудови точних розв'язків рівняння (1) використовується підстановка

$$p(x) = w_1(t)\varphi(u), \tag{3}$$

де $p(x)$ є розв'язком рівняння

$$(p')^2 = Ap^2 + B, \quad A \neq 0, \quad B \neq 0.$$

Підставимо (3) в рівняння (1):

$$-\frac{w_1'}{w_1}\frac{\varphi}{\varphi'} = \frac{1}{w_1^2}\left(-FB\frac{\varphi''}{(\varphi')^3} + F'B\frac{1}{(\varphi')^2}\right) + \left(-FA\frac{\varphi^2\varphi''}{(\varphi')^3} + F'A\frac{\varphi^2}{(\varphi')^2} + FA\frac{\varphi}{\varphi'} + H\right). \tag{4}$$

Для визначення функцій $F(u)$ і $\varphi(u)$ отримаємо таку систему рівнянь:

$$-F\frac{\varphi''}{(\varphi')^3} + F'\frac{1}{(\varphi')^2} = \lambda_1\frac{\varphi}{\varphi'}, \tag{5}$$

$$-FA\frac{\varphi^2\varphi''}{(\varphi')^3} + F'A\frac{\varphi^2}{(\varphi')^2} + FA\frac{\varphi}{\varphi'} + H = \lambda_2\frac{\varphi}{\varphi'}, \tag{6}$$

де $\lambda_1, \lambda_2 \in \mathbb{R}$. Нехай $F'(u) \neq 0$. Інтегруючи рівняння (5), яке є лінійним відносно функції $F = F(u)$, знаходимо

$$F = \left(\lambda_1\int\varphi du + C_1\right)\varphi, \tag{7}$$

де тут і далі $C, C_1, C_2, \ldots$ — довільні сталі інтегрування. Підставивши (5), (6) в рівняння (4), отримуємо рівняння для визначення функції $w_1(t)$:

$$\frac{w_1'}{w_1} + \lambda_1 B\frac{1}{w_1^2} + \lambda_2 = 0. \tag{8}$$

З рівнянь (5), (6) знаходимо

$$H = \frac{1}{\varphi'}(-\lambda_1 A\varphi^3 - AF\varphi + \lambda_2\varphi). \tag{9}$$

У підсумку отримаємо таку теорему:

Теорема 1. *Якщо рівняння* (1) *допускає підстановку вигляду* (3) *і* $F'(u) \neq 0$, *то функції* $F(u)$ *і* $H(u)$ *визначаються формулами* (7) *і* (9) *відповідно, а функція* $w_1(t)$ *є розв'язком рівняння* (8).

Отримані розв'язки рівняння (1) можна узагальнити, використовуючи підстановки:

$$\varphi(u) = w_1(t)\,\mathrm{ch}(k\,(x + C_3)) + w_2(t)\,\mathrm{sh}(k\,(x + C_3)), \tag{10}$$

якщо $A = k^2 > 0$,

$$\varphi(u) = w_1(t)\cos(k\,(x + C_3)) + w_2(t)\sin(k\,(x + C_3)), \tag{11}$$

якщо $A = -k^2 < 0$.

Розглянемо, наприклад, підстановку (10). Якщо функції $F(u)$ і $H(u)$ визначаються за формулами (7) і (9) відповідно і $A = k^2 > 0$, то підстановка (10) редукує рівняння (1) до системи

$$w_1' = \left(-\lambda_1 k^2 w_1^2 + \lambda_1 k^2 w_2^2\right) w_1 + \lambda_2 w_1, \tag{12}$$
$$w_2' = \left(-\lambda_1 k^2 w_1^2 + \lambda_1 k^2 w_2^2\right) w_2 + \lambda_2 w_2. \tag{13}$$

Нехай $w_1 \neq 0$. З рівнянь (12), (13) випливає, що $w_2 = Cw_1$. Рівняння (12) набуває вигляду

$$w_1' = \lambda_1 k^2 \left(C^2 - 1\right) w_1^3 + \lambda_2 w_1. \tag{14}$$

Якщо $\lambda_2 \neq 0$, то розв'язком рівняння (14) є функція

$$w_1^2 = \left(\frac{C_2}{\lambda_2}\exp(-2\lambda_2 t) - \frac{\lambda_1}{\lambda_2}k^2\left(C^2 - 1\right)\right)^{-1},$$

де $C_2 \neq 0$. Маємо такий розв'язок рівняння (1):

$$\varphi(u) = \pm\left(\frac{C_2}{\lambda_2}\exp(-2\lambda_2 t) - \frac{\lambda_1}{\lambda_2}k^2\left(C^2 - 1\right)\right)^{-1/2} \times [\operatorname{ch}(k\,(x + C_3)) + w_2(t)\operatorname{sh}(k\,(x + C_3))].$$

Якщо $\lambda_2 = 0$, то розв'язком рівняння (14) є функція

$$w_1^2 = [-2\lambda_1 k^2\left(C^2 - 1\right) t + C_2]^{-1}, \quad \lambda_2 \neq 0.$$

У підсумку отримуємо такий розв'язок рівняння (1):

$$\varphi(u) = [-2\lambda_1 k^2\left(C^2 - 1\right) t + C_2]^{-1/2} \times [\operatorname{ch}(k(x + C_3)) + w_2(t)\operatorname{sh}(k(x + C_3))].$$

Випадок $w_1 = 0$ зводиться до інтегрування рівняння

$$w_2' = \lambda_1 k^2 w_2^3 + \lambda_2 w_2.$$

Отже, якщо $\lambda_2 \neq 0$, то маємо такий розв'язок рівняння (1):

$$\varphi(u) = \left(\frac{C_2}{\lambda_2} \exp(-2\lambda_2 t) - \frac{\lambda_1}{\lambda_2} k^2 \right)^{-1/2} \operatorname{sh}(k(x + C_3)),$$

де $C_2 \neq 0$, а у випадку $\lambda_2 = 0$ — розв'язок

$$\varphi(u) = \left(-2\lambda_1 k^2 \left(C^2 - 1\right) t + C_2\right)^{-1/2} \operatorname{sh}(k\,(x + C_3)).$$

Аналогічно, підстановка (11) редукує рівняння (1) до системи

$$w_1' = (\lambda_1^2 k^2 w_1^2 + \lambda_1^2 k^2 w_2^2) w_1 + \lambda_2 w_1, \tag{15}$$
$$w_2' = (\lambda_1^2 k^2 w_1^2 + \lambda_1^2 k^2 w_2^2) w_2 + \lambda_2 w_2. \tag{16}$$

Проінтегрувавши (15), (16), отримуємо такі розв'язки рівняння (1):

$$\varphi(u) = \left(\frac{C_2}{\lambda_2} \exp(-2\lambda_2 t) - \frac{\lambda_1}{\lambda_2} k^2 \left(1 + C^2\right) \right)^{-1/2} \times [\cos(k\,(x + C_3)) + C \sin(k\,(x + C_3))],$$

де $C_2 \neq 0$, $\lambda_2 \neq 0$;

$$\varphi(u) = (-2\lambda_1 k^2 \left(C^2 + 1\right) t + C_2)^{-1/2} \times [\cos(k(x + C_3)) + C \sin(k(x + C_3))],\ \lambda_1 \neq 0,$$

де $\lambda_1 \neq 0$, $\lambda_2 = 0$;

$$\varphi(u) = \left(\frac{C_2}{\lambda_2} \exp(-2\lambda_2 t) - \frac{\lambda_1}{\lambda_2} k^2 \right)^{-1/2} \sin(k\,(x + C_3)),$$

де $C_2 \neq 0$, $\lambda_2 \neq 0$;

$$\varphi(u) = (-2\lambda_1 k^2 t + C_2)^{-1/2} \sin(k(x + C_3)),$$

де $\lambda_1 \neq 0$, $\lambda_2 = 0$.

3. Розв'язки рівняння (1), що виражаються через еліптичні функції Якобі. Опишемо рівняння виду (1) і їх точні розв'язки, які допускають підстановку

$$p(x) = w_1(t)\varphi(u), \tag{17}$$

де $p(x)$ є розв'язком рівняння

$$(p')^2 = Ap^4 + Bp^2 + C, \quad A \neq 0, \quad C \neq 0. \tag{18}$$

Підставивши в рівняння (1), отримуємо

$$\begin{aligned} -\frac{w_1'}{w_1}\frac{\varphi}{\varphi'} = w_1^2 \left(2AF\frac{\varphi^3}{\varphi'} - AF\frac{\varphi^4\varphi''}{(\varphi')^3} + AF'\frac{\varphi^4}{(\varphi')^2}\right) \\ + \frac{1}{w_1^2}\left(-CF\frac{\varphi''}{(\varphi')^3} + CF'\frac{1}{(\varphi')^2}\right) \\ + \left(-BF\frac{\varphi^2\varphi''}{\varphi'^3} + BF'\frac{\varphi^2}{\varphi'^2} + BF\frac{\varphi}{\varphi'} + H(u)\right). \end{aligned} \tag{19}$$

З рівняння (19) отримуємо систему

$$-F\frac{\varphi''}{(\varphi')^3} + F'\frac{1}{(\varphi')^2} = \lambda_1\frac{\varphi}{\varphi'}, \tag{20}$$

$$2AF\frac{\varphi^3}{\varphi'} + A\varphi^4\left(-F\frac{\varphi''}{\varphi'^3} + F'\frac{1}{(\varphi')^2}\right) = \lambda_2\frac{\varphi}{\varphi'}, \tag{21}$$

$$-BF\frac{\varphi^2\varphi''}{(\varphi')^3} + BF'\frac{\varphi^2}{(\varphi')^2} + BF\frac{\varphi}{\varphi'} + H(u) = \lambda_3\frac{\varphi}{\varphi'}, \tag{22}$$

де $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$. Підставивши (20) в (21), знаходимо

$$F = \frac{\lambda_2}{2A}\frac{1}{\varphi^2} - \frac{\lambda_1}{2}\varphi^2. \tag{23}$$

З рівняння (22)

$$H = -B\varphi^2\left(-F\frac{\varphi''}{\varphi'^3} + F'\frac{1}{(\varphi')^2}\right) - BF\frac{\varphi}{\varphi'} + \lambda_3\frac{\varphi}{\varphi'},$$

а тому на підставі (20) і (23):

$$H(u) = -\frac{\lambda_1 B}{2}\frac{\varphi^3}{\varphi'} + \lambda_3\frac{\varphi}{\varphi'} - \frac{\lambda_2 B}{2A}\frac{1}{\varphi\varphi'}. \tag{24}$$

Підставивши (23) в (20), знаходимо рівняння для визначення функції $\varphi = \varphi(u)$:

$$\varphi'' = \left(\frac{\lambda_2}{A} + 2\lambda_1\varphi^4\right)\left(\frac{\lambda_1}{2}\varphi^5 - \frac{\lambda_2}{2A}\varphi\right)^{-1}(\varphi')^2. \tag{25}$$

Підставивши (20)–(22) в (19), отримуємо рівняння для визначення функції $w_1 = w_1(t)$:

$$\frac{w_1'}{w_1} + \lambda_2 w_1^2 + \frac{\lambda_1 C}{w_1^2} + \lambda_3 = 0. \tag{26}$$

У підсумку отримаємо таку теорему:

Теорема 2. *Якщо рівняння* (1) *допускає підстановку* (17)*, то функції* $F(u)$ *і* $H(u)$ *визначаються формулами* (23) *і* (24) *відповідно, а функції* φ *та* $w_1(t)$ *є розв'язками звичайних диференціальних рівнянь* (25) *та* (26)*.*

Таким чином, побудову точних розв'язків виду (17) рівняння (1) зведено до інтегрування рівнянь (25), (26).

Розглянемо два випадки.

I) Випадок $\boldsymbol{\lambda_2 = 0}$. Рівняння (25) набуває вигляду

$$\varphi'' = \frac{4}{\varphi}(\varphi')^2. \tag{27}$$

Інтегруючи рівняння (27), знаходимо

$$\varphi = (C_1 u + C_2)^{-1/3},$$

$C_1 \neq 0$, і на підставі (23), (25)

$$F = -\frac{\lambda_1}{2}(C_1 u + C_2)^{-2/3},$$
$$H = \frac{3\lambda_1 B}{2C_1}(C_1 u + C_2)^{1/3} - \frac{3\lambda_3}{C_1}(C_1 u + C_2).$$

Рівняння (1) набуває вигляду

$$u_t = \left(-\frac{\lambda_1}{2}(C_1 u + C_2)^{-2/3} u_x\right)_x + \frac{3\lambda_1 B}{2C_1}(C_1 u + C_2)^{1/3} - \frac{3\lambda_3}{C_1}(C_1 u + C_2), \tag{28}$$

і підстановкою

$$v = \varphi(u) = (C_1 u + C_2)^{-1/3}$$

зводиться до виду

$$v_t = -\frac{\lambda_1}{2} v^2 v_{xx} + \lambda_1 v (v_x)^2 - \frac{\lambda_1}{2} B v^3 + \lambda_3 v. \tag{29}$$

Інтегруючи рівняння (26) у випадку $\lambda_2 = 0$, знаходимо

$$w_1^2 = C_3 \exp(-2\lambda_3 t) - \frac{\lambda_1}{\lambda_3} C, \quad C_3 \neq 0, \quad \text{якщо } \lambda_3 \neq 0,$$
$$w_1^2 = -2\lambda_1 C t + C_3, \quad \text{якщо } \lambda_3 = 0.$$

У підсумку отримуємо такі розв'язки рівнянь (28), (29):

а) Якщо $A = k^2$, $B = -\big(1 + k^2\big)$, $C = 1$, то

$$v = \varphi(u) = \left(C_3 \exp(-2\lambda_3 t) - \frac{\lambda_1}{\lambda_3} \right)^{-1/2} \operatorname{sn}(x; k), \ \lambda_3 \neq 0,$$
$$v = \varphi(u) = (-2\lambda_1 t + C_3)^{-1/2} \operatorname{sn}(x; k), \ \lambda_3 = 0.$$

б) Якщо $A = -k^2$, $B = 2k^2 - 1$, $C = 1 - k^2$, то

$$v = \varphi(u) = \left(C_3 \exp(-2\lambda_3 t) - (1 - k^2) \frac{\lambda_1}{\lambda_3} \right)^{-1/2} \operatorname{cn}(x; k), \ \lambda_3 \neq 0,$$
$$v = \varphi(u) = \left(-2\lambda_1 \left(1 - k^2\right) t + C_3 \right)^{-1/2} \operatorname{cn}(x; k), \ \lambda_3 = 0.$$

в) Якщо $A = -1$, $B = 2 - k^2$, $C = -1 + k^2$, то

$$v = \varphi(u) = \left(C_3 \exp(-2\lambda_3 t) - (-1 + k^2) \frac{\lambda_1}{\lambda_3} \right)^{-1/2} \operatorname{dn}(x; k), \ \lambda_3 \neq 0,$$
$$v = \varphi(u) = \left(-2\lambda_1 (-1 + k^2) t + C_3 \right)^{-1/2} \operatorname{dn}(x; k), \ \lambda_3 = 0.$$

II) Випадок $\boldsymbol{\lambda_1 = 0}$. Рівняння (25) набуває вигляду

$$\varphi'' = -\frac{2}{\varphi} (\varphi')^2. \tag{30}$$

Інтегруючи рівняння (30), знаходимо

$$\varphi = (C_1 u + C_2)^{1/3},$$

де $C_1 \neq 0$, і на підставі (23), (25)

$$F = \frac{\lambda_2}{2A} (C_1 u + C_2)^{-2/3},$$

$$H = \frac{3\lambda_3}{2A}(C_1u + C_2) - \frac{3\lambda_2 B}{2AC_1}(C_1u + C_2)^{1/3}.$$

Рівняння (1) набуває вигляду

$$u_t = \left(\frac{\lambda_2}{2A}(C_1u + C_2)^{-2/3}u_x\right)_x + \frac{3\lambda_3}{C_1}(C_1u + C_2) - \frac{3\lambda_2 B}{2AC_1}(C_1u + C_2)^{1/3}, \tag{31}$$

і підстановкою

$$v = \varphi(u) = (C_1u + C_2)^{1/3}$$

зводиться до виду

$$v_t = \frac{\lambda_2}{2A}v^{-2}v_{xx} + \lambda_3 v - \frac{\lambda_2 B}{2A}\frac{1}{v}. \tag{32}$$

Підставивши (20)–(22) в (19), отримуємо рівняння для визначення функції $w_1 = w_1(u)$:

$$\frac{w_1'}{w_1} + \lambda_2 w_1^2 + \lambda_3 = 0. \tag{33}$$

Інтегруючи рівняння (33), знаходимо

$$w_1^{-2} = C_3\exp(2\lambda_3 t) - \frac{\lambda_2}{\lambda_3},\ C_3 \neq 0, \text{якщо } \lambda_3 \neq 0;$$
$$w_1^{-2} = 2\lambda_2 t + C_3, \text{якщо } \lambda_3 = 0.$$

У підсумку отримуємо такі розв'язки рівнянь (31), (32):

а) Якщо $A = k^2$, $B = -\big(1 + k^2\big)$, $C = 1$, то рівняння (32) має вигляд

$$v_t = \frac{\lambda_2}{2k^2}v^{-2}v_{xx} + \lambda_3 v + \frac{\lambda_2\big(1 + k^2\big)}{2k^2}\frac{1}{v}. \tag{34}$$

Розв'язки рівняння (34):

$$v = \left(C_3\exp(2\lambda_3 t) - \frac{\lambda_2}{\lambda_3}\right)^{1/2}\operatorname{sn}(x;k),\ \text{якщо } \lambda_3 \neq 0;$$

$$v = (2\lambda_2 t + C_3)^{1/2}\,\mathrm{sn}(x;k),\ \text{якщо}\ \lambda_3 = 0.$$

б) Якщо $A = -k^2$, $B = 2k^2 - 1$, $C = 1 - k^2$, то рівняння (32) має вигляд

$$v_t = -\frac{\lambda_2}{2k^2} v^{-2} v_{xx} + \lambda_3 v + \frac{\lambda_2(2k^2-1)}{2k^2}\frac{1}{v}. \tag{35}$$

Розв'язки рівняння (35):

$$v = \left(C_3\exp(2\lambda_3 t) - \frac{\lambda_2}{\lambda_3}\right)^{1/2}\mathrm{cn}(x;k),\ \text{якщо}\ \lambda_3 \neq 0;$$
$$v = (2\lambda_2 t + C_3)^{1/2}\,\mathrm{cn}(x;k),\ \text{якщо}\ \lambda_3 = 0.$$

в) Якщо $A = -1$, $B = 2 - k^2$, $C = -1 + k^2$, то рівняння (32) має вигляд

$$v_t = -\frac{\lambda_2}{2} v^{-2} v_{xx} + \lambda_3 v + \frac{\lambda_2(2-k^2)}{2}\frac{1}{v}. \tag{36}$$

Розв'язки рівняння (36):

$$v = \left(C_3\exp(2\lambda_3 t) - \frac{\lambda_2}{\lambda_3}\right)^{1/2}\mathrm{dn}(x;k),\ \text{якщо}\ \lambda_3 \neq 0;$$
$$v = (2\lambda_2 t + C_3)^{1/2}\,\mathrm{dn}(x;k),\ \text{якщо}\ \lambda_3 = 0.$$

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Збірник праць Інституту математики НАН України 2019, т. 16, № 1, 16–21

УДК 517.912:512.816

(1+1)-вимірні нелінійні еволюційні рівняння другого порядку з максимальними ліївськими симетріями

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Знайдено явний вигляд перетворень, що пов'язують нелінійні (1+1)-вимірні еволюційні рівняння другого порядку з максимальними семивимірними алгебрами ліївських симетрій.

We have established the explicit forms of the transformations that connect nonlinear (1+1)-dimensional evolution equations of the second order with maximal seven-dimensional Lie symmetry algebras.

Розглянемо клас (1+1)-вимірних еволюційних рівнянь порядку n

$$u_t = F(t, x, u, u_1, \ldots, u_n), \tag{1}$$

де $u_t = \frac{\partial u}{\partial t}$, $u_i = \frac{\partial^i u}{\partial x^i}$, $i = 0, \ldots, n$, $u_0 \equiv u$, $n \geq 2$, F — довільна гладка функція. Також будемо використовувати позначення $u_x, u_{xx}, \ldots$ для похідних за змінною x.

Симетрійним властивостям рівнянь з класу (1) присвячено багато досліджень. Крім того, у багатьох випадків саме еволюційні рівняння з класу (1), як правило, виступають базовими прикладами в симетрійному аналізі диференціальних рівнянь (див., наприклад, монографії [6, 9, 15, 17]).

Відповідно до результатів В.В. Соколова [18, p. 173] та Б.А. Магадєєва [12, p. 346] (див. також статтю Р.З. Жданова [19]) контактні перетворення, які зберігають вигляд еволюційних рівнянь (1), вичерпуються перетвореннями

$$\widetilde{t} = \varkappa(t), \quad \widetilde{x} = \phi(t, x, u, u_x), \quad \widetilde{u} = \psi(t, x, u, u_x),$$

де функції ϕ та ψ задовольняють умову контактності

$$\phi_{u_x}(u_x\psi_u + \psi_x) = \psi_{u_x}(u_x\phi_u + \phi_x).$$

Б.А. Магадєєвим [12, теорема 0.1] доведено, що розмірність алгебри контактних симетрій (Cont) (1+1)-вимірних еволюційних рівнянь (1) не перевищує $n+5$ або дорівнює ∞. В останньому випадку еволюційні рівняння зводяться до лінійних за допомогою контактних перетворень. У цій же роботі автором отримано повний перелік алгебр скінченновимірних контактних симетрій еволюційних рівнянь та показано, як описати еволюційні рівняння, які допускають задану алгебру контактної симетрії.

Зокрема, згідно з [12, теорема 3.5], будь-яке рівняння з класу (1) з максимальною $(n+5)$-вимірною алгеброю контактних симетрій, еквівалентне рівнянню

$$u_t = u_n^{\frac{1-n}{1+n}}.$$

При цьому відповідна алгебра контактних симетрій має вигляд [12, див. доведення теореми 3.5 та додаток]:

$$\begin{aligned}\mathfrak{g}_{\mathrm{M1}} = \big\langle 1,\, x,\, \dots,\, x^k,\, u_x,\, -\tfrac{n-1}{2}u + xu_x,\, x^2u_x - (n-1)xu,\\ u_t,\, tu_t + \lambda u\big\rangle,\end{aligned} \tag{2}$$

де $k = 1, \dots, n-1$, $\lambda \neq 0$, $\lambda = -\frac{n+1}{2n}$, $\varphi = \{1, x, \dots, tu_t + \lambda u\}$ — $(n+5)$-компонентна генеруюча функція інфінітезимального оператора

$$\begin{aligned}Q = \tau(t)\partial_t + \xi(t,x,u,u_x)\partial_x + \eta(t,x,u,u_x)\partial_u\\ + \zeta(t,x,u,u_x)\partial_{u_t} + \rho(t,x,u,u_x)\partial_{u_x}\end{aligned}$$

з коефіцієнтами τ, ξ, η, ζ, ρ, які визначаються наступним чином [11, 19]:

$$\begin{aligned}&\tau = -\phi_{u_t}, \quad \xi = -\phi_{u_x}, \quad \eta = \phi - u_t\phi_{u_t} - u_x\phi_{u_x},\\ &\zeta = \phi_t + u_t\phi_u, \quad \rho = \phi_x + u_x\phi_u.\end{aligned}$$

Для довільного $n \geq 2$ всі базисні елементи алебри (2) є продовженнями відповідних ліївських симетрій (тобто алебра (2) є тривіальною алгеброю контактних симетрій). Зокрема, для $n = 2$ ця алгебра має вигляд

$$\mathfrak{g}_{n=2} = \langle \partial_t,\, \partial_x,\, \partial_u,\, 2x\partial_x + u\partial_u + u_t\partial_{u_t} - u_x\partial_{u_x},$$

$$
\begin{aligned}
&x\partial_u + \partial_{u_x},\, 4t\partial_t + 3u\partial_u - u_t\partial_{u_t} + 3u_x\partial_{u_x},\\
&x^2\partial_x + xu\partial_u + xu_t\partial_{u_t} - xu_x\partial_{u_x}\rangle
\end{aligned}
$$

і є продовженням алгебри ліївських (точкових) симетрій рівняння $u_t = u_{xx}^{-1/3}$ (див. реалізацію (5) нижче). Умови на функцію F, при яких клас (1) допускає лише тривіальні контактні перетворення, отримано в роботі [13].

У роботі [19] Р.З. Ждановим встановлено зв'язок між потенціальними та контактними симетріями еволюційних рівнянь (1), а також запропоновано підхід до класифікації таких рівнянь.

Свіжий огляд та останні результати щодо некласичних симетрій еволюційних рівнянь можна знайти в роботі [8].

Значне місце в літературі приділяється знаходженню ліївських симетрій еволюційних рівнянь. Крім того, вивчаються симетрійні властивості різноманітних підкласів класу (1) при $n = 2, 3$. У роботі [5], І.Ш. Ахатов, Р.К. Газізов та Н.Х. Ібрагімов розглянули локальні та нелокальні симетрії для деяких класів еволюційних рівнянь другого порядку, а саме для рівнянь нелінійної теплопровідності, нелінійної фільтрації та газової динаміки. Зокрема, у цій роботі знайдено групу еквівалентності та виконано повну групову класифікацію класу $u_t = H(u_{xx})$. Якщо виключити з розгляду лінійний випадок, то при довільній функції H цей клас допускає п'ятивимірну алгебру ліївських симетрій. Крім того, існує 5 нееквівалентних випадків розширення цієї п'ятивимірної алгебри. У випадку степеневої, логарифмічної та експоненціальної нелінійності, алгебра інваріантності — шестивимірна, а семивимірну алгебру допускають два наступні рівняння [5]:

$$u_t = u_{xx}^{-1/3}, \tag{3}$$

$$u_t = u_{xx}^{1/3}. \tag{4}$$

Згідно з [5], максимальні ліївські алгебри інваріантності рівнянь (3) та (4) наступні:

$$
\begin{aligned}
\mathfrak{g}_{\mathrm{AGI1}} = \langle &\partial_t,\, \partial_x,\, \partial_u,\, 2t\partial_t + x\partial_x + 2u\partial_u,\, x\partial_u,\, 4t\partial_t + 3u\partial_u,\\
&x^2\partial_x + xu\partial_u\rangle,
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
\mathfrak{g}_{\mathrm{AGI2}} = \langle &\partial_t,\, \partial_x,\, \partial_u,\, 2t\partial_t + x\partial_x + 2u\partial_u,\, x\partial_u,\\
&2t\partial_t + 3u\partial_u,\, u\partial_x\rangle.
\end{aligned} \tag{6}
$$

У роботах [4, 7] вивчено симетрійні властивості класу

$$u_t + uu_x = F(u_n).$$

Зокрема, показано, що рівняння

$$u_t + uu_x = u_{xx}^{1/3} \tag{7}$$

допускає семивимірну алгебру Лі

$$\mathfrak{g}_{\mathrm{BF}} = \langle \partial_t,\ \partial_x,\ t\partial_x + \partial_u,\ 4t\partial_t + 5x\partial_x + u\partial_u,\\ u\partial_x,\ (2t - x)\partial_x + u\partial_u,\ (tu - x)(t\partial_x + \partial_u)\rangle. \tag{8}$$

У роботах [1, 2] за допомогою техніки розгалуженого розщеплення виконано повну групову класифікацію ліїських симетрій підкласів $u_t + uu_x = H(u_n)$ та $u_t = H(u_n)$ відповідно, де $n \geq 3$. Див. [16] та список літератури в цій роботі щодо методу розгалуженого розщеплення та інших сучасних алгебраїчних технік симетрійної класифікації диференціальних рівнянь.

Оскільки, згідно з результатом Б.А. Магадєєва [12], існує єдине з точністю до контактних перетворень еквівалентності (1+1)-вимірне еволюційне рівняння другого порядку з семивимірною максимальною алгеброю контактних симетрій, то основна мета цієї роботи полягає в наступному: знайти перетворення, що пов'язують нелінійні (1+1)-вимірні еволюційні рівняння (3), (4) та (7).

Відомо, що рівняння (4) зводиться до рівняння (3) за допомогою контактного перетворення (див. [10, 14])

$$t = -\tilde{t}, \quad x = \tilde{u}_{\tilde{x}}, \quad u = \tilde{x}\tilde{u}_{\tilde{x}} - \tilde{u}, \quad u_t = \tilde{u}_{\tilde{t}}, \quad u_{xx} = \frac{1}{\tilde{u}_{\tilde{x}\tilde{x}}}, \tag{9}$$

де $\tilde{u}$ — нова залежна змінна та $\tilde{t}$, $\tilde{x}$ — нові незалежні змінні.

Зауважимо, що рівняння (4) інваріантне щодо перетворення годографа [3, с. 409]

$$t = \tilde{t}, \quad x = \tilde{u}(\tilde{t}, \tilde{x}), \quad u(t, x) = \tilde{x}, \quad u_t = -\tilde{u}_{\tilde{t}}, \quad u_{xx} = -\frac{1}{\tilde{u}_{\tilde{x}\tilde{x}}}.$$

Таким чином, рівняння нелінійної теплопровідності (4) — ще один приклад годограф-інваріантного еволюційного рівняння другого порядку поряд з рівняннями швидкої дифузії $u_t = u_{xx}u_x^{-1}$ та фільтрації $u_t = u_{xx}\big(1 + u_x^2\big)^{-1}$.

Нами знайдено модифіковане перетворення годографа

$$
\begin{gathered}
t=\tilde{t}, \quad x=\tilde{u}(\tilde{t}, \tilde{x})+\tilde{x}\tilde{t}, \quad u(t,x)=\tilde{x}, \\
u_t=-\frac{\tilde{u}_{\tilde{x}}+\tilde{x}}{\tilde{u}_{\tilde{x}}+\tilde{t}}, \quad u_x=\frac{1}{\tilde{u}_{\tilde{x}}+\tilde{t}}, \quad u_{xx}=-\frac{\tilde{u}_{\tilde{x}\tilde{x}}}{(\tilde{u}_{\tilde{x}}+\tilde{t})^3},
\end{gathered}
\tag{10}
$$

яке зводить рівняння (7) до рівняння (4).

Отже, нелінійні рівняння (4) та (7) з семивимірними максимальними алгебрами інваріантності зводяться до нелінійного рівняння теплоровідності (3) з класифікації Б.А. Магадєєва за допомогою контактного перетворення (9) та узагальненого перетворення годографа (10), а відповідні алгебри (6) та (8) ізоморфні з точністю до контактних перетворень алгебрі (5).

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Збірник праць Інституту математики НАН України 2019, т. 16, № 1, 22–43

УДК 517.912:512.816

Formality morphism as the mechanism of ⋆-product associativity: how it works

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Морфізм $\boldsymbol{\mathcal{F}} = \{\mathcal{F}_n,\ n \geqslant 1\}$, що забезпечує властивість формальності алгебр в деформаційному квантуванні за Концевичем, задано набором відображень тензорних степеней диференціальної градуйованої алгебри Лі (dgLa) мультивекторних полів в dgLa полідиференціальних операторів на скінченновимірних афінних многовидах. Хоча перший член $\mathcal{F}_1$ сам по собі не є морфізмом алгебр Лі, послідовність $\boldsymbol{\mathcal{F}}$ в цілому є L_∞-морфізмом. На його основі будується відображення елементів Маурера–Картана, яке ставить у відповідність пуасоновим бівекторам деформації $\mu_A \mapsto \star_{A[[\hbar]]}$, що добудовують звичайне множення функцій до асоціативних некомутативних $\star$-добутків на просторі степеневих рядів по $\hbar$. При цьому асоциативність $\star$-добутків забезпечено — на мові графів Концевича, що представляють полідиференціальні оператори, — диференціальними наслідками тотожності Якобі. Мета роботи — проілюструвати цей алгебраїчний механізм для $\star$-добутків Концевича (зокрема, з гармонічними пропагаторами).

The formality morphism $\boldsymbol{\mathcal{F}} = \{\mathcal{F}_n,\ n \geqslant 1\}$ in Kontsevich's deformation quantization is a collection of maps from tensor powers of the differential graded Lie algebra (dgLa) of multivector fields to the dgLa of polydifferential operators on finite-dimensional affine manifolds. Not a Lie algebra morphism by its term $\mathcal{F}_1$ alone, the entire set $\boldsymbol{\mathcal{F}}$ is an L_∞-morphism instead. It induces a map of the Maurer–Cartan elements, taking Poisson bi-vectors to deformations $\mu_A \mapsto \star_{A[[\hbar]]}$ of the usual multiplication of functions into associative noncommutative $\star$-products of power series in $\hbar$. The associativity of $\star$-products is then realized, in terms of the Kontsevich graphs which encode polydifferential operators, by differential consequences of the Jacobi identity. The aim of this paper is to illustrate the work of this algebraic mechanism for the Kontsevich $\star$-products (in particular, with harmonic propagators).

1. Introduction. The Kontsevich formality morphism $\boldsymbol{\mathcal{F}}$ relates two differential graded Lie algebras (dgLa). Its domain of definition is the shifted-graded vector space $T_{\text{poly}}^{\downarrow[1]}(M^r)$ of multivectors on an affine real finite-dimensional manifold M^r; the graded Lie algebra structure is the Schouten bracket $[\![\,,\,]\!]$ and the differential is set to (the bracket with) zero by definition. On the other hand, the target space of the formality morphism $\boldsymbol{\mathcal{F}}$ is the graded vector space $D_{\text{poly}}^{\downarrow[1]}(M^r)$ of polydifferential operators on M^r; the graded Lie algebra structure is the Gerstenhaber bracket $[\,,\,]_G$ and the differential $\mathrm{d}_H = [\mu_A, \cdot]$ is induced by using the multiplication μ_A in the algebra $A := C^\infty(M^r)$ of functions on M^r. It is readily seen that w.r.t. the above notation, Poisson bi-vectors $\mathcal{P}$ satisfying the Jacobi identity $[\![\mathcal{P},\mathcal{P}]\!] = 0$ on M^r are the Maurer–Cartan elements (indeed, $(d \equiv 0)(\mathcal{P}) + \frac{1}{2}[\![\mathcal{P},\mathcal{P}]\!] = 0$). Likewise, for a (non)commutative star-product $\star = \mu_{A[[\hbar]]} + \langle \text{tail} =: B\rangle$, which deforms the usual multiplication $\mu = \mu_{A[[\hbar]]}$ in $A[[\hbar]] = C^\infty(M^r) \otimes_{\mathbb{R}} \mathbb{R}[[\hbar]]$ by a tail B w.r.t. a formal parameter $\hbar$, the requirement that $\star$ be associative again is the Maurer–Cartan equation,

$$[\mu, B]_G + \tfrac{1}{2}[B,B]_G = 0 \quad \Longleftrightarrow \quad \tfrac{1}{2}[\mu + B, \mu + B]_G = 0.$$

Here, the leading order equality $[\mu,\mu]_G = 0$ expresses the given associativity of the product μ itself.

The Kontsevich formality mapping $\boldsymbol{\mathcal{F}} = \{\mathcal{F}_n \colon T_{\text{poly}}^{\otimes n} \to D_{\text{poly}},\ n \geqslant 1\}$ in [15, 16] is an L_∞-morphism which induces a map that takes Maurer–Cartan elements $\mathcal{P}$, i.e., formal Poisson bi-vectors $\tilde{\mathcal{P}} = \hbar\mathcal{P} + \bar{o}(\hbar)$ on M^r, to Maurer–Cartan elements[1], i.e., the tails B in solutions $\star$ of the associativity equation on $A[[\hbar]]$.

The theory required to build the Kontsevich map $\boldsymbol{\mathcal{F}}$ is standard, well reflected in the literature (see [15, 16], as well as [9, 11] and references therein); a proper choice of signs is analysed in [2, 20]. The framework of homotopy Lie algebras and L_∞-morphisms, introduced by Schlessinger–Stasheff [19], is available from [17], cf. [10] in the context of present paper.

So, the general fact of (existence of) factorization,

$$\mathrm{Assoc}(\star)(\mathcal{P})(f,g,h) = \Diamond\big(\mathcal{P}, [\![\mathcal{P},\mathcal{P}]\!]\big)(f,g,h), \quad f,g,h \in A[[\hbar]], \quad (1)$$

is known to the expert community. Indeed, this factorization is immediate from the construction of L_∞-morphism in [16, Section 6.4].

[1] In fact, the morphism $\boldsymbol{\mathcal{F}}$ is a quasi-isomorphism (see [16, Theorem 6.3]), inducing a bijection between the sets of gauge-equivalence classes of Maurer–Cartan elements.

We shall inspect how this mechanism works in practice, i.e., how precisely the $\star$-product is made associative in its perturbative expansion whenever the bi-vector $\mathcal{P}$ is Poisson, thus satisfying the Jacobi identity $\mathrm{Jac}(\mathcal{P}) := \frac{1}{2}[\![\mathcal{P},\mathcal{P}]\!] = 0$. To the same extent as our paper [6] justifies a similar factorization, $[\![\mathcal{P},\mathcal{Q}(\mathcal{P})]\!] = \Diamond\bigl(\mathcal{P},[\![\mathcal{P},\mathcal{P}]\!]\bigr)$, of the Poisson cocycle condition for universal deformations $\dot{\mathcal{P}} = \mathcal{Q}(\mathcal{P})$ of Poisson structures,[2] we presently motivate the findings in [5] for $\star$ mod $\bar{o}(\hbar^3)$, proceeding to the next order $\star$ mod $\bar{o}(\hbar^4)$ from [7] (and higher orders, recently available from [3]).[3] Let us emphasize that the theoretical constructions and algorithms (contained in the computer-assisted proof scheme under study and in the tools for graph weight calculation) would still work at arbitrarily high orders of expansion $\star$ mod $\bar{o}(\hbar^k)$ as $k\to\infty$. Explicit factorization (1) up to $\bar{o}(\hbar^k)$ helps us build the star-product $\star$ mod $\bar{o}(\hbar^k)$ by using a self-starting iterative process, because the Jacobi identity for $\mathcal{P}$ is the only obstruction to the associativity of $\star$. Specifically, the Kontsevich weights of graphs on fewer vertices (yet with a number of edges such that they do not show up in the perturbative expansion of $\star$) dictate the coefficients of Leibniz orgraphs in operator $\Diamond$ at higher orders in $\hbar$. These weights in the r.h.s. of (1) constrain the higher-order weights of the Kontsevich orgraphs in the expansion of $\star$-product itself. This is important also in the context of a number-theoretic open problem about the (ir)rational value $(\mathrm{const}\in\mathbb{Q}\setminus\{0\})\cdot\zeta(3)^2/\pi^6+(\mathrm{const}\in\mathbb{Q})$ of a graph weight at $\hbar^7$ in $\star$ (see [12] and [3]).

Our paper is structured as follows. First, we fix notation and recall some basic facts from relevant theory. Secondly, we provide three examples which illustrate the work of formality morphism in solving Eq. (1). Specifically, we read the operators $\Diamond_k = \Diamond$ mod $\bar{o}(\hbar^k)$ satisfying

$$\mathrm{Assoc}(\star)(\mathcal{P})(f,g,h) \bmod \bar{o}(\hbar^k) = \Diamond_k\left(\mathcal{P},[\![\mathcal{P},\mathcal{P}]\!]\right)(f,g,h) \qquad (1')$$

at $k=2$, 3, and 4. This corresponds to the expansions $\star$ mod $\bar{o}(\hbar^k)$ in [16], [5], and [7], respectively. One can then continue with $k=5,6$; these expansions are in [3]. Independently, one can probe such factorizations using other stable formality morphisms: for instance, the ones

[2]Universal w.r.t. all Poisson brackets on all finite-dimensional affine manifolds, such infinitesimal deformations were pioneered in [15]; explicit examples of these flows $\dot{\mathcal{P}} = \mathcal{Q}(\mathcal{P})$ are given in [4, 6, 8].

[3]Note that both the approaches – to noncommutative associative $\star$-products and deformations of Poisson structures – rely on the same calculus of oriented graphs by Kontsevich [13, 14, 15, 16].

which correspond to a different star-product, the weights in which are determined by a logarithmic propagator instead of the harmonic one (see [1, 18]).

2. Two differential graded Lie algebra structures. Let M^r be an r-dimensional affine real manifold (we set $\Bbbk = \mathbb{R}$ for simplicity). In the algebra $A := C^\infty(M^r)$ of smooth functions, denote by μ_A (or equivalently, by the dot $\cdot$) the usual commutative, associative, bi-linear multiplication. The space of formal power series in $\hbar$ over A will be $A[[\hbar]]$ and the $\hbar$-linear multiplication in it is μ (instead of $\mu_{A[[\hbar]]}$). Consider two differential graded Lie algebra structures. First, we have that the shifted-graded space $T^{\downarrow[1]}_{\text{poly}}(M^r)$ of multivector fields on M^r is equipped with the shifted-graded skew-symmetric Schouten bracket $[\![\,,\,]\!]$ (itself bi-linear by construction and satisfying the shifted-graded Jacobi identity); the differential is set to zero. Secondly, the vector space $D^{\downarrow[1]}_{\text{poly}}(M^r)$ of polydifferential operators (linear in each argument but not necessarily skew over the set of arguments or a derivation in any of them) is graded by using the number of arguments m: by definition, let $\deg(\theta(m \text{ arguments})) := m-1$. For instance, $\deg(\mu_A) = 1$. The Lie algebra structure on $D^{\downarrow[1]}_{\text{poly}}(M^r)$ is the Gerstenhaber bracket $[\,,\,]_G$; for two homogeneous operators Φ_1 and Φ_2 it equals $[\Phi_1, \Phi_2]_G = \Phi_1 \mathbin{\overleftarrow{\circ}} \Phi_2 - (-1)^{\deg\Phi_1\cdot\deg\Phi_2}\Phi_2 \mathbin{\overleftarrow{\circ}} \Phi_1$, where the directed, non-associative insertion product is, by definition

$$(\Phi_1 \mathbin{\overleftarrow{\circ}} \Phi_2)(a_0,\ldots,a_{k_1+k_2}) = \sum_{i=0}^{k_1}(-1)^{ik_2}\Phi_1\big(a_0\otimes\cdots\otimes a_{i-1} \otimes \Phi_2(a_i\otimes\cdots\otimes a_{i+k_2})\otimes a_{i+k_2+1}\otimes\cdots\otimes a_{k_1+k_2}\big).$$

In the above, $\Phi_i\colon A^{\otimes(k_i+1)} \to A$ so that $a_j \in A$.

Example 1. The associativity of the product μ_A in the algebra of functions $A = C^\infty(M^r)$ is the statement that

$$\mu_A^{(1)}\big(\mu_A^{(2)}(a_0,a_1),a_2\big) + (-1)^{(i=1)\cdot(\deg\mu_A=1)}\mu_A^{(1)}\big(a_0,\mu_A^{(2)}(a_1,a_2)\big) - (-1)^{(\deg\mu_A^{(1)}=1)\cdot(\deg\mu_A^{(2)}=1)}\big\{\mu_A^{(1)}\big(\mu_A^{(1)}(a_0,a_1),a_2\big) - \mu_A^{(2)}\big(a_0,\mu_A^{(1)}(a_1,a_2)\big)\big\} = 2\big\{(a_0\cdot a_1)\cdot a_2 - a_0\cdot(a_1\cdot a_2)\big\} = 0.$$

So, the associator $\operatorname{Assoc}(\mu_A)(a_0,a_1,a_2) = \frac{1}{2}[\mu_A,\mu_A]_G\,(a_0,a_1,a_2) = 0$ for any $a_j \in A$.

Like $[\![\cdot,\cdot]\!]$, the Gerstenhaber bracket satisfies the shifted-graded Jacobi identity. The Hochshild differential on $D^{\downarrow[1]}_{\text{poly}}(M^r)$ is $\mathrm{d}_H = [\mu_A, \cdot]_G$; indeed, its square vanishes, $\mathrm{d}_H^2 = 0$, due to the Jacobi identity for $[\,,\,]_G$ into which one plugs the equality $[\mu_A, \mu_A]_G = 0$.

3. The Maurer–Cartan elements. In every differential graded Lie algebra with a Lie bracket $[\,,\,]$, the Maurer–Cartan (MC) elements are solutions of degree 1 for the Maurer–Cartan equation

$$\mathrm{d}\alpha + \tfrac{1}{2}[\alpha, \alpha] = 0, \tag{2}$$

where d is the differential (equal, we recall, to $\mathrm{d}_H = [\mu_A, \cdot]_G$ on $D^{\downarrow[1]}_{\text{poly}}(M^r)$ and zero identically on $T^{\downarrow[1]}_{\text{poly}}(M^r)$. Likewise, the Lie algebra structure $[\cdot,\cdot]$ is the Gerstenhaber bracket $[\cdot,\cdot]_G$ and the Schouten bracket $[\![\cdot,\cdot]\!]$, respectively.)

Now tensor the degree-one parts of both dgLa structures with $\hbar\cdot\Bbbk[[\hbar]]$, i.e., with formal power series starting at $\hbar^1$, and, preserving the notation (that is, extending the brackets and the differentials by $\hbar$-linearity), consider the same Maurer–Cartan equation (2). Let us study its formal power series solutions $\alpha = \hbar^1\alpha_1 + \cdots$.

So far, in the Poisson world we have that the Maurer–Cartan bivectors are formal Poisson structures $0 + \hbar\mathcal{P}_1 + \bar{o}(\hbar)$ satisfying (2), which is $[\![\hbar\mathcal{P}_1 + \bar{o}(\hbar), \hbar\mathcal{P}_1 + \bar{o}(\hbar)]\!] = 0$ with zero differential. In the world of associative structures, the Maurer–Cartan elements are the tails B in expansions $\star = \mu + B$, so that the associativity equation $[\star,\star]_G = 0$ reads (for $[\mu,\mu]_G = 0$)

$$[\mu, B]_G + \tfrac{1}{2}[B, B]_G = 0,$$

which is again (2).

4. The L_∞-morphisms. Our goal is to have (and use) a morphism $T^{\downarrow[1]}_{\text{poly}}(M^r) \to D^{\downarrow[1]}_{\text{poly}}(M^r)$ which would induce a map that takes Maurer–Cartan elements in the Poisson world to Maurer–Cartan elements in the associative world.

The leading term $\mathcal{F}_1$, i.e., the first approximation to the morphism which we consider, is the Hochschild–Kostant–Rosenberg (HKR) map (obviously, extended by linearity),

$$\mathcal{F}\colon\ \xi_1 \wedge \cdots \wedge \xi_m \mapsto \frac{1}{m!} \sum_{\sigma \in S_m} (-1)^\sigma \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(m)},$$

which takes a split multi-vector to a polydifferential operator (in fact, an m-vector). More explicitly, we have that

$$\mathcal{F}_1\colon\ (\xi_1 \wedge \cdots \wedge \xi_m) \mapsto \left(a_1 \otimes \cdots \otimes a_m \mapsto \frac{1}{m!} \sum_{\sigma \in S_m} (-1)^\sigma \prod_{i=1}^{m} \xi_{\sigma(i)}(a_i)\right), \qquad (3)$$

here $a_j \in A := C^\infty(M^r)$. For zero-vectors $h \in A$, one has $\mathcal{F}_1\colon h \mapsto (1 \mapsto h)$.

Claim 1 ([16, Section 4.6.2])**.** *The leading term, map $\mathcal{F}_1$, is not a Lie algebra morphism (which, if it were, would take the Schouten bracket of multivectors to the Gerstenhaber bracket of polydifferential operators).*

Proof (by counterexample). Take two bi-vectors; their Schouten bracket is a tri-vector, but the Gerstenhaber bracket of two bi-vectors is a differential operator which has homogeneous components of differential orders (2,1,1) and (1,1,2). And in general, those components do not vanish. □

The construction of not a single map $\mathcal{F}_1$ but of an entire collection $\boldsymbol{\mathcal{F}} = \{\mathcal{F}_n,\ n \geqslant 1\}$ of maps does nevertheless yield a well-defined mapping of the Maurer–Cartan elements from the two differential graded Lie algebras.[4]

Theorem 2 ([16, Main Theorem])**.** *There exists a collection of linear maps $\boldsymbol{\mathcal{F}} = \bigl\{\mathcal{F}_n\colon T_{\text{poly}}^{\downarrow[1]}(M^r)^{\otimes n} \to D_{\text{poly}}^{\downarrow[1]}(M^r),\ n \geqslant 1\bigr\}$ such that $\mathcal{F}_1$ is the HKR map (3) and $\boldsymbol{\mathcal{F}}$ is an L_∞-morphism of the two differential graded Lie algebras: $\bigl(T_{\text{poly}}^{\downarrow[1]}(M^r),\ [\![\cdot,\cdot]\!],\ d = 0\bigr) \to \bigl(D_{\text{poly}}^{\downarrow[1]}(M^r),\ [\cdot,\cdot]_G,\ \mathrm{d}_H = [\mu_A,\cdot]_G\bigr)$. Namely,*

(1) *each component $\mathcal{F}_n$ is homogeneous of own grading $1-n$,*

(2) *each morphism $\mathcal{F}_n$ is graded skew-symmetric, i.e.,*

$$\mathcal{F}_n(\ldots,\xi,\eta,\ldots) = -(-1)^{\deg(\xi)\cdot\deg(\eta)}\mathcal{F}_n(\ldots,\eta,\xi,\ldots)$$

for ξ, η homogeneous,

[4]The name ‘formality’ for the collection $\boldsymbol{\mathcal{F}}$ of maps is motivated by Theorem 4.10 in [16] and by the main theorem in *loc. cit.*

(3) *for each* $n \geqslant 1$ *and (homogeneous) multivectors* $\xi_1, \ldots, \xi_n \in T_{\text{poly}}^{\downarrow[1]}(M^r)$, *we have that (cf.* [11, Section 3.6])

$$
\begin{aligned}
&\mathrm{d}_H(\mathcal{F}_n(\xi_1,\ldots,\xi_n)) - (-1)^{n-1}\sum_{i=1}^{n}(-1)^u \mathcal{F}_n(\xi_1,\ldots,d\xi_i,\ldots,\xi_n)\\
&+\tfrac{1}{2}\sum_{\substack{p+q=n\\ p,q>0}}\sum_{\sigma\in S_{p,q}}(-1)^{pn+t}\bigl[\mathcal{F}_p(\xi_{\sigma(1)},\ldots,\xi_{\sigma(p)}), \qquad (4)\\
&\mathcal{F}_q(\xi_{\sigma(p+1)},\ldots,\xi_{\sigma(n)})\bigr]_G\\
&= (-1)^n\sum_{i<j}(-1)^s\mathcal{F}_{n-1}\bigl([\xi_i,\xi_j],\xi_1,\ldots,\widehat{\xi_i},\ldots,\widehat{\xi_j},\ldots,\xi_n\bigr). \qquad (5)
\end{aligned}
$$

In the above formula, σ *runs through the set of* (p,q)*-shuffles, i.e., all permutations* $\sigma \in S_n$ *such that* $\sigma(1) < \cdots < \sigma(p)$ *and independently* $\sigma(p+1) < \cdots < \sigma(n)$*; the exponents* t *and* s *are the numbers of transpositions of odd elements which we count when passing* (t) *from* $(\mathcal{F}_p, \mathcal{F}_q, \xi_1, \ldots, \xi_n)$ *to* $(\mathcal{F}_p, \xi_{\sigma(1)}, \ldots, \xi_{\sigma(p)}, \mathcal{F}_q, \xi_{\sigma(p+1)}, \ldots, \xi_{\sigma(n)})$*, and* (s) *from* $(\xi_1, \ldots, \xi_n)$ *to* $(\xi_i, \xi_j, \xi_1, \ldots, \widehat{\xi_1}, \ldots, \widehat{\xi_j}, \ldots, \xi_n)$.[5]

Remark 1. Let $n := 1$, then equality (5) in Theorem 2 is

$$
\begin{aligned}
&\mathrm{d}_H \circ \mathcal{F}_1 - (-1)^{1-1} \cdot (-1)^{u=0 \text{ from } (d,\xi_1)\mapsto(d,\xi_1)} F_1 \circ d = 0\\
&\quad \Longleftrightarrow \mathrm{d}_H \circ \mathcal{F}_1 = \mathcal{F}_1 \circ d,
\end{aligned}
$$

whence $\mathcal{F}_1$ is a morphism of complexes.

- Let $n := 2$, then for any homogeneous multivectors ξ_1 and ξ_2,

$$
\begin{aligned}
&\mathcal{F}_1\bigl([\![\xi_1,\xi_2]\!]\bigr) - \bigl[\mathcal{F}_1(\xi_1),\mathcal{F}_1(\xi_2)\bigr]_G = \mathrm{d}_H\bigl(\mathcal{F}_2(\xi_1,\xi_2)\bigr)\\
&\quad + \mathcal{F}_2\bigl((d=0)(\xi_1),\xi_2\bigr) + (-1)^{\deg\xi_1}\mathcal{F}_2\bigl(\xi_1,(d=0)(\xi_2)\bigr),
\end{aligned}
$$

so that in our case $\mathcal{F}_1$ is “almost” a Lie algebra morphism but for the discrepancy which is controlled by the differential of the (value of the) succeeding map $\mathcal{F}_2$ in the sequence $\boldsymbol{\mathcal{F}} = \{\mathcal{F}_n,\ n \geqslant 1\}$. Big formula (5) shows in precisely which sense this is also the case for higher homotopies $\mathcal{F}_n$, $n \geqslant 2$ in the L_∞-morphism $\boldsymbol{\mathcal{F}}$. Indeed, an L_∞-morphism is a map between dgLas which, in every term, almost preserves the bracket up to a homotopy $\mathrm{d}_H \circ \{\ldots\}$ provided by the next term.

[5] The exponent u is not essential for us now because the differential d on $T_{\text{poly}}^{\downarrow[1]}(M^r)$ is set equal to zero identically, so that the entire term with u does not contribute (recall $\mathcal{F}_n$ is linear).

Even though neither $\mathcal{F}_1$ nor the entire collection $\boldsymbol{\mathcal{F}} = \{\mathcal{F}_n,\, n \geqslant 1\}$ is a dgLa morphism, their defining property (5) guarantees that $\boldsymbol{\mathcal{F}}$ gives us a well defined mapping of the Maurer–Cartan elements (which, we recall, are formal Poisson bi-vectors and tails B of associative (non)commutative multiplcations $\star = \mu + B$ on $A[[\hbar]]$, respectively).

Corollary 3. *The natural $\hbar$-linear extension of $\boldsymbol{\mathcal{F}}$, now acting on the space of formal power series in $\hbar$ with coefficients in $T^{\downarrow[1]}_{\text{poly}}(M^r)$ and with zero free term by the rule*

$$\xi \mapsto \sum_{n\geqslant 1} \frac{1}{n!}\mathcal{F}_n(\xi,\ldots,\xi),$$

takes the Maurer–Cartan elements $\tilde{\mathcal{P}} = \hbar\mathcal{P} + \bar{o}(\hbar)$ to the Maurer–Cartan elements $B = \sum_{n\geqslant 1}\frac{1}{n!}\mathcal{F}_n(\tilde{\mathcal{P}},\ldots,\tilde{\mathcal{P}}) = \hbar\tilde{\mathcal{P}} + \bar{o}(\hbar)$. (Note that the HKR map $\mathcal{F}_1$, extended by $\hbar$-linearity, still is an identity mapping on multivectors, now viewed as special polydifferential operators.)

In plain terms, for a bivector $\mathcal{P}$ itself Poisson, formal Poisson structures $\tilde{\mathcal{P}} = \hbar\mathcal{P} + \bar{o}(\hbar)$ satisfying $[\![\tilde{\mathcal{P}},\tilde{\mathcal{P}}]\!] = 0$ are mapped by $\boldsymbol{\mathcal{F}}$ to the tails $B = \hbar\mathcal{P} + \bar{o}(\hbar)$ such that $\star = \mu + B$ is associative and its leading order deformation term is a given Poisson structure $\mathcal{P}$.

Proof of Corollary 3. Let us presently consider the restricted case when $\tilde{\mathcal{P}} = \hbar\mathcal{P}$, without any higher order tail $\bar{o}(\hbar)$. The Maurer–Cartan equation in $D^{\downarrow[1]}_{\text{poly}}(M^r) \otimes \hbar\Bbbk[[\hbar]]$ is $[\mu, B]_G + \frac{1}{2}[B,B]_G = 0$, where

$$B = \sum_{n\geqslant 1}\frac{1}{n!}\mathcal{F}_n\bigl(\tilde{\mathcal{P}},\ldots,\tilde{\mathcal{P}}\bigr)$$

and we let $\tilde{\mathcal{P}} = \hbar\mathcal{P}$, so that $B = \sum_{n\geqslant 1}\frac{\hbar^n}{n!}\mathcal{F}_n(\mathcal{P},\,\ldots,\,\mathcal{P})$. Let us plug this formal power series in the l.h.s. of the above equation. Equating the coefficients at powers $\hbar^n$ and multiplying by $n!$, we obtain the expression

$$[\mu, \mathcal{F}_n(\mathcal{P},\ldots,\mathcal{P})]_G + \tfrac{1}{2}\sum_{\substack{p+q=n\\ p,q>0}}\frac{n!}{p!q!}\bigl[\mathcal{F}_p(\mathcal{P},\ldots,\mathcal{P}), \mathcal{F}_q(\mathcal{P},\ldots,\mathcal{P})\bigr]_G.$$

It is readily seen that now the sum $\sum_{\sigma\in S_{p,q}}$ in (5) over the set of (p,q)-shuffles of $n = p+q$ identical copies of an object $\mathcal{P}$ just counts the number of ways to pick p copies going first in an ordered string of length n. To balance the signs, we note at once that by item (2) in Theorem 2, see above, $\mathcal{F}_p(\ldots,\mathcal{P}^{(\alpha)},\mathcal{P}^{(\alpha+1)},\ldots) = \mathcal{F}_p(\ldots,\mathcal{P}^{(\alpha+1)},\mathcal{P}^{(\alpha)},\ldots)$ because bi-vector's shifted degree is $+1$, so that no (p,q)-shuffles of $(\mathcal{P},\ldots,\mathcal{P})$ contribute with any sign factor. The only sign contribution that remains stems from the symbol $\mathcal{F}_q$ of grading $1-q$ transported along p copies of odd-degree bi-vector $\mathcal{P}$; this yields $t = (1-q)\cdot p$ and $(-1)^{pn+t} = (-1)^{p\cdot(p+q)}\cdot(-1)^{(1-q)\cdot p} = (-1)^{p\cdot(p+1)} = +$.

The left-hand side of the Maurer–Cartan equation (2) is, by the above, expressed by the left-hand side of (5) which the L_∞-morphism $\boldsymbol{\mathcal{F}}$ satisfies. In the right-hand side of (5), we now obtain (with, actually, whatever sign factors) the values of linear mappings $\mathcal{F}_{n-1}$ at twice the Jacobiator $[\![\tilde{\mathcal{P}},\tilde{\mathcal{P}}]\!]$ as one of the arguments. All these values are therefore zero, which implies that the right-hand side of the Maurer–Cartan equation (2) vanishes, so that the tail B indeed is a Maurer–Cartan element in the Hochschild cochain complex (in other words, the star-product $\star = \mu + B$ is associative).

This completes the proof in the restricted case when $\tilde{\mathcal{P}} = \hbar\mathcal{P}$. Formal power series bi-vectors $\tilde{\mathcal{P}} = \hbar\mathcal{P} + \bar{o}(\hbar)$ refer to the same count of signs as above, yet the calculation of multiplicities at $\hbar^n$ (for all possible lexicographically ordered p- and q-tuples of n arguments) is an extensive exercise in combinatorics. $\square$

Corollary 4. *Because the right-hand side of* (2) *in the above reasoning is determined by the right-hand side of* (5)*, we read off an explicit formula of the operator* $\Diamond$ *that solves the factorization problem*

$$\operatorname{Assoc}(\star)(\mathcal{P})(f,g,h) = \Diamond\big(\mathcal{P},[\![\mathcal{P},\mathcal{P}]\!]\big)(f,g,h),\quad f,g,h\in A[[\hbar]]. \quad (1)$$

Indeed, the operator is

$$\Diamond = 2\cdot\sum_{n\geqslant 1}\frac{\hbar^n}{n!}\cdot c_n\cdot\mathcal{F}_{n-1}\big([\![\mathcal{P},\mathcal{P}]\!],\mathcal{P},\ldots,\mathcal{P}\big). \quad (6)$$

But what are the coefficients $c_n\in\mathbb{R}$ equal to? Let us find it out.

5. Explicit construction of the formality morphism $\boldsymbol{\mathcal{F}}$. The first explicit formula for the formality morphism $\boldsymbol{\mathcal{F}}$ which we study in this paper was discovered by Kontsevich in [16, Section 6.4], providing an expansion of every term $\mathcal{F}_n$ using weighted decorated graphs:

$$\boldsymbol{\mathcal{F}} = \Big\{\mathcal{F}_n = \sum_{m\geqslant 0}\sum_{\Gamma\in G_{n,m}} W_\Gamma\cdot\mathcal{U}_\Gamma\Big\}.$$

Here Γ belongs to the set $G_{n,m}$ of oriented graphs on n internal vertices (i.e., arrowtails), m sinks (from which no arrows start), and $2n+m-2 \geqslant 0$ edges, such that at every internal vertex there is an ordering of outgoing edges. By decorating each edge with a summation index that runs from 1 to r, by viewing each edge as a derivation $\partial/\partial x^\alpha$ of the arrowhead vertex content, by placing n multivectors from an ordered tuple of arguments of $\mathcal{F}_n$ into the respective vertices, now taking the sum over all indices of the resulting products of the content of vertices, and skew-symmetrizing over the n-tuple of (shifted-)graded multivectors, we realize each graph at hand as a polydifferential operator $T^{\downarrow[1]}_{\text{poly}}(M^r)^{\otimes n} \to D^{\downarrow[1]}_{\text{poly}}(M^r)$ whose arguments are multivectors. Note that the *value* $\mathcal{F}_n(\xi_1, \ldots, \xi_n)$ itself is, by construction, a differential operator w.r.t. the contents of sinks of the graph Γ. All of this is discussed in detail in [13, 14, 15, 16] or [4, 5, 7].

The formula for the harmonic weights $W_\Gamma \in \mathbb{R}$ is given in [16, Section 6.2]; it is

$$W_\Gamma = \left(\prod_{k=1}^{n} \frac{1}{\#\text{Star}(k)!} \right) \cdot \frac{1}{(2\pi)^{2n+m-2}} \int_{\bar{C}^+_{n,m}} \bigwedge_{e\in E_\Gamma} \mathrm{d}\phi_e,$$

where $\#\,\text{Star}(k)$ is the number of edges *star*ting from vertex k, $\mathrm{d}\varphi_e$ is the "harmonic angle" differential 1-form associated to the edge e, and the integration domain $\bar{C}^+_{n,m}$ is the connected component of $\bar{C}_{n,m}$ which is the closure of configurations where points q_j, $1 \leqslant j \leqslant m$ on $\mathbb{R}$ are placed in increasing order: $q_1 < \cdots < q_m$. For convenience, let us also define

$$w_\Gamma = \left(\prod_{k=1}^{n} \#\text{Star}(k)! \right) \cdot W_\Gamma.$$

The convenience is that by summing over labelled graphs Γ, we actually sum over the equivalence classes $[\Gamma]$ (i.e., over unlabeled graphs) with multiplicities $(w_\Gamma/W_\Gamma) \cdot n!/\#\text{Aut}(\Gamma)$. The division by the volume $\#\text{Aut}(\Gamma)$ of the symmetry group eliminates the repetitions of graphs which differ only by a labeling of vertices but, modulo such, do not differ by the labeling of ordered edge tuples (issued from the vertices which are matched by a symmetry).

Let us remember that the integrand in the formula of W_Γ is defined in terms of the harmonic propagator; other propagators (e.g., logarithmic, or other members of the family interpolating between harmonic and logarithmic [1, 18]) would give other formality morphisms. A path integral realization of the $\star$-product itself and of the components $\mathcal{F}_n$ in the formality morphism is proposed in [10].

To calculate the graph weights W_Γ in practice, we employ methods which were outlined in [7], as well as [12, Appendix E] (about the cyclic weight relations), and [3] that puts those real values in the context of Riemann multiple zeta functions and polylogarithms.[6] Examples of such decorated oriented graphs Γ and their weights W_Γ will be given in the next section.

5.1. Sum over equivalence classes. The sum in Kontsevich's formula is over *labeled* graphs: internal vertices are numbered from 1 to n, and the edges starting from each internal vertex k are numbered from 1 to $\#\mathrm{Star}(k)$. Under a re-labeling $\sigma\colon \Gamma \mapsto \Gamma^\sigma$ of internal vertices and edges it is seen from the definitions that the operator $\mathcal{U}_\Gamma$ and the weight W_Γ enjoy the same skew-symmetry property (as remarked in [16, Section 6.5]), whence $W_\Gamma \cdot \mathcal{U}_\Gamma = W_{\Gamma^\sigma} \cdot \mathcal{U}_{\Gamma^\sigma}$. It follows that the sum over labeled graphs can be replaced by a sum over equivalence classes $[\Gamma]$ of graphs, modulo labeling of internal vertices and edges. For this it remains to count the size of an equivalence class: the edges can be labeled in $\prod_{k=1}^n \#\mathrm{Star}(k)!$ ways, while the n internal vertices can be labeled in $n!/\#\mathrm{Aut}(\Gamma)$ ways.

Example 2. The double wedge on two ground vertices has only *one* possible labeling of vertices, due to the automorphism that interchanges the wedges.

We denote by $M_\Gamma = \bigl(\prod_{k=1}^n \#\mathrm{Star}(k)!\bigr) \cdot n!/\#\mathrm{Aut}(\Gamma)$ the *multiplicity* of the graph Γ, and let $\bar{G}_{n,m}$ be the set of equivalence classes $[\Gamma]$ modulo labeling of $\Gamma \in G_{n,m}$. The formula for the formality morphism can then be rewritten as

$$\boldsymbol{\mathcal{F}} = \Bigl\{ \mathcal{F}_n = \sum_{m\geqslant 0} \sum_{[\Gamma]\in\bar{G}_{n,m}} M_\Gamma \cdot W_\Gamma \cdot \mathcal{U}_\Gamma \Bigr\};$$

here the Γ in $M_\Gamma \cdot W_\Gamma \cdot \mathcal{U}_\Gamma$ is *any* representative of $[\Gamma]$. Any ambiguity in signs (due to the choice of representative) in the latter two factors is cancelled in their product. Note that the factor $\bigl(\prod_{k=1}^n \#\mathrm{Star}(k)!\bigr)$ in M_Γ kills the corresponding factor in W_Γ, as remarked in [16, Section 6.5].

[6] It is the values w_Γ instead of W_Γ which are calculated by software [3].

5.2. The coefficient of a graph in the ⋆-product. The ⋆-product associated with a Poisson structure $\mathcal{P}$ is given by Corollary 3:

$$\begin{aligned}\star &= \mu + \sum_{n\geqslant 1} \frac{\hbar^n}{n!} \mathcal{F}_n(\mathcal{P},\dots,\mathcal{P}) \\ &= \mu + \sum_{n\geqslant 1} \frac{\hbar^n}{n!} \sum_{[\Gamma]\in \bar{G}_{n,2}} M_\Gamma \cdot W_\Gamma \cdot \mathcal{U}_\Gamma(\mathcal{P},\dots,\mathcal{P}).\end{aligned}$$

For a graph $\Gamma \in G_{n,2}$ such that each internal vertex has two outgoing edges (these are the only graphs that contribute, because we insert bi-vectors) we have $M_\Gamma = 2^n \cdot n!/\#\mathrm{Aut}(\Gamma)$. In total, the coefficient of $\mathcal{U}_\Gamma(\mathcal{P},\dots,\mathcal{P})$ at $\hbar^n$ is $2^n/\#\mathrm{Aut}(\Gamma)\cdot W_\Gamma = w_\Gamma/\#\mathrm{Aut}(\Gamma)$. The skew-symmetrization *without prefactor* of bi-vector coefficients in $\mathcal{U}_\Gamma(\mathcal{P},\dots,\mathcal{P})$ provides an extra factor 2^n.

Example 3 (at $\hbar^1$)**.** The coefficient of the wedge graph is $1/2$ and the operator is $2\mathcal{P}$, hence we recover $\mathcal{P}$.

5.3. The coefficient of a Leibniz graph in the associator. The factorizing operator $\Diamond$ for $\mathrm{Assoc}(\star)$ is given by Corollary 4:

$$\begin{aligned}\Diamond &= 2\cdot \sum_{n\geqslant 1} \frac{\hbar^n}{n!} \cdot c_n \cdot \mathcal{F}_{n-1}\big([\![\mathcal{P},\mathcal{P}]\!],\mathcal{P},\dots,\mathcal{P}\big) \\ &= 2\cdot \sum_{n\geqslant 1} \frac{\hbar^n}{n!} \cdot c_n \cdot \sum_{[\Gamma]\in \bar{G}_{n-1,3}} M_\Gamma \cdot W_\Gamma \cdot \mathcal{U}_\Gamma\big([\![\mathcal{P},\mathcal{P}]\!],\mathcal{P},\dots,\mathcal{P}\big).\end{aligned}$$

For a graph $\Gamma \in G_{n-1,3}$ where one internal vertex has three outgoing edges and the rest have two, we have $M_\Gamma = 3!\cdot 2^{n-2}\cdot (n-1)!/\#\mathrm{Aut}(\Gamma)$. In total, the coefficient of $\mathcal{U}_\Gamma([\![\mathcal{P},\mathcal{P}]\!],\mathcal{P},\dots,\mathcal{P})$ at $\hbar^n$ is

$$\left[2\cdot \frac{1}{n!}\cdot c_n \cdot 3! \cdot 2^{n-2}\cdot (n-1)!\right] \cdot \frac{W_\Gamma}{\#\mathrm{Aut}(\Gamma)} = \left[2\cdot \frac{c_n}{n}\right]\cdot \frac{w_\Gamma}{\#\mathrm{Aut}(\Gamma)}.$$

The skew-symmetrization *without prefactor* of bi- and tri-vector coefficients in the operator $\mathcal{U}_\Gamma([\![\mathcal{P},\mathcal{P}]\!],\mathcal{P},\dots,\mathcal{P})$ provides an extra factor $3!\cdot 2^{n-2}$.

Example 4 (at $\hbar^2$)**.** The coefficient of the tripod graph is $c_2 \cdot \frac{1}{3!}$ and the operator is $3!\cdot [\![\mathcal{P},\mathcal{P}]\!]$, hence we recover $c_2[\![\mathcal{P},\mathcal{P}]\!] = \frac{2}{3}\,\mathrm{Jac}(\mathcal{P})$. (The right-hand side is known from the associator, e.g., from [5].) This yields $c_2 = 1/3$. In addition, we see that the HKR map $\mathcal{F}_1$ acts here by the identity on $[\![\mathcal{P},\mathcal{P}]\!]$.

In the next section, we shall find that at $\hbar^n$, the coefficients of our Leibniz graphs (with $\text{Jac}(\mathcal{P})$ inserted instead of $[\![\mathcal{P},\mathcal{P}]\!]$) are

$$\frac{[\![P,P]\!]}{\text{Jac}(\mathcal{P})} \cdot \left[3! \cdot 2^{n-2}\right] \cdot \left[2 \cdot \frac{c_n}{n}\right] \cdot \frac{w_\Gamma}{\#\text{Aut}(\Gamma)} = 2^n \cdot \frac{w_\Gamma}{\#\text{Aut}(\Gamma)},$$

so $3! \cdot 2^n \cdot \frac{c_n}{n} = 2^n$. We deduce that $c_n = n/3! = n/6$ in all our experiments.

Conjecture. *For all* $n \geqslant 2$*, the coefficients in* (6) *are* $c_n = n/3! = n/6$ (*hence, the coefficients of markers* Γ *for equivalence classes* $[\Gamma]$ *of the Leibniz graphs in* (6) *are* $2^n \cdot w_\Gamma/\#\text{Aut}(\Gamma)$), *although it still remains to be explained how exactly this follows from the* L_∞ *condition* (5).

6. Examples. Let $\mathcal{P}$ be a Poisson bi-vector on an affine manifold M^r. We inspect the associativity of the star-product $\star = \mu + \sum_{n\geqslant 1} \frac{\hbar^n}{n!}\mathcal{F}_n(\mathcal{P}, \ldots, \mathcal{P})$ given by Corollary 3 by illustrating the work of the factorization mechanism from Corollary 4. The powers of deformation parameter $\hbar$ provide a natural filtration $\hbar^2 \cdot \mathsf{A}^{(2)} + \hbar^3 \cdot \mathsf{A}^{(3)} + \hbar^4 \cdot \mathsf{A}^{(4)} + \bar{o}(\hbar^4)$ so that we verify the vanishing of $\text{Assoc}(\star)(\mathcal{P})(\cdot,\cdot,\cdot) \mod \bar{o}(\hbar^4)$ for $\star \mod \bar{o}(\hbar^4)$ order by order.

At $\hbar^0$ there is nothing to do (indeed, the usual multiplication is associative). All contribution to the associator of $\star$ at $\hbar^1$ cancels out because the leading deformation term $\hbar\mathcal{P}$ in the star-product $\star = \mu + \hbar P + \bar{o}(\hbar)$ is a bi-derivation. The order $\hbar^2$ was discussed in Example 4 in Section 5.3.

Remark 2. In all our reasoning at any order $\hbar^{n\geqslant 2}$, the Jacobiator in Leibniz graphs is expanded (w.r.t. the three cyclic permutations of its arguments) into the Kontsevich graphs, built of wedges, in such a way that the internal edge, connecting two Poisson bi-vectors in $\text{Jac}(\mathcal{P})$, is proclaimed Left by construction. Specifically, the algorithm to expand each Leibniz graphs is as follows:

1. Split the trivalent vertex with ordered targets (a, b, c) into two wedges: the first wedge stands on a and b (in that order), and the second wedge stands on the first wedge-top and c (in that order), so that the internal edge of the Jacobiator is marked Left, preceding the Right edge towards c.

2. Re-direct the edges (if any) which had the tri-valent vertex as their target, to one of the wedge-tops; take the sum over all possible combinations (this is the iterated Leibniz rule).

3. Take the sum over cyclic permutations of the targets of the edges which (initially) have (a, b, c) as their targets (this is the expansion of the Jacobiator).

6.1. The order $\hbar^3$. To factorize the next order expansion of the associator, $\text{Assoc}(\star)(\mathcal{P}) \bmod \bar{o}(\hbar^3) = \hbar^2 \cdot \mathsf{A}^{(2)} + \hbar^3 \cdot \mathsf{A}^{(3)} + \bar{o}(\hbar^3)$, at $\hbar^3$ in the operator $\Diamond$ in the right-hand side of (1), we use graphs on $n - 1 = 2$ vertices, $m = 3$ sinks, and $2(n-1) + m - 2 = 5$ edges.

At $\hbar^3$, two internal vertices in the Leibniz graphs in the r.h.s. of factorization (1) are manifestly different: one vertex, containg the bi-vector $\mathcal{P}$, is a source of two outgoing edges, and the other, with $[\![\mathcal{P}, \mathcal{P}]\!]$, of three. Therefore, the automorphism groups of such Leibniz graphs (under relabellings of internal vertices of the same valency but with the sinks fixed) can only be trivial, i.e., one-element. (This will not necessarily be the case of Leibniz graphs on $(n-2)+1$ internal vertices at $\hbar^{\geqslant 4}$: compare Examples 8 vs 9 on p. 39 below, where the weight of a graph is divided further by the size of its automorphism group.)

The coefficient of $\hbar^3$ in the factorizing operator $\Diamond$,

$$\text{coeff}\left(\Diamond, \hbar^3\right) = 2 \cdot \frac{1}{3!} \cdot c_3 \cdot \sum_{[\Gamma] \in \bar{G}_{2,3}} M_\Gamma \cdot W_\Gamma \cdot \mathcal{U}_\Gamma\left([\![\mathcal{P}, \mathcal{P}]\!], \mathcal{P}, \ldots, \mathcal{P}\right),$$

expands into a sum of $\leqslant 24$ admissible oriented graphs. Indeed, there are six essentially different oriented graph topologies, filtered by the number of sinks on which the tri-vector $[\![\mathcal{P}, \mathcal{P}]\!]$ and bi-vector $\mathcal{P}$ stand; the ordering of sinks in the associator then yields $3 + 3 + 3 \times 2 + 3 \times 2 + 3 = 24$ oriented graphs. (None of them is a zero orgraph.) As we recall from [5], only thirteen of them actually occur with nonzero coefficients in the term $\mathsf{A}^{(3)} \sim \hbar^3$ in $\text{Assoc}(\star)(\mathcal{P})$), the remaining eleven have zero weights.[7] The weights of 15 relevant oriented Leibniz graphs from [5] are listed in Table 1.[8]

Here we let by definition

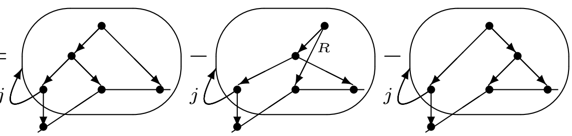
$$I_f := \partial_j\left(\text{Jac}(\mathcal{P})(\mathcal{P}^{ij}, g, h)\right) \partial_i f$$

[7] Yet, these seemingly 'unnecessary' graphs can contribute to the cyclic weight relations (see [12, Appendix E]): zero values of some of such graph weights can simplify the system of linear relations between nonzero weights.

[8] To get the values, one uses the software [3] by Banks–Panzer–Pym or, independently, exact symbolic or approximate numeric methods from [7], also taking into account the cyclic weight relations from [12, Appendix E].

Table 1. Weights w_Γ of oriented Leibniz graphs Γ in coeff $(\Diamond, \hbar^3)$.

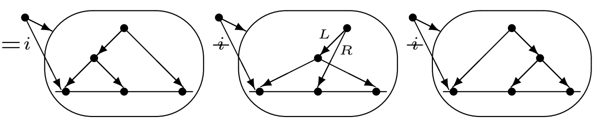
$(S_f)_{221}$	=	[01; 012]	$\frac{1}{12}$	$(S_g)_{122}$	=	[12; 012]	$\frac{1}{12}$
$(I_f)_{112}$	=	[02; 312]	$\frac{1}{48}$	$(I_g)_{112}$	=	[12; 032]	$\frac{1}{48}$
$(S_f)_{211}$	=	[04; 012]	$\frac{1}{24}$	$(I_g)_{211}$	=	[10; 032]	$\frac{-1}{48}$
$(I_f)_{111}$	=	[04; 312]	$\frac{1}{48}$	$(I_h)_{111}$	=	[24; 013]	$\frac{-1}{48}$
$(S_g)_{111}$	=	[14; 012]	0	$(I_f)_{121}$	=	[01; 312]	$\frac{1}{24}$
$(S_h)_{212}$	=	[20; 012]	$\frac{-1}{12}$				
$(S_h)_{112}$	=	[24; 012]	$\frac{-1}{24}$				
$(I_h)_{211}$	=	[20; 013]	$\frac{-1}{48}$				
$(I_g)_{111}$	=	[14; 032]	0				
$(I_h)_{121}$	=	[21; 013]	$\frac{-1}{24}$				

=  = 0.

Likewise, $I_g := \partial_j\big(\mathrm{Jac}(\mathcal{P})(f, \mathcal{P}^{ij}, h)\big) \cdot \partial_i g$ and $I_h := \partial_j\big(\mathrm{Jac}(\mathcal{P})(f, g, \mathcal{P}^{ij}\big) \cdot \partial_i h$, respectively.[9]

We also set

$$S_f := \mathcal{P}^{ij} \partial_j \, \mathrm{Jac}(\mathcal{P})(\partial_i f, g, h)$$

=  = 0.

Similarly, we let $S_g := \mathcal{P}^{ij} \partial_j \, \mathrm{Jac}(\mathcal{P})(f, \partial_i g, h) = 0$ and define $S_h := \mathcal{P}^{ij} \partial_j \, \mathrm{Jac}(\mathcal{P})(f, g, \partial_i h) = 0$. Note that after all the Leibniz rules are reworked, each of the six graphs $I_f, \ldots, S_h$ – with the Jacobiator $\mathrm{Jac}(\mathcal{P}) = \frac{1}{2}[\![\mathcal{P}, \mathcal{P}]\!]$ at the tri-valent vertex – splits into several homogeneous components, like $(I_f)_{111}$ or $(S_h)_{212}$; taken alone, each of the components encodes a zero polydifferential operator of respective orders.

[9] In [5], the indices i and j were interchanged in the definitions of both I_g and I_h (compare the expression of I_f); that typo is now corrected in the above formulae.

Claim 5. *Multiplied by a common factor* $\bigl([\![\mathcal{P},\mathcal{P}]\!]/\operatorname{Jac}(\mathcal{P})\bigr)\cdot 2^{k-1} = 2\cdot 4 = 8$*, the Leibniz graph weights from Table 1 at* $\hbar^3$ *fully reproduce the factorization which was found in the main Claim in* [5]*, namely:*

$$\begin{aligned}
&\mathsf{A}^{(3)}_{221} = \tfrac{2}{3}(S_f)_{221}, \quad \mathsf{A}^{(3)}_{122} = \tfrac{2}{3}(S_g)_{122}, \quad \mathsf{A}^{(3)}_{212} = -\tfrac{2}{3}(S_h)_{212},\\
&\mathsf{A}^{(3)}_{111} = \tfrac{1}{6}(I_f - I_h)_{111}, \quad \mathsf{A}^{(3)}_{112} = \bigl(\tfrac{1}{6}I_f + \tfrac{1}{6}I_g - \tfrac{1}{3}S_h\bigr)_{112},\\
&\mathsf{A}^{(3)}_{121} = \tfrac{1}{3}(I_f - I_h)_{121}, \quad \mathsf{A}^{(3)}_{211} = \bigl(\tfrac{1}{3}S_f - \tfrac{1}{6}I_g - \tfrac{1}{6}I_h\bigr)_{211}.
\end{aligned}$$

Otherwise speaking, the sum of these Leibniz oriented graphs with these weights (times $2{\cdot}4 = 8$), when expanded into the sum of 39 weighted Kontsevich graphs (built only of wedges), equals identically the $\hbar^3$-proportional term in the associator $\operatorname{Assoc}(\star)(\mathcal{P})(f,g,h)$.

Proof scheme. The encodings of weighted Kontsevich-graph expansions of the homogeneous components of the weighted Leibniz graphs $I_f,\ldots,S_h$, which show up in the associator at $\hbar^3$ and which are processed according to the algorithm in Remark 2, are listed in Appendix A. Reducing that collection modulo skew symmetry at internal vertices, we reproduce, as desired, the entire term $\mathsf{A}^{(3)}$ in the expansion $\hbar^2\cdot\mathsf{A}^{(2)} + \hbar^3\cdot\mathsf{A}^{(3)} + \bar{o}(\hbar^3)$ of the associator $\operatorname{Assoc}(\star)(\mathcal{P})$ mod $\bar{o}(\hbar^3)$. □

Three examples, corresponding to the leftmost column of equalities in Claim 5, illustrate this scheme at order $\hbar^3$. The three cases differ in that for $\mathsf{A}^{(3)}_{221}$ in Example 5, there is just one Leibniz graph without any arrows acting on the Jacobiator vertex. In the other Example 6 for $\mathsf{A}^{(3)}_{121}$, there are two Leibniz graphs still without Leibniz-rule actions on the Jacobiators in them, so that we aim to show how similar terms are collected.[10] Finally, in Example 7 about $\mathsf{A}^{(3)}_{111}$ there are two Leibniz graphs with one Leibniz rule action per either graph: an arrow targets the two internal vertices in the Jacobiator.

Example 5. Take the Leibniz graph $(S_f)_{221} = [01; 012]$. Its weight is $1/12$. Multiplying the Leibniz graph by 8 times its weight and expanding

[10]To collect and compare the Kontsevich orgraphs (built of wedges, i.e., ordered edge pairs issued from internal vertices), we can bring every such graph to its normal form, that is, represent it using the *minimal* base-(# sinks + # internal vertices) number, encoding the graph as the list of ordered pairs of target vertices, by running over all the relabellings of internal vertices. (The labelling of ordered sinks is always $0 \prec 1 \prec \cdots \prec m-1$.)

the Jacobiator (there are no Leibniz rules to expand) yields the sum of three Kontsevich graphs: $\frac{2}{3}\big([\mathsf{01;01;42}] + [\mathsf{01;12;40}] + [\mathsf{01;20;41}]\big)$. This is identically equal to the differential order $(2,2,1)$ homogeneous part $\mathsf{A}^{(3)}_{221}$ of $\text{Assoc}(\star)(\mathcal{P})$ at $\hbar^3$. For instance, these terms are listed in [7, Appendix D].

Example 6. Take the Leibniz graphs $(I_f)_{121} = [\mathsf{01;312}]$ and $(I_h)_{121} = [\mathsf{21;013}]$. Their weights are $1/24$ and $-1/24$, respectively; multiply them by 8. Expanding the Jacobiator in the linear combination $\frac{1}{3}(I_f - I_h)_{121}$ yields the sum of Kontsevich graphs $\frac{1}{3}\big([\mathsf{01;31;42}] + [\mathsf{01;12;43}] + [\mathsf{01;23;41}] - [\mathsf{21;01;43}] - [\mathsf{21;13;40}] - [\mathsf{21;30;41}]\big)$. The two Leibniz graphs have a Kontsevich graph in common: $[\mathsf{01;12;43}] = [\mathsf{21;01;43}]$ (recall that internal vertex labels can be permuted at no cost and the swap $L \rightleftarrows R$ at a wedge costs a minus sign). This gives one cancellation; the remaining four terms equal $\mathsf{A}^{(3)}_{121}$ as listed in [7, Appendix D].

Example 7. Take the Leibniz graphs $(I_f)_{111} = [\mathsf{04;312}]$ and $(I_h)_{111} = [\mathsf{24;013}]$. Their weights are $1/48$ and $-1/48$, respectively; multiply them by 8. Expanding the Jacobiator and the Leibniz rule in the linear combination $\frac{1}{6}(I_f - I_h)_{111}$ yields the sum of Kontsevich graphs:

$$\begin{aligned}\tfrac{1}{6}\big([\mathsf{04;31;42}] &+ [\mathsf{04;12;43}] + [\mathsf{04;23;41}] + [\mathsf{05;31;42}]\\ &+ [\mathsf{05;12;43}] + [\mathsf{05;23;41}] - [\mathsf{24;01;43}] - [\mathsf{24;13;40}]\\ &- [\mathsf{24;30;41}] - [\mathsf{25;01;43}] - [\mathsf{25;13;40}] - [\mathsf{25;30;41}]\big).\end{aligned}$$

Two pairs of graphs cancel; namely $[\mathsf{05;31;42}] = [\mathsf{25;30;41}]$ and the pair $[\mathsf{05;23;41}] = [\mathsf{25;13;40}]$. The remaining eight terms equal $\mathsf{A}^{(3)}_{111}$ as listed in [7, Appendix D].

6.2. The order $\hbar^4$. Let us proceed with the term $\mathsf{A}^{(4)}$ at $\hbar^4$ in $\text{Assoc}(\star)(\mathcal{P})(\cdot,\cdot,\cdot) \bmod \bar{o}(\hbar^4)$. The numbers of Kontsevich oriented graphs in the star-product expansion grow as fast as

$$\begin{aligned}\star = \hbar^0 \cdot (\#\text{graphs} = 1) + \hbar^1 \cdot (\# = 1) + \hbar^2 \cdot (\# = 4) + \hbar^3 \cdot (\# = 13)\\ + \hbar^4 \cdot (\# = 247) + \hbar^5 \cdot (\# = 2356) + \hbar^6 \cdot (\# = 66041) + \bar{o}(\hbar^6);\end{aligned}$$

here we report the count of all nonzero-weight Kontsevich oriented graphs. Counting them modulo automorphisms (which may also swap the sinks), Banks, Panzer, and Pym obtain the numbers ($\hbar^0 : 1$, $\hbar^1 : 1$,

$\hbar^2 : 3$, $\hbar^3 : 8$, $\hbar^4 : 133$, $\hbar^5 : 1209$, $\hbar^6 : 33268$). This shows that at orders $\hbar^{k\geqslant 4}$, the use of graph-processing software is indispensible in the task of verifying factorization (1) using weighted graph expansion (6) of the operator $\Diamond$.

Specifically, the number of Kontsevich oriented graphs at $\hbar^k$ in the left-hand side of the factorization problem $\text{Assoc}(\star)(\mathcal{P})(\cdot,\cdot,\cdot) = \Diamond\big(\mathcal{P}, [\![\mathcal{P},\mathcal{P}]\!]\big)(\cdot,\cdot,\cdot)$, and the number of Leibniz graphs which assemble with nonzero coefficients to a solution $\Diamond$ in the right-hand side is presented in Table 2. At $\hbar^4$, the expansion of $\text{Assoc}(\star)(\mathcal{P}) \bmod \bar{o}\big(\hbar^4\big)$ requires 241

Table 2. Number of graphs in either side of the factorization.

k	2	3	4	5	6
LHS: # K. orgraphs	3 (Jac)	39	740	12464	290305
RHS: # L. orgraphs, coeff $\neq 0$	1 (Jac)	13	241	?	?
Reference	§5.3, [16]	§6.1, [5]	§6.2, [7]	[3]	

nonzero coefficients of Leibniz graphs on 3 sinks, $2 = n - 1$ internal vertices for bi-vectors $\mathcal{P}$ and one internal vertex for the tri-vector $[\![\mathcal{P},\mathcal{P}]\!]$, and therefore, $2(n-1)+3 = 2n+3-2 = 7$ oriented edges.

Remark 3. Again, this set of Leibniz graphs is well structured. Indeed, it is a disjoint union of homogeneous differential operators arranged according to their differential orders w.r.t. the sinks, e.g., $(1,1,1)$, $(2,1,1)$, $(1,2,1)$, $(1,1,2)$, etc., up to $(3,3,1)$.

Example 8. The Leibniz graph $L_{331} := [\mathsf{01;01;012}]$ of differential orders $(3,3,1)$ has the weight $1/24$ according to [3]. Multiplied by a universal (for all graphs at $\hbar^4$) factor $2^4 = 16$ and the factor $1/(\#\,\text{Aut}(L_{331})) = 1/2$ due to this graph's symmetry ($\mathsf{3} \rightleftarrows \mathsf{4}$), it expands to $\frac{1}{3}\big([\mathsf{01;01;01;52}] + [\mathsf{01;01;12;50}] + [\mathsf{01;01;20;51}]\big)$ by the definition of Jacobi's identity. This sum of three weighted Kontsevich orgraphs reproduces exactly $\mathsf{A}^{(4)}_{331}$, which is known from [7, Table 8 in Appendix D].

Example 9. The Leibniz graph $L_{322} := [\mathsf{01;02;012}]$ of differential orders $(3,2,2)$ has the weight $1/24$ according to [3]. Multiplied now by a universal (for all graphs at $\hbar^4$) factor $2^4 = 16$ and the factor $1/(\#\,\text{Aut}(L_{322})) = 1$, it expands to $\frac{2}{3}\big([\mathsf{01;02;01;52}] + [\mathsf{01;02;12;50}] + [\mathsf{01;02;20;51}]\big)$. This sum reproduces $\mathsf{A}^{(4)}_{322}$ (again, see [7, Table 8 in Appendix D]).

Example 10. Consider at the differential order $(1,3,2)$ at $\hbar^4$ the three Leibniz graphs $L^{(1)}_{132} := [12;13;012]$, $L^{(2)}_{132} := [12;12;014]$, and $L^{(3)}_{132} := [12;01;412]$. They have no symmetries, i.e., their automorphism groups are one-element, and their weights are $W\bigl(L^{(1)}_{132}\bigr) = 1/72$, $W\bigl(L^{(2)}_{132}\bigr) = 1/48$, and $W\bigl(L^{(3)}_{132}\bigr) = 1/48$, respectively. Pre-multiplied by their weights and universal factor $2^4 = 16$, these Leibniz graphs expand to

$$\begin{aligned}&\tfrac{2}{9}\bigl([12;13;01;52] + [12;13;12;50] + [12;13;20;51]\bigr)\\ &\quad + \tfrac{1}{3}\bigl([12;12;01;54] + [12;12;14;50] + [12;12;40;51]\bigr)\\ &\quad + \tfrac{1}{3}\bigl([12;01;41;52] + [12;01;12;54] + [12;01;24;51]\bigr).\end{aligned}$$

There is one cancellation, since $[12;01;12;54] = -[12;12;01;54]$. The remaining seven terms reproduce exactly $\mathsf{A}^{(4)}_{132}$; that component is known from [7, Table 8 in Appendix D]. Actually, there was another Leibniz graph at this homogeneity order, $L^{(4)}_{132} := [12;15;012]$, but its weight is zero and hence it does not contribute. (Indeed, we get an independent verification of this by having already balanced the entire homogeneous component at differential orders $(1,3,2)$ in the associator.)

Intermediate conclusion. We have experimentally found the constants c_k in Corollary 4 which balance the Kontsevich graph expansion of the $\hbar^k$-term $\mathsf{A}^{(k)}$ in the associator against an expansion of the respective term at $\hbar^k$ in the r.h.s. of (1) using the weighted Leibniz graphs. Namely, we conjecture $c_k = k/6$ in Section 5.3. The origin of these constants, in particular how they arise from the sum over $i < j$ in the L_∞ condition (5) (perhaps, in combination with different normalizations of the objects which we consider) still remains to be explained, similar to the reasoning in [2, 20] where the signs are fixed. Note that both in the associator, which is quadratic w.r.t. the weights of Kontsevich graphs in $\star$, and in the operator $\diamond$, which is linear in the Kontsevich weights of Leibniz graphs, the weight values are provided simultaneously, by using identical techniques (for instance, from [3]). Indeed, the weights are provided by the integral formula which is universal with respect to all the graphs under study [16].

A. Encodings of weighted Kontsevich-graph expansions for (p,q,r)-homogeneous components $(I_f,\ldots,S_h)_{pqr}$.

```
# 2/3 (S_f)_{221}
3 3 1 0 1 0 1 4 2 2/3
```

```
3 3 1 0 1 1 2 4 0 2/3
3 3 1 0 1 2 0 4 1 2/3
# 2/3 (S_g)_{122}
3 3 1 1 2 0 1 4 2 2/3
3 3 1 1 2 1 2 4 0 2/3
3 3 1 1 2 2 0 4 1 2/3
# -2/3 (S_h)_{212}
3 3 1 2 0 0 1 4 2 -2/3
3 3 1 2 0 1 2 4 0 -2/3
3 3 1 2 0 2 0 4 1 -2/3
# 1/6 (I_f)_{111}
3 3 1 0 4 3 1 4 2 1/6
3 3 1 0 4 1 2 4 3 1/6
3 3 1 0 4 2 3 4 1 1/6
3 3 1 0 5 3 1 4 2 1/6
3 3 1 0 5 1 2 4 3 1/6
3 3 1 0 5 2 3 4 1 1/6
# -1/6 (I_h)_{111}
3 3 1 2 4 0 1 4 3 -1/6
3 3 1 2 4 1 3 4 0 -1/6
3 3 1 2 4 3 0 4 1 -1/6
3 3 1 2 5 0 1 4 3 -1/6
3 3 1 2 5 1 3 4 0 -1/6
3 3 1 2 5 3 0 4 1 -1/6
# 1/6 (I_f)_{112}
3 3 1 0 2 3 1 4 2 1/6
3 3 1 0 2 1 2 4 3 1/6
3 3 1 0 2 2 3 4 1 1/6
# 1/6 (I_g)_{112}
3 3 1 1 2 0 3 4 2 1/6
3 3 1 1 2 3 2 4 0 1/6
3 3 1 1 2 2 0 4 3 1/6
# -1/3 (S_h)_{112}
3 3 1 2 4 0 1 4 2 -1/3
3 3 1 2 4 1 2 4 0 -1/3
3 3 1 2 4 2 0 4 1 -1/3
3 3 1 2 5 0 1 4 2 -1/3
3 3 1 2 5 1 2 4 0 -1/3
3 3 1 2 5 2 0 4 1 -1/3
```

```
# 1/3 (I_f)_{121}
3 3 1 0 1 3 1 4 2 1/3
3 3 1 0 1 1 2 4 3 1/3
3 3 1 0 1 2 3 4 1 1/3
# -1/3 (I_h)_{121}
3 3 1 2 1 0 1 4 3 -1/3
3 3 1 2 1 1 3 4 0 -1/3
3 3 1 2 1 3 0 4 1 -1/3
# 1/3 (S_f)_{211}
3 3 1 0 4 0 1 4 2 1/3
3 3 1 0 4 1 2 4 0 1/3
3 3 1 0 4 2 0 4 1 1/3
3 3 1 0 5 0 1 4 2 1/3
3 3 1 0 5 1 2 4 0 1/3
3 3 1 0 5 2 0 4 1 1/3
# -1/6 (I_g)_{211}
3 3 1 1 0 0 3 4 2 -1/6
3 3 1 1 0 3 2 4 0 -1/6
3 3 1 1 0 2 0 4 3 -1/6
# -1/6 (I_h)_{211}
3 3 1 2 0 0 1 4 3 -1/6
3 3 1 2 0 1 3 4 0 -1/6
3 3 1 2 0 3 0 4 1 -1/6
```

Acknowledgements. The first author thanks the Organisers of international workshop ‘Symmetries and Integrability of Equations of Mathematical Physics’ (December 22–24, 2018, Kyiv) for helpful discussions and warm atmosphere during the meeting. A part of this research was done while RB was visiting at RUG and AVK was visiting at JGU Mainz (supported by IM JGU via project 5020 and JBI RUG project 106552). The research of AVK is supported by the IHÉS (partially, by the Nokia Fund).

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Збірник праць Інституту математики НАН України 2019, т. 16, № 1, 44–49

УДК 517.958

Точні розв'язки рівнянь Фішера з коефіцієнтами, що залежать від часової змінної

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Метод еквівалентності, а також метод перетворень між класами диференціальних рівнянь, запропонований у роботі [O. Vaneeva et al. *Acta Appl. Math.* **106** (2009), 1–46], застосовано для побудови точних розв'язків рівнянь Фішера з коефіцієнтами, що залежать від часової змінної.

The equivalence method and the method of mapping between classes of differential equations proposed in [O. Vaneeva et al. *Acta Appl. Math.* **106** (2009), 1–46] are used for construction of exact solutions for Fisher equations with time-dependent coefficients.

Рівняння Фішера,

$$u_t = ku_{xx} + mu(1-u), \quad km \neq 0, \tag{1}$$

запропоноване Р.Е. Фішером у 1937 році [5], є класичною детерміністичною моделлю популяційної генетики, що описує динаміку частоти появи мутантного гену у популяції, який володіє селективною перевагою. Залежна змінна u — частота появи мутантного гену у популяції, що однорідно розташована у лінійному середовищі проживання, наприклад, на береговій лінії, стала m — інтенсивність селекції на перевагу мутантного гену, k — коефіцієнт дифузії. Максимальна алгебра ліївської інваріантності рівняння (1) є двовимірною. Базисними операторами цієї алгебри є оператори зсувів за часовою та просторовою змінною ∂_t та ∂_x, що дозволяє побудувати для цього рівняння розв'язки типу біжучої хвилі. Такі розв'язки було побудовано у роботах [1, 3, 4, 8, 9]. Теореми існування та єдиності обмежених розв'язків більш загального класу рівнянь $u_t = u_{xx} + F(t,x,u)$ доведено А.М. Комогоровим, І.Г. Петровським та М.С. Піскуновим [7].

Пізніше було запропоновано розглянути узагальнену модель вигляду

$$u_t = g(t)u_{xx} + f(t)u(1-u), \quad gf \neq 0, \tag{2}$$

де дифузійний коефіцієнт g і коефіцієнт селективної переваги f залежать від часової змінної [6, 11]. Завдяки таким коефіцієнтам можна взяти до уваги вплив довготермінової зміни клімату або короткострокової сезонності.

Групову класифікацію рівнянь (2) було виконано у роботі [17], однак задача пошуку точних розв'язків таких рівнянь там не розглядалася. У цій роботі для побудови точних розв'язків рівнянь Фішера зі змінними коефіцієнтами застосовано методи, що базуються на використанні невироджених точкових перетворень, а саме метод еквівалентності та метод перетворень між класами диференціальних рівнянь. У результаті побудовано декілька сімей точних розв'язків для певних підкласів класу (2).

Метод еквівалентності. Під методом еквівалентності для побудови точних розв'язків ми розуміємо використання невироджених точкових перетворень з групи еквівалентності заданого класу та точних розв'язків, що є відомими для деяких рівнянь з цього класу. Якщо два рівняння пов'язані між собою невиродженим точковим перетворенням, то, за термінологією Л.В. Овсяннікова, вони називаються подібними [10]. Тоді подібними відносно цього ж перетворення є і відповідні набори точних розв'язків, симетрій, законів збереження цих рівнянь. Для класів зі змінними коефіцієнтами найбільш ефективне використання методу еквівалентності полягає у зведенні певного рівняння зі змінними коефіцієнтами з досліджуваного класу до рівняння зі сталими коефіцієнтами з того ж класу. Наступним кроком є побудова точних розв'язків для першого з цих рівнянь шляхом розмноження відомих розв'язків другого рівняння перетвореннями еквівалентності.

У роботі [17] отримано критерій звідності рівнянь зі змінними коефіцієнтами з класу (2) до рівняння Фішера зі сталими коефіцієнтами (1). Рівняння з класу (2) можна звести до рівняння вигляду (1) тоді і тільки тоді, коли для деякої додатної сталої λ коефіцієнти f і g задовольняють умову

$$\lambda g^2 - 2\frac{g_{tt}}{g} + 3\frac{g_t^2}{g^2} = f^2 - 2\frac{f_{tt}}{f} + 3\frac{f_t^2}{f^2}. \tag{3}$$

Умова (3) виконується тоді і тільки тоді, коли функцію g можна виразити через функцію f за формулою:

$$g(t)=\frac{\lambda\Delta f(t)\mathrm{e}^{\int f(t)\,\mathrm{d}t}}{\left(\alpha\mathrm{e}^{\int f(t)\,\mathrm{d}t}+\beta\right)\left(\gamma\mathrm{e}^{\int f(t)\,\mathrm{d}t}+\delta\right)},$$

де λ — додатна стала, а пари сталих (α,β) і (γ,δ) визначено з точністю до ненульового сталого множника, при цьому $\Delta=\alpha\delta-\beta\gamma\neq 0$. Для компактності запису введемо позначення $h(t)=\mathrm{e}^{\int f(t)\,\mathrm{d}t}$.

Отже, клас рівнянь Фішера зі змінними коефіцієнтами вигляду

$$u_t=\frac{\lambda\Delta f(t)h(t)}{(\alpha h(t)+\beta)(\gamma h(t)+\delta)}u_{xx}+f(t)u(1-u), \tag{4}$$

що є підкласом класу (2), зводиться точковими перетвореннями до класичного рівняння Фішера зі сталими коефіцієнтами,

$$u_t=u_{xx}+u(1-u). \tag{5}$$

Для того, щоб знайти точкові перетворення, що реалізують подібність рівнянь (4) та (5), знайдемо спочатку групу еквівалентності.

Теорема 1. *Репараметризований клас* (2) *з новим довільним елементом* $h(t)$*, що задовольняє рівняння* $h_t=fh$*, є нормалізованим відносно своєї узагальненої групи еквівалентності* $\hat{G}^{\sim}$*. Група* $\hat{G}^{\sim}$ *складається з перетворень*

$$\tilde t=T(t),\quad \tilde x=\delta_1x+\delta_2,\quad \tilde u=\frac{(\alpha h+\beta)(\gamma h+\delta)}{h\Delta}u-\gamma\frac{\alpha h+\beta}{\Delta},$$
$$\tilde f=\frac{h\Delta}{T_t(\alpha h+\beta)(\gamma h+\delta)}f,\quad \tilde g=\frac{{\delta_1}^2}{T_t}g,\quad \tilde h=\frac{\alpha h+\beta}{\gamma h+\delta},$$

де $T(t)$ *— довільна гладка функція, що задовольняє умову* $T_t\neq 0$*,* δ_1 *і* δ_2 *— довільні сталі, причому* $\delta_1\neq 0$*, пари сталих* (α,β) *і* (γ,δ) *є визначеними з точністю до ненульового сталого множника і* $\Delta=\alpha\delta-\beta\gamma\neq 0$.

Узагальнена група еквівалентності $\hat{G}^{\sim}$ для репараметризованого класу (2), набір довільних елементів якого формально містить функцію $h(t)$, є розширеною узагальненою групою еквівалентності для вихідного класу (2). Означення узагальненої та розширеної узагальненої груп еквівалентності і нормалізованості класу наведено, зокрема, у [13, 14].

З теореми 1 знаходимо перетворення, що відображають рівняння (4) у рівняння (5). Такі перетворення мають вигляд

$$\tilde{t} = \ln \frac{\alpha h(t) + \beta}{\gamma h(t) + \delta} + c_1, \quad \tilde{x} = \frac{x}{\sqrt{\lambda}} + c_2, \tag{6}$$

$$\tilde{u} = \frac{(\alpha h(t) + \beta)(\gamma h(t) + \delta)}{h(t)\Delta} u - \gamma \frac{\alpha h(t) + \beta}{\Delta},$$

де c_1, c_2 — довільні сталі. З допомогою цих перетворень отримуємо розв'язки рівняння (4) з відомих розв'язків класичного рівняння Фішера (5). Побудовано сім'ю точних розв'язків рівнянь (4):

$$u = \frac{h\Delta \exp\left(\frac{5}{3}\tilde{t} + \frac{\sqrt{6}}{3}\tilde{x}\right) \wp\left(\exp\left(\frac{5}{6}\tilde{t} + \frac{\sqrt{6}}{6}\tilde{x}\right) + \tilde{C}, 0, \hat{C}\right)}{(\alpha h + \beta)(\gamma h + \delta)} + \frac{\gamma h}{\gamma h + \delta},$$

частинний випадком якої в елементарних функціях є

$$u = \frac{h\Delta}{(\alpha h + \beta)(\gamma h + \delta)} \frac{1}{\left(C \exp\left(\frac{\sqrt{6}}{6}\tilde{x} - \frac{5}{6}\tilde{t}\right) \pm 1\right)^2} + \frac{\gamma h}{\gamma h + \delta}.$$

В отриманих розв'язках $\tilde{t}$ та $\tilde{x}$ визначено у (6), $\wp(z, k_1, k_2)$ — еліптична функція Вейєрштраса, c_1, c_2, C, $\tilde{C}$, $\hat{C}$ — довільні сталі, $C \neq 0$.

Оскільки рівняння Фішера допускають дискретне перетворення симетрії $x \mapsto -x$, всі отримані розв'язки з протилежними знаками x також задовольняють рівняння (4). Ще одне перетворення симетрії $u \mapsto 1 - u$ також дозволяє додатково розмножити знайдені розв'язки.

Метод перетворень між класами диференціальних рівнянь. Окрім перетворень еквівалентності, що не змінюють структуру класу диференціальних рівнянь, а лише переводять одне рівняння з класу в інше рівняння з цього ж класу, можливо також розглянути невироджені точкові перетворення між класами диференціальних рівнянь. Цей метод було запропоновано у роботі [16] для виконання групової класифікації квазілінійних рівнянь реакції-дифузії зі змінними коефіцієнтами та степеневою нелінійністю. Пізніше цим методом було досліджено з симетрійної точки зору й інші класи рівнянь (див., [15], а також [18] та наведені там посилання). У цій роботі метод перетворень між класами диференціальних рівнянь застосовано для побудови точних розв'язків.

Доведено, що сім'я точкових перетворень, параметризованих довільним елементом $f(t)$ класу (2),

$$\tilde{t} = \int f(t)\mathrm{e}^{\int f(t)\mathrm{d}t}\mathrm{d}t, \quad \tilde{x} = x, \quad \tilde{u} = -\mathrm{e}^{-\int f(t)\mathrm{d}t}u, \tag{7}$$

відображає клас (2) у клас квазілінійних рівнянь реакції–дифузії з квадратичною нелінійністю та одним довільним елементом, що залежить від змінної часу:

$$\tilde{u}_{\tilde{t}} = \tilde{g}(\tilde{t})\tilde{u}_{\tilde{x}\tilde{x}} + \tilde{u}^2, \quad \tilde{g} \neq 0. \tag{8}$$

Довільні елементи класів (2) та (8) пов'язані формулою

$$\tilde{g} = \frac{g(t)}{f(t)}\mathrm{e}^{-\int f(t)\mathrm{d}t}.$$

Для рівняння $u_t = u_{xx} + u^2$ відомі декілька точних розв'язків (див. [2, 9] та [12, с. 157]). Використовуючи їх та перетворення (7), знаходимо нові точні розв'язки рівняння Фішера зі змінними коефіцієнтами

$$u_t = f(t)\mathrm{e}^{\int f(t)\mathrm{d}t}u_{xx} + f(t)u(1-u)\colon$$

$$u = \frac{12\big(4\pm\sqrt{6}\big)x(x+c_1) + 120(12\pm 5\sqrt{6})\Theta + 12\big(2\pm\sqrt{6}\big)c_2 + 6c_1^2}{\mathrm{e}^{-\int f(t)\mathrm{d}t}\big(x^2 + c_1x + 10(3\pm\sqrt{6})\Theta + c_2\big)^2},$$

$$u = \mathrm{e}^{\int f(t)\mathrm{d}t}\wp\left(\frac{x}{\sqrt{6}}, 0, \hat{C}\right),$$

де $\Theta = \int f(t)\mathrm{e}^{\int f(t)\mathrm{d}t}\mathrm{d}t$, c_1, c_2, $\hat{C}$ — довільні сталі.

Авторка вдячна професору Р.О. Поповичу за цінні поради.

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Збірник праць Інституту математики НАН України 2019, т. 16, № 1, 50–65

УДК 517.912:512.816

Differential invariants for a class of diffusion equations

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Знайдено повну групу еквівалентності класу (1+1)-вимірних еволюційних рівнянь другого порядку, яка виявилася нескінченновимірною. Методологію екваріантних рухомих реперів застосовано у регулярному випадку процедури нормалізації до побудови рухомого репера групи, пов'язаної з групою еквівалентності в контексті перетворень еквівалентності між рівняннями класу. За допомогою побудованого рухомого репера описано алгебру диференціальних інваріантів цієї групи через отримання мінімальної генеруючої множини диференціальних інваріантів і повної множини операторів інваріантного диференціювання.

We find the complete equivalence group of a class of (1+1)-dimensional second-order evolution equations, which is infinite-dimensional. The equivariant moving frame methodology is invoked to construct, in the regular case of the normalization procedure, a moving frame for a group related to the equivalence group in the context of equivalence transformations among equations of the class under consideration. Using the moving frame constructed, we describe the algebra of differential invariants of the former group by obtaining a minimum generating set of differential invariants and a complete set of independent operators of invariant differentiation.

1. Introduction. Invariants and differential invariants of transformation groups, in particular, point symmetry groups admitted by systems of differential equations have a wide range of applications and are therefore an intensively investigated subject. Differential invariants play a central role in the invariant parameterization problem [1, 2, 30] and in the problem of invariant discretization [3, 5, 7]. They are also used to construct invariant differential equations and invariant variational problems [22, 23], as well as in computer vision, integrable systems, classical invariant theory and the calculus of variations [6, 22, 24].

Rather recently, finding differential invariants in problems related to group classification became a research topic of interest. The idea is to compute the differential invariants not for the point symmetry group of a single system of differential equations but for the equivalence group admitted by a class of such systems. The primary motivation for such a survey is to study the equivalence of systems of differential equations. Exploring equivalence, it is possible to explicitly determine point transformations among systems from a class [28]. Such a mapping between two systems of differential equations is especially helpful if wide sets of exact solutions are known for one of the systems involved. These solutions then can be mapped to solutions of the equivalent system. Another case of particular interest is the mapping between nonlinear and linear elements of a class of systems of differential equations [19]. For the solution of the equivalence problem, finding differential invariants for the equivalence group is a main ingredient. There are a number of papers where some low-order differential invariants of the equivalence groups of various physically relevant classes of systems of differential equations were computed using the Lie infinitesimal method; see, e.g., [11, 12, 13, 14, 15, 17, 32, 33, 34, 35] and references therein.

In the present paper we will be concerned with differential invariants for a group[1] related to the equivalence group of the class of diffusion equations

$$u_t = u_{xx} + f(u, u_x) \tag{1}$$

in the context of equivalence transformations among equations of this class. This subject was originally considered in [32], using the infinitesimal method and restricting the order of differential invariants up to two. We revisit the construction of differential invariants for the class (1) from the very beginning, analyzing differential invariants of which group should be found. Then, we apply the method of equivariant moving frames in the formulation originally proposed and formulated by Fels and Olver [9, 10], which was later generalized to infinite-dimensional Lie (pseudo)groups in [6, 25, 26], and this is the setting that is needed

[1] In fact, this object and the "equivalence group" of the class (1) are Lie pseudogroups of locally defined point transformations. We use the term "group" for brevity since this does not lead to any confusion.

to study differential invariants for the class (1). The advantage of moving frames is that they allow for a canonical process of *invariantization*, which associates to each object, such as functions, differential functions, differential forms and total differentiation operators, its invariant counterpart. For the problem of finding differential invariants of a Lie transformation (pseudo)group, this property is especially convenient. The invariantization of the jet-space coordinate functions yields the so-called *normalized differential invariants*. The invariantized coordinate functions whose transformed counterparts were involved in the construction of the corresponding moving frame via the *normalization procedure* are equal to the respective constants chosen in the course of normalization. This is why these objects are called *phantom normalized differential invariants*. The *non-phantom normalized differential invariants* constitute a complete set of functionally independent differential invariants. As a further asset, the method of moving frames also permits to study the algebra of differential invariants by deriving relations, called *syzygies*, between invariant derivatives of non-phantom normalized differential invariants. Finding syzygies can aid in the establishment of a *minimum generating set* of differential invariants. See e.g. [6, 8, 22, 25, 26] for more details and an extensive discussion on the computation of differential invariants for both finite-dimensional Lie symmetry groups and for infinite-dimensional Lie (pseudo)groups using moving frames.

The further organization of this paper is as follows. In Section 2 we compute the equivalence group and the equivalence algebra of the class (1). Section 3 is devoted to the selection of a group to be considered and a preliminary analysis of equivariant moving frames associated with this group. The structure of the algebra of differential invariants is determined in the main Section 4. This includes a description of a minimum generating set of differential invariants and a complete set of independent operators of invariant differentiation, which serve to exhaustively describe the set of differential invariants. Moreover, for each $k \in \mathbb{N}_0$ we explicitly present a functional basis of differential invariants of order not greater than k.

2. The equivalence group. The auxiliary system for the class (1), which is satisfied by the arbitrary element f, is $f_t = f_x = f_{u_t} = f_{u_{tt}} = f_{u_{tx}} = f_{u_{xx}} = 0$. By definition [27, 28, 29, 31], the (usual) equivalence group $G^{\sim}$ of the class (1) consists of the point transformations in the space with coordinates $(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, f)$ that have the following properties:

- they are projectable to the space with the coordinates (t, x, u),
- their components for derivatives of u are found by prolongation using the chain rule, and
- they map every equation from the class (1) to an equation from the same class.

To begin finding the group $G^{\sim}$, we fix an arbitrary equation of the class (1), $u_t = u_{xx} + f(u, u_x)$, and aim to find point transformations in the space with coordinates (t, x, u),

$$\tilde{t} = T(t, x, u), \quad \tilde{x} = X(t, x, u), \quad \tilde{u} = U(t, x, u), \tag{2}$$

that transform the fixed equation to an equation of the same class,

$$\tilde{u}_{\tilde{t}} = \tilde{u}_{\tilde{x}\tilde{x}} + \tilde{f}(\tilde{u}, \tilde{u}_{\tilde{x}}). \tag{3}$$

A preliminary simplification is obtained from noting that the class (1) is a subclass of the class of second-order (1+1)-dimensional semi-linear evolution equations. Any point transformation between two equations from the latter class satisfies the constraints $T_x = T_u = X_u = 0$, i.e., $\tilde{t} = T(t)$, $\tilde{x} = X(t, x)$, and $T_t X_x U_u \neq 0$. See [16, 18, 21] for further details. After taking into account the above constraints, the required transformed derivatives read

$$\tilde{u}_{\tilde{t}} = \frac{1}{T_t}\left(\mathrm{D}_t U - \frac{X_t}{X_x}\mathrm{D}_x U\right), \quad \tilde{u}_{\tilde{x}} = \frac{1}{X_x}\mathrm{D}_x U,$$
$$\tilde{u}_{\tilde{x}\tilde{x}} = \left(\frac{1}{X_x}\mathrm{D}_x\right)^2 U,$$

where D_t and D_x are the usual total derivative operators with respect to t and x, respectively. Substituting these expressions and $u_t = u_{xx} + f$ into Eq. (3), we split the resulting equation with respect to u_{xx} yielding $T_t = X_x^2$. The remaining equation is

$$f = \frac{T_t}{U_u}\tilde{f} - U_t + \frac{X_t}{X_x}(U_x + U_u u_x) + U_{xx} + 2U_{xu}u_x + U_{uu}u_x^2. \tag{4}$$

The differential consequences of Eq. (4) that are obtained by separate differentiations with respect to t and x can be split with respect to derivatives of $\tilde{f}$ since they are regarded as independent for equivalence transformations. This yields the equations

$$T_{tt} = X_{xt} = X_{tt} = U_t = U_x = 0.$$

The equation (4) itself gives the f-component of equivalence transformations.

The arbitrary element f in fact depends only on u and u_x. The space with coordinates (t, x, u, u_x, f) is preserved by all elements of $G^\sim$. This is why we can assume this space as the underlying space for $G^\sim$ and present merely the transformation components for its coordinates.

As a result, we have proved the following theorem.

Theorem 1. *The equivalence group $G^\sim$ of the class* (1) *is constituted by the transformations*

$$\begin{aligned} &\tilde t = C_1^2 t + C_0, \quad \tilde x = C_1 x + C_1C_2 t + C_3, \quad \tilde u = \varphi(u), \\ &\tilde u_{\tilde x} = C_1^{-1}\varphi' u_x, \quad \tilde f = C_1^{-2}\left(\varphi' f - C_2\varphi' u_x - \varphi'' u_x^2\right), \end{aligned} \tag{5}$$

where $C_0, C_1, C_2, C_3 \in \mathbb{R}$, φ is an arbitrary smooth function of u and $C_1\varphi' \neq 0$.

The infinitesimal generators of one-parameter subgroups of $G^\sim$, which constitute the equivalence algebra $\mathfrak{g}^\sim$ of the class (1), can be derived from (5) by differentiation, cf. the proof of Corollary 11 in [20] or the proof of Corollary 6 in [4]. These generators coincide with those determined in [32]. As we will later need them for the description of the algebra of differential invariants of a group related to $G^\sim$ in the context of the $G^\sim$-equivalence among equations of the class (1), we present them here. The general element of $\mathfrak{g}^\sim$ is

$$Q = \tau\partial_t + \xi\partial_x + \phi\partial_u + \eta\partial_{u_x} + \theta\partial_f,$$

where the components are of the form

$$\begin{aligned} &\tau = 2c_1 t + c_0, \quad \xi = c_1 x + c_2 t + c_3, \quad \phi = \phi(u), \\ &\eta = (\phi' - c_1)u_x, \quad \theta = (\phi' - 2c_1)f - c_2 u_x - \phi'' u_x^2, \end{aligned}$$

in which c_0, c_1, c_2 and c_3 are arbitrary real constants, and ϕ is an arbitrary smooth function of u. In other words, the equivalence algebra $\mathfrak{g}^\sim$ of the class (1) is spanned by the vector fields

$$\begin{aligned} &\partial_t, \quad 2t\partial_t + x\partial_x - u_x\partial_{u_x} - 2f\partial_f, \quad t\partial_x - u_x\partial_f, \\ &\phi\partial_u + \phi' u_x\partial_{u_x} + (\phi' f - \phi'' u_x^2)\partial_f, \end{aligned}$$

where ϕ runs through the set of smooth functions of u.

3. Preliminary analysis of moving frames. Let us first clarify the space of independent and dependent variables to be used and the group to be considered. While formally the arbitrary element f is a smooth function on the second-order jet space with coordinates $(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx})$, practically it explicitly depends only on u and u_x. This is why subsequently we will only consider the projection of the equivalence transformations to the space with coordinates (u, u_x, f). As a shorthand, we denote $v := u_x$ and $\tilde{v} := \tilde{u}_{\tilde{x}} = V(u, v) := C_1^{-1}\varphi'(u)v$. In other words, we will in fact study differential invariants of the projection G_1 of $G^\sim$ to the space with coordinates (u, v, f), where u and v are the independent variables and f is the dependent variable. The infinitesimal counterpart of G_1 is the projection $\mathfrak{g}_1$ of $\mathfrak{g}^\sim$ to the space with coordinates (u, v, f).

In order to describe the algebra of differential invariants of the group G_1, we now construct a moving frame for this group. Since it is infinite-dimensional, we have to use the machinery developed for Lie pseudo-groups, see [6, 25] for an extensive description of this subject.

The first step in the construction of the moving frame is the computation of the lifted horizontal coframe, the dual of which yields the implicit total differentiation operators $\mathrm{D}_{\tilde{u}}$ and $\mathrm{D}_{\tilde{v}}$. For the equivalence transformations (5), the lifted horizontal coframe is

$$\mathrm{d_h}\tilde{u} = (\mathrm{D}_u U)\,\mathrm{d}u + (\mathrm{D}_v U)\,\mathrm{d}v = \varphi'\,\mathrm{d}u,$$
$$\mathrm{d_h}\tilde{v} = (\mathrm{D}_u V)\,\mathrm{d}u + (\mathrm{D}_v V)\,\mathrm{d}v = \frac{\varphi''}{C_1}v\,\mathrm{d}u + \frac{\varphi'}{C_1}\,\mathrm{d}v.$$

Computing the dual, we derive that

$$\mathrm{D}_{\tilde{u}} = \frac{1}{\varphi'}\mathrm{D}_u - \frac{\varphi''}{(\varphi')^2}v\mathrm{D}_v, \quad \mathrm{D}_{\tilde{v}} = \frac{C_1}{\varphi'}\mathrm{D}_v \tag{6}$$

are the required implicit differentiation operators. Acting with them on the transformation component for f, we find that

$$\tilde{f}_{ij} = \frac{\partial^{i+j}\tilde{f}}{\partial\tilde{u}^i\partial\tilde{v}^j} = \mathrm{D}_{\tilde{u}}^i\mathrm{D}_{\tilde{v}}^j F,$$

where $i, j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and

$$\tilde{f}_{00} = \tilde{f} = F := \frac{1}{C_1^2}(\varphi' f - C_2\varphi' v - \varphi'' v^2)$$

is the f-component of equivalence transformations. In particular, the derivatives up to order 2 are exhausted by

$$\tilde f_{10} = \frac{1}{C_1^2\varphi'}\left(\varphi' f_u + \varphi''(f - vf_v) - \varphi''' v^2 + 2\frac{(\varphi'')^2}{\varphi'}v^2\right),$$

$$\tilde f_{01} = \frac{1}{C_1\varphi'}\left(\varphi' f_v - C_2\varphi' - 2\varphi'' v\right),$$

$$\tilde f_{20} = \frac{1}{C_1^2\varphi'}\left(f_{uu} - \frac{\varphi''}{\varphi'}(f_u - 2vf_{uv}) + \left(\frac{\varphi''}{\varphi'}\right)^2 v^2 f_{vv} + \left(\frac{\varphi''}{\varphi'}\right)'(f - vf_v) - (\varphi')^2\left(\frac{1}{\varphi'}\left(\frac{1}{\varphi'}\right)''\right)' v^2\right),$$

$$\tilde f_{11} = \frac{1}{C_1\varphi'^2}\left(\varphi' f_{uv} - \varphi'' v f_{vv} - 2\varphi''' v + 4\frac{\varphi''^2}{\varphi'}v\right),$$

$$\tilde f_{02} = \frac{1}{\varphi'^2}(\varphi' f_{vv} - 2\varphi'').$$

There are a relative invariant and a relative conditional invariant which play a significant role in the following consideration. By taking the difference $\tilde f_{00} - \tilde v\tilde f_{01}$ we exclude the inessential constant C_2, which only arises in $\tilde f_{00}$ and $\tilde f_{01}$,

$$\tilde f_{00} - \tilde v\tilde f_{01} = \frac{1}{C_1^2}\left(\varphi'(f - vf_v) + \varphi'' v^2\right).$$

Combining further $2(\tilde f_{00} - \tilde v\tilde f_{01}) + \tilde v^2\tilde f_{02}$ to exclude φ'', we obtain

$$\tilde W = \frac{1}{C_1^2}W, \quad \text{where} \quad \begin{array}{l} W = 2f - 2vf_v + v^2 f_{vv}, \\ \tilde W = 2\tilde f - 2\tilde v\tilde f_{\tilde v} + \tilde v^2\tilde f_{\tilde v\tilde v}, \end{array}$$

i.e., W is a relative invariant of G_1. In other words, the condition $W = 0$ is preserved by any equivalence transformation in the class (1). Analogously, the combination $2\tilde f_{10} - v\tilde f_{11}$ gives

$$\tilde S = \frac{1}{C_1^2}S + \frac{1}{C_1^2}\frac{\varphi''}{\varphi'}W, \quad \text{where} \quad \begin{array}{l} S = 2f_u - vf_{uv}, \\ \tilde S = 2\tilde f_{\tilde u} - \tilde v\tilde f_{\tilde u\tilde v}. \end{array} \tag{7}$$

This means that S is a relative invariant of G_1 if the condition $W = 0$ is satisfied. Values of the differential functions W and S determine which normalization conditions should be chosen.

We next find appropriate normalization conditions, which form the basis for the construction of an equivariant moving frame. As φ arises only in U, we can set U to any value including zero. The value of V can be set to any constant excluding zero, and all these possibilities are equivalent. We find it convenient to put $V=1$ and express $\varphi'=C_1/v$. The constraint $W=0$ singles out the *singular* case for the moving frame construction, which has to be investigated separately. Within this singular case, there is the *ultra-singular* subcase associated with the constraint $S=0$. Indeed, under the constraint $W=0$ the equation (7) can be solved for C_1 if and only if $S\neq 0$.

4. Differential invariants for the regular case. In this paper, we only consider the *regular* case for moving frames of G_1, where $W\neq 0$. In this case, the following normalization conditions can be used to determine a complete moving frame

$$
\begin{aligned}
&\tilde u=0,\quad \tilde v=1,\quad \tilde f=1,\quad \tilde f_{01}=0,\quad \tilde f_{02}=0,\\
&\tilde f_{i0}=-\frac{v^2\varphi^{(i+2)}}{{C_1}^2(\varphi')^i}+\frac{1}{{C_1}^2}\sum_{i'=0}^{i}\binom{i}{i'}\frac{1}{(\varphi')^{i'}}\left(\frac{\varphi''}{(\varphi')^2}\right)^{i-i'}f_{i',i-i'}\\
&\qquad+\cdots=0,\ i\in\mathbb{N}.
\end{aligned}
\tag{8}
$$

In the expression for $\tilde f_{i0}$, we presented only the summands with the highest-order derivatives of φ and f, which are $\varphi^{(i+2)}$ and $f_{i',i-i'}$, $i'=0,\dots,i$, respectively. We solve the first five equations with respect to C_1, C_2, φ, φ' and φ'' and substitute the obtained expressions into the other equations. For each fixed $i\in\mathbb{N}$, we solve the modified equation $\tilde f_{i0}=0$ in view of the similar equations with lower values of i and thus find an expression for $\varphi^{(i+2)}$, the explicit form of which is essential for further consideration only for $i=3$. This yields the following complete moving frame:

$$
\begin{aligned}
&C_1=\frac{W}{2v},\quad C_2=f_v-vf_{vv},\\
&\varphi=0,\quad \varphi'=\frac{W}{2v^2},\quad \varphi''=\frac{W}{4v^2}f_{vv},\\
&\varphi'''=\frac{W}{4v^4}\Big(2f_u+(f-vf_v+v^2f_{vv})f_{vv}\Big),\\
&\varphi^{(i+2)}=\frac{W}{2v^1}\sum_{i'=0}^{i}\binom{i}{i'}\left(\frac{v^2}{W}\right)^{i-i'}f_{i',i-i'}+\cdots,\quad i=2,3,\dots.
\end{aligned}
\tag{9}
$$

In the expression for $\varphi^{(i+2)}$, we presented only the summands with the highest-order derivatives $f_{i',i-i'}$, $i'=0,\dots,i$. The invariantization $I^{ij}=\iota(f_{ij})$ of the derivatives f_{ij} of $\tilde f$ that are not involved in the normalization conditions (8) gives rise to a complete set of functionally independent differential invariants of G_1. The lowest-order non-phantom normalized differential invariant is I^{11}, and it reads

$$I^{11}=-2v^2\frac{4f_u-2vf_{uv}+(2f-2vf_v+v^2f_{vv})f_{vv}}{(2f-2vf_v+v^2f_{vv})^2}.$$

This differential invariant is of second order. For each tuple (i,j) with $i+j\geqslant 3$ and $j\neq 0$, the maximal orders of derivatives of f and φ appearing in the expression for $\tilde f_{ij}$ are $i+j$ and $i+2$, respectively. This is why the maximal order of derivatives of f in the expression for $\tilde f_{ij}$ cannot be lowered in the course of the invariantization, i.e., the order of the normalized differential invariant I^{ij} is $i+j$. Therefore, there are precisely $\frac12 k(k+1)-2$ functionally independent differential G_1-invariants of order not greater than $k\geqslant 2$. They are given by the functions I^{11} and I^{ij} with $3\leqslant i+j\leqslant k$ and $j\neq 0$.

Apart from finding the complete set of functionally independent differential invariants of G_1 for each fixed order by successively invariantizing all the derivatives f_{ij}, the moving frame (9) can be used to determine the operators of invariant differentiation. They are found upon invariantizing the operators of total differentiation (6) and read

$$\mathrm{D}^{\mathrm i}_u=\frac{2v^2}{2f-2vf_v+v^2f_{vv}}\left(\mathrm{D}_u-\frac12 vf_{vv}\mathrm{D}_v\right),\quad \mathrm{D}^{\mathrm i}_v=v\mathrm{D}_v. \tag{10}$$

We now aim to investigate the structure of the algebra of differential invariants of G_1. The starting point for this investigation is the universal recurrence relation, which relates the differentiated invariantized differential functions or differential forms with the invariantization of the respective differentiated objects. This universal recurrence relation reads [25]

$$\mathrm{d}\iota(\Omega)=\iota\big(\mathrm{d}\Omega+Q^{(\infty)}(\Omega)\big). \tag{11}$$

The first step in our study is the evaluation of (11) for the independent variables u and v and the derivatives f_{ij}, $i,j\in\mathbb N_0$,

$$\mathrm{d_h}\iota(u)=\omega^1+\iota(\phi),\quad \mathrm{d_h}\iota(v)=\omega^2+\iota(\eta),$$

$$\begin{aligned}\mathrm{d_h}I^{ij} &= \mathrm{d_h}\iota(f_{ij}) = \iota(f_{i+1,j}\mathrm{d}u + f_{i,j+1}\mathrm{d}v + \theta^{ij})\\ &= I^{i+1,j}\omega^1 + I^{i,j+1}\omega^2 + \iota(\theta^{ij}),\end{aligned}$$

where $\omega^1 = \iota(\mathrm{d}u)$, $\omega^2 = \iota(\mathrm{d}v)$, and

$$\begin{aligned}\theta^{ij} &= \mathrm{D}_u^{\,i}\mathrm{D}_v^{\,j}(\theta - \phi f_{10} - \eta f_{01}) + \phi f_{i+1,j} + \eta f_{i,j+1}\\ &= (j-2)c_1 f_{ij} - (j-1)\sum_{i'=0}^{i}\binom{i}{i'}\phi^{(i'+1)}f_{i-i',j}\\ &\quad - \sum_{i'=1}^{i}\binom{i}{i'}\Big(\phi^{(i')}f_{i-i'+1,j} + v\phi^{(i'+1)}f_{i-i',j+1}\Big)\\ &\quad - c_2\delta_{0i}(\delta_{0j}v + \delta_{1j}) - \phi^{(i+2)}(\delta_{0j}v^2 + 2\delta_{1j}v + 2\delta_{2j})\end{aligned}$$

is the f_{ij}-component of the infinite prolongation of the vector field $\phi\partial_u + \eta\partial_v + \theta\partial_f$. Here δ_{ij} is the Kronecker delta. The respective recurrence relations then split into two kinds, the first being the so-called phantom recurrence relations. For a well-defined moving frame cross-section, they can be uniquely solved for the invariantized Maurer–Cartan forms, which arise due to the presence of the correction term $\iota\big(Q^{(\infty)}(\Omega)\big)$ in (11). Then, plugging these invariantized Maurer–Cartan forms into the second kind of recurrence relations, the non-phantom ones, gives a complete description of the relation between the normalized and differentiated differential invariants, see [6, 25] for more details. For the chosen cross-section (8), the phantom recurrence relations read

$$\begin{aligned}&0 = \mathrm{d_h}\iota(u) = \omega^1 + \iota(\phi) = \omega^1 + \hat\phi,\\ &0 = \mathrm{d_h}\iota(v) = \omega^2 + \iota(\eta) = \omega^2 + \hat\phi' - \hat c_1,\\ &0 = \mathrm{d_h}I^{00} = \iota(\theta) = \hat\phi' - 2\hat c_1 - \hat c_2 - \hat\phi'',\\ &0 = \mathrm{d_h}I^{01} = I^{11}\omega^1 + \iota(\theta^{01}) = I^{11}\omega^1 - \hat c_2 - 2\hat\phi'',\\ &0 = \mathrm{d_h}I^{02} = I^{12}\omega^1 + I^{03}\omega^2 + \iota(\theta^{02}) = I^{12}\omega^1 + I^{03}\omega^2 - 2\hat\phi'',\\ &0 = \mathrm{d_h}I^{i0} = I^{i1}\omega^2 + \iota(\theta^{i0})\\ &\quad = I^{i1}\omega^2 + \hat\phi^{(i+1)} - \hat\phi^{(i+2)} - \sum_{i'=1}^{i-1}\binom{i}{i'}I^{i-i',1}\hat\phi^{(i'+1)}, \quad i\in\mathbb{N},\end{aligned}$$

where the forms $\hat c_1$, $\hat c_2$ and $\hat\phi^{(i)}$, $i\in\mathbb{N}_0$, are the invariantizations of the parameters c_1, c_2 and $\phi^{(i)}$ of the infinitely prolonged general element of the algebra $\mathfrak{g}_1$, respectively, $\hat c_1 = \iota(c_1)$, $\hat c_2 = \iota(c_2)$ and $\hat\phi^{(i)} = \iota(\phi^{(i)})$.

More rigorously, here the parameters c_1, c_2 and $\phi^{(i)}$, $i \in \mathbb{N}_0$, are interpreted as the coordinate functions on the infinite prolongation of $\mathfrak{g}_1$. Recall that under the prolongation we consider u and v to be the independent variables and f to be the dependent variable. In other words, these coefficients are first-order differential forms in the jet space $\mathrm{J}^\infty(u,v|\,f)$. Hence their invariantizations are also forms, which are called *invariantized Maurer–Cartan forms.*

The above system can be solved to yield the following invariantized Maurer–Cartan forms

$$
\begin{aligned}
&\hat{c}_1 = \left(\frac12 I^{12} - I^{11}\right)\omega^1 + \left(\frac12 I^{03} - 1\right)\omega^2,\\
&\hat{c}_2 = (I^{11} - I^{12})\omega^1 - I^{03}\omega^2,\\
&\hat{\phi} = -\omega^1, \quad \hat{\phi}' = \left(\frac12 I^{12} - I^{11}\right)\omega^1 + \left(\frac12 I^{03} - 2\right)\omega^2,\\
&\hat{\phi}'' = \frac12 I^{12}\omega^1 + \frac12 I^{03}\omega^2,\\
&\hat{\phi}^{(i+2)} = \hat{\phi}^{(i+1)} - \sum_{i'=1}^{i-1}\binom{i}{i'} I^{i-i',1}\hat{\phi}^{(i'+1)} + I^{i1}\omega^2, \quad i \in \mathbb{N}.
\end{aligned}
\tag{12}
$$

The explicit expression for the invariantized form $\hat{\phi}^{(i+2)}$, $i \in \mathbb{N}$, as a combination of ω^1 and ω^2 with coefficients being polynomials of normalized differential invariants is obtained by expanding the above expression when successively going over the values of i. In particular,

$$
\begin{aligned}
&\hat{\phi}''' = \frac12 I^{12}\omega^1 + \left(I^{11} + \frac12 I^{03}\right)\omega^2,\\
&\hat{\phi}^{(4)} = \left(\frac12 I^{12} - I^{11}I^{12}\right)\omega^1 + \left(I^{11} + \frac12 I^{03} + I^{21} - I^{11}I^{03}\right)\omega^2.
\end{aligned}
$$

For $i \geqslant 3$, the greatest value of $i'+j'$ for the normalized differential invariants $I^{i'j'}$ that are involved in $\hat{\phi}^{(i+2)}$ is $i+1$, and $I^{i1}\omega^2$ is the only summand with this value.

The non-phantom recurrence relations are

$$
\begin{aligned}
\mathrm{d_h} I^{11} &= I^{21}\omega^1 + I^{12}\omega^2 + \iota(\theta^{11})\\
&= \big(I^{21} + 2(I^{11})^2 - I^{11}I^{12} - I^{12}\big)\omega^1\\
&\quad + \big(I^{12} - I^{11}I^{03} + I^{11} - I^{03}\big)\omega^2,
\end{aligned}
$$

$$\mathrm{d_h} I^{ij} = I^{i+1,j}\omega^1 + I^{i,j+1}\omega^2 + \iota(\theta^{ij}), \quad i+j \geqslant 3,\ j \neq 0,$$

with

$$\begin{aligned}\iota(\theta^{ij}) = (j-2)I^{ij}\hat{c}_1 - (j-1)\sum_{i'=0}^{i}\binom{i}{i'} I^{i-i',j}\hat{\phi}^{(i'+1)} \\ - \sum_{i'=1}^{i}\binom{i}{i'}\Big(I^{i-i'+1,j}\hat{\phi}^{(i')} + I^{i-i',j+1}\hat{\phi}^{(i'+1)}\Big) \\ - \delta_{0i}(\delta_{0j}+\delta_{1j})\hat{c}_2 - (\delta_{0j} + 2\delta_{1j} + 2\delta_{2j})\hat{\phi}^{(i+2)}.\end{aligned}$$

The first non-phantom recurrence relation splits into

$$\begin{aligned}\mathrm{D}^{\mathrm{i}}_u I^{11} &= I^{21} + 2(I^{11})^2 - I^{11}I^{12} - I^{12}, \\ \mathrm{D}^{\mathrm{i}}_v I^{11} &= I^{12} - I^{11}I^{03} + I^{11} - I^{03}.\end{aligned}$$

Therefore, the normalized differential invariants I^{12} and I^{21} are expressed in terms of invariant derivatives of I^{11} and I^{03},

$$\begin{aligned}I^{12} &= \mathrm{D}^{\mathrm{i}}_v I^{11} + I^{11}I^{03} - I^{11} + I^{03}, \\ I^{21} &= \mathrm{D}^{\mathrm{i}}_u I^{11} - 2(I^{11})^2 \\ &\quad + (I^{11}+1)(\mathrm{D}^{\mathrm{i}}_v I^{11} + I^{11}I^{03} - I^{11} + I^{03}).\end{aligned} \tag{13}$$

In view of the above discussion on the invariantize forms $\hat{\phi}^{(i')}$, $i \in \mathbb{N}$, the expression for $\iota(\theta^{ij})$ with $i+j \geqslant 3$ and $j \neq 0$ implies that the greatest value of $i'+j'$ for $I^{i'j'}$ involved in $\iota(\theta^{ij})$ is $i+j$. Hence splitting the recurrence relation with $\mathrm{d_h} I^{ij}$ leads to expressions for $I^{i+1,j}$ and $I^{i,j+1}$ in terms of invariant derivatives of $I^{i'j'}$ with $i'+j' \leqslant i+j$. For example, from the non-phantom recurrence relation

$$\begin{aligned}\mathrm{d_h} I^{03} &= I^{13}\omega^1 + I^{04}\omega^2 + \iota(\theta^{03}) \\ &= \left(I^{13} + I^{11}I^{03} - \frac{I^{12}I^{03}}{2}\right)\omega^1 + \left(I^{04} + I^{03} - \frac{(I^{03})^2}{2}\right)\omega^2\end{aligned}$$

we derive

$$\begin{aligned}\mathrm{D}^{\mathrm{i}}_u I^{03} &= I^{13} + I^{11}I^{03} - \frac{1}{2}I^{12}I^{03}, \\ \mathrm{D}^{\mathrm{i}}_v I^{03} &= I^{04} - \frac{1}{2}(I^{03})^2 + I^{03}.\end{aligned}$$

This implies by induction, where the expressions (13) for I^{12} and I^{21} give the base case, that any non-phantom normalized differential invariant can be expressed in terms of invariant derivatives of I^{11} and I^{03}.

To find a minimum generating set of differential invariants for the projected group G_1, we should additionally check whether I^{03} can be expressed in terms of invariant derivatives of I^{11}. We use (11) to compute the commutator between the operators of invariant differentiation. This is done upon evaluating (11) for the basis horizontal forms du and dv,

$$\begin{aligned}
\mathrm{d_h}\iota(\mathrm{d}u) &= \iota(\phi'\mathrm{d}u) = \iota(\phi')\wedge\iota(\mathrm{d}u)\\
&= \left(2-\frac{1}{2}I^{03}\right)\omega^1\wedge\omega^2 = -Y^1_{12}\,\omega^1\wedge\omega^2,\\
\mathrm{d_h}\iota(\mathrm{d}v) &= \iota\big(\phi''v\mathrm{d}u + (\phi'-c_1)\mathrm{d}v\big)\\
&= \iota(\phi''v)\wedge\iota(\mathrm{d}u) = -\frac{1}{2}I^{03}\omega^1\wedge\omega^2 = -Y^2_{12}\,\omega^1\wedge\omega^2.
\end{aligned}$$

The commutation relation then evaluates as

$$[\mathrm{D}^{\mathrm{i}}_u,\mathrm{D}^{\mathrm{i}}_v] = Y^1_{12}\mathrm{D}^{\mathrm{i}}_u + Y^2_{12}\mathrm{D}^{\mathrm{i}}_v = \left(\frac{1}{2}I^{03}-2\right)\mathrm{D}^{\mathrm{i}}_u + \frac{1}{2}I^{03}\mathrm{D}^{\mathrm{i}}_v,$$

see [25] for details of the technique applied. Evaluating $[\mathrm{D}^{\mathrm{i}}_u,\mathrm{D}^{\mathrm{i}}_v]I^{11}$, we can derive the following expression for I^{03}:

$$I^{03} := \frac{2v^3f_{vvv}}{2f-2vf_v+v^2f_{vv}} = 2\frac{2\mathrm{D}^{\mathrm{i}}_uI^{11}+[\mathrm{D}^{\mathrm{i}}_u,\mathrm{D}^{\mathrm{i}}_v]I^{11}}{\mathrm{D}^{\mathrm{i}}_uI^{11}+\mathrm{D}^{\mathrm{i}}_vI^{11}}.$$

As a result, we have proved the following theorem.

Theorem 2. *The algebra of differential invariants of the group G_1, which is the projection of the equivalence group $G^\sim$ of the class of diffusion equations* (1) *to the space with coordinates (u,v,f), is generated by the single differential invariant*

$$I^{11} = -2v^2\frac{4f_u-2vf_{uv}+(2f-2vf_v+v^2f_{vv})f_{vv}}{(2f-2vf_v+v^2f_{vv})^2}$$

along with the two operators of invariant differentiation

$$\mathrm{D}^{\mathrm{i}}_u = \frac{2v^2}{2f-2vf_v+v^2f_{vv}}\left(\mathrm{D}_u-\frac{1}{2}vf_{vv}\mathrm{D}_v\right),\quad \mathrm{D}^{\mathrm{i}}_v = v\mathrm{D}_v.$$

All other differential invariants are functions of I^{11} and invariant derivatives thereof.

Corollary 1. *A functional basis of differential invariants of order not greater than $k \in \mathbb{N}_0$ in terms of invariant derivatives of non-phantom normalized differential invariants is exhausted by*

$$\left(\mathrm{D}_u^{\mathrm{i}}\right)^i \left(\mathrm{D}_v^{\mathrm{i}}\right)^j I^{11},\ i+j \leqslant k-2, \quad \left(\mathrm{D}_v^{\mathrm{i}}\right)^{j'} I^{03},\ j' \leqslant k-3.$$

Acknowledgements. This research was undertaken, in part, thanks to funding from the Canada Research Chairs program, the NSERC Discovery Grant program and the InnovateNL LeverageR&D program. AB is a recipient of an APART Fellowship of the Austrian Academy of Sciences. The research of ROP and EMDSCB was supported by the Austrian Science Fund (FWF), projects P25064 and P29177.

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Збірник праць Інституту математики НАН України 2019, т. 16, № 1, 66–77

УДК 517.912:512.816

Розв'язки системи пов'язаних рівнянь ейконалу

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Наведено короткий огляд методу отримання загального розв'язку системи пов'язаних рівнянь ейконалу, що базується на використанні перетворень годографа та контактних перетворень. Використана процедура дозволила також знайти загальний розв'язок системи рівнянь ейконалу та Гамільтона–Якобі.

We review the approach to obtaining the general solution for a coupled system of eikonal equations that based on using hodograph and contact transformations of the initial system. The procedure used allowed also finding of the general solution for a coupled system of the eikonal and Hamilton–Jacobi equation.

1. Вступ. Ми розглядаємо перевизначену систему, яка складається з двох рівнянь ейконалу для двох функцій від $(1+n)$ незалежних змінних, та ще одного рівняння, що пов'язує ці дві функції:

$$u_\mu u_\mu = 0, \quad v_\mu v_\mu = 0, \quad u_\mu v_\mu = 1, \tag{1}$$

де $u = u(x_0, x_1, \ldots, x_n)$, $v = v(x_0, x_1, \ldots, x_n)$.

Якщо не зазначено інше, індекси у незалежних змінних x_μ можуть приймати значення від 0 до n, $\mu = 0, 1, \ldots, n$; нижні індекси у залежних змінних означають похідні за відповідними змінними x_μ, і пара індексів, що повторюються, означає підсумовування за цими індексами від 0 до n в просторі Мінковського:

$$x_\mu x_\mu = x_0 x_0 - x_1 x_1 - \cdots - x_n x_n.$$

Ми також будемо вважати, що всі функції, які ми розглядаємо, є достатньо гладкими для існування та неперервності всіх потрібних похідних, та що всі залежні та незалежні змінні приймають значення у просторі дійсних чисел.

Відзначимо, що система (1) є спеціальним випадком більш загальної системи пов'язаних рівнянь ейконалу

$$u_\mu u_\mu = 0, \quad v_\mu v_\mu = 0, \quad u_\mu v_\mu = h(u, v) \tag{2}$$

з довільною функцією $h(u, v)$.

Система (2) може бути отримана в результаті локальних перетворень системи

$$u_\mu u_\mu = \rho(u, v), \quad v_\mu v_\mu = \sigma(u, v), \quad u_\mu v_\mu = \tau(u, v) \tag{3}$$

з довільними функціями ρ, σ та τ, де $\rho\sigma - \tau^2 < 0$. Така система виникла в нашій роботі [5] як частина умов редукції багатовимірного нелінійного хвильового рівняння $\Box\phi = F(\phi)$ з застосуванням анзацу з двома новими незалежними змінними $\phi = \phi(\omega_1, \omega_2)$.

Загальний вигляд системи типу (3), яка може бути редукована до системи вигляду (1), є наступним

$$\begin{aligned}
u_\mu u_\mu &= 2A_a(a, b)A_b(a, b),\\
v_\mu v_\mu &= 2B_a(a, b)B_b(a, b),\\
u_\mu v_\mu &= A_a(a, b)B_b(a, b) + B_a(a, b)A_b(a, b),
\end{aligned}$$

де $a = a(u, v)$, $b = b(u, v)$ — це довільні достатньо гладкі функції.

Прикладом системи такого вигляду є

$$u_\mu u_\mu = 1, \quad v_\mu v_\mu = -1, \quad u_\mu v_\mu = 0.$$

Проте, системи пов'язаних рівнянь ейконалу є цікавими і з точки зору безпосереднього практичного застосування у геометричній оптиці, розпізнаванні образів, механіці суцільного середовища та інших галузях.

Найменша розмірність, коли для системи (1) можна отримати нетривіальні розв'язки — це $n = 2$, тобто це буде система з однією часовою та двома просторовими змінними. У випадку однієї просторової змінної ми матимемо лише тривіальний лінійний розв'язок $u = a(x_0 \pm x_1) + c_1$, $v = 1/2a(x_0 \mp x_1) + c_2$, де $a = \mathrm{const} \neq 0$, c_1 та c_2 — довільні дійсні сталі.

У роботі [7] нами був знайдений параметричний загальний розв'язок для системи (1) та двох просторових змінних (ми виключали спеціальні випадки в процесі знаходження)

$$u = \frac{x_1 + \frac{x_2 z}{\sqrt{1-z^2}} - k'(z)}{g'(z)},$$

$$v = \frac{gx_2}{\sqrt{1-z^2}} + \frac{p(z)}{g'(z)}\left[x_1 + \frac{x_2 z}{\sqrt{1-z^2}} - k'(z)\right] + r(z),$$
$$0 = x_0 - x_1 z + x_2\sqrt{1-z^2} + \frac{g(z)}{g'(z)}\left(x_1 + \frac{x_2 z}{\sqrt{1-z^2}} - k'(z)\right) - k(z).$$

Тут

$$r' = -k''\bigl(zg + (1-z^2)g'\bigr), \quad p = \tfrac{1}{2}\bigl(-g'^2 + (g - zg')^2\bigr).$$

Метод, який ми використовуємо, був розроблений на основі ідей, представлених в роботах Р.З. Жданова, І.В. Ревенка та В.І. Фущича [3, 4] щодо загального розв'язку системи д'Аламбера–Гамільтона.

2. Застосування перетворень годографа та контактних перетворень. У цьому параграфі розглянемо лише частковий випадок $n = 2$, і функції

$$u = u(x_0, x_1, x_2), \quad v = v(x_0, x_1, x_2) \tag{4}$$

та будемо вважати, що $u_{x_0} \neq 0$ (в іншому випадку перше рівняння системи (1) матиме лише сталі розв'язки).

Ми переходимо від початкової пари (4) до нової пари залежних змінних w та v, та нових незалежних змінних y_0, y_1, y_2:

$$u = y_0, \quad x_0 = w, \quad x_1 = y_1, \quad x_2 = y_2. \tag{5}$$

Вирази для похідних початкової пари функцій:

$$u_{x_0} = \frac{1}{w_{y_0}}, \quad u_{x_1} = -\frac{w_{y_1}}{w_{y_0}}, \quad u_{x_2} = -\frac{w_{y_2}}{w_{y_0}},$$
$$v_{x_0} = \frac{v_{y_0}}{w_{y_0}}, \quad v_{x_1} = v_{y_1} - v_{y_0}\frac{w_{y_1}}{w_{y_0}}, \quad v_{x_2} = v_{y_2} - v_{y_0}\frac{w_{y_2}}{w_{y_0}}. \tag{6}$$

Зазначимо, що у нових рівняннях, отриманих після застосування перетворення годографа, ми будемо позначати похідні за змінними y_μ як v_{y_μ} та w_{y_μ}.

Підстановка формул для похідних (6) до першого рівняння системи (1) дає наступні вирази:

$$-\frac{w_{y_1}^2}{w_{y_0}^2} - \frac{w_{y_2}^2}{w_{y_0}^2} + \frac{1}{w_{y_0}^2} = 0.$$

Ми використовуємо припущення $w_{y_0} \neq 0$, і тому знайдене рівняння є еквівалентним наступному:

$$w_{y_1}^2 + w_{y_2}^2 = 1.$$

Підстановка формул для похідних до другого рівняння системи (1) дає

$$v_{y_1}^2 + v_{y_0}^2 \frac{w_{y_1}^2}{w_{y_0}^2} - 2\frac{v_{y_0} v_{y_1} w_{y_1}}{w_{y_0}} + v_{y_2}^2 + v_{y_0}^2 \frac{w_{y_2}^2}{w_{y_0}^2} - 2\frac{v_{y_0} v_{y_2} w_{y_2}}{w_{y_0}} = \frac{v_{y_0}^2}{w_{y_0}^2},$$

і в результаті отримуємо

$$v_{y_1}^2 + v_{y_2}^2 - 2(v_{y_1} w_{y_1} + v_{y_2} w_2)\frac{v_{y_0}}{w_{y_0}} = 0. \tag{7}$$

Підстановка до третього рівняння (1) дає

$$\frac{v_{y_0}}{w_{y_0}^2} + \frac{w_{y_1}}{w_{y_0}}\left(v_{y_1} + v_{y_0}\frac{w_{y_1}}{w_{y_0}}\right) + \frac{w_{y_2}}{w_{y_0}}\left(v_{y_2} + v_{y_0}\frac{w_{y_2}}{w_{y_0}}\right) = 1.$$

Враховуючи, що $w_{y_0} \neq 0$, ми приходимо до виразу

$$v_{y_1} w_{y_1} + v_{y_2} w_{y_2} = w_{y_0}.$$

Рівняння (7) перетворюється на наступне:

$$v_{y_1}^2 + v_{y_2}^2 = 2v_{y_0}.$$

У результаті перетворень отримуємо таку систему рівнянь:

$$\begin{aligned} &w_{y_1}^2 + w_{y_2}^2 = 1, \\ &v_{y_1}^2 + v_{y_2}^2 = 2v_{y_0}, \\ &v_{y_1} w_{y_1} + v_{y_2} w_{y_2} = w_{y_0}. \end{aligned} \tag{8}$$

Зазначимо, що система (8) включає рівняння ейконалу та рівняння Гамільтона–Якобі, подібні до умов редукції для рівняння Шрьодінгера, які розглядались в роботі [6].

Для застосування контактних перетворень, розглядаємо наступний набір нових незалежних змінних $z_0 = y_0$, $z_1 = w_{y_1}$, $z_2 = y_2$.

Розглянемо нові залежні змінні

$$H(z_0, z_1, z_2) = y_1 w_{y_1} - w, \quad v = v(z_0, z_1, z_2) \tag{9}$$

та випишемо співвідношення для похідних за новими незалежними змінними:

$$
\begin{aligned}
&H_{z_0} = -w_{y_0}, \quad H_{z_1} = y_1, \quad H_{z_2} = -w_{y_2},\\
&v_{y_0} = v_{z_0} + v_{z_1} w_{y_0 y_1}, \quad v_{y_1} = v_{z_1} + w_{y_1 y_1}, \quad v_{y_2} = v_{z_2} + v_{z_1} w_{y_1 y_2},\\
&w_{y_1 y_1} = \frac{1}{H_{z_1 z_1}}, \quad w_{y_1 y_2} = -\frac{H_{z_1 z_2}}{H_{z_1 z_1}}, \quad w_{y_0 y_1} = -\frac{H_{z_0 z_1}}{H_{z_1 z_1}},\\
&w_{y_0 y_2} = -\frac{\begin{vmatrix} H_{z_1 z_1} & H_{z_1 z_2} \\ H_{z_0 z_1} & H_{z_0 z_2} \end{vmatrix}}{H_{z_1 z_1}}.
\end{aligned}
\tag{10}
$$

Після відповідної підстановки у систему (8) приходимо до наступної системи рівнянь:

$$
z_1^2 + H_{z_2}^2 = 1, \tag{11}
$$

$$
\left(\frac{v_{z_1}}{H_{z_1 z_1}}\right)^2 + \left(v_{z_2} - v_{z_1}\frac{H_{z_1 z_2}}{H_{z_1 z_1}}\right)^2 = 2\left(v_{z_0} - v_{z_1}\frac{H_{z_1 z_2}}{H_{z_1 z_1}}\right), \tag{12}
$$

$$
v_{z_1}\frac{z_1}{H_{z_1 z_1}} - H_{z_2}\left(v_{z_2} - v_{z_1}\frac{H_{z_1 z_2}}{H_{z_1 z_1}}\right) = -H_{z_0}. \tag{13}
$$

Перше рівняння (11) цієї системи має загальний розв'язок для функції H:

$$
H = z_2\sqrt{1 - z_1^2} + G(z_0, z_1), \tag{14}
$$

де G — функція своїх аргументів, явний вигляд якої ми знайдемо нижче з інших рівнянь цієї системи.

Випадок

$$
z_1 = w_{y_1} = -\frac{u_{x_1}}{u_{x_0}} = \pm 1
$$

— це спеціальний випадок, який дає тривіальний розв'язок, і ми будемо його розглядати окремо.

З виразу для функції H (14) ми отримуємо, що

$$
\begin{aligned}
&H_{z_0} = G_{z_0}, \quad H_{z_1} = -\frac{z_1 z_2}{\sqrt{1 - z_1^2}}, \quad H_{z_2} = \sqrt{1 - z_1^2},\\
&H_{z_0 z_1} = G_{z_0 z_1}, \quad H_{z_1 z_2} = -\frac{z_1}{\sqrt{1 - z_1^2}},
\end{aligned}
\tag{15}
$$

$$H_{z_1 z_1} = -\frac{z_2}{\sqrt{1-z_1^2}} - \frac{z_1^2 z_2}{\left(1-z_1^2\right)^{\frac{3}{2}}} + G_{z_1 z_1} = -\frac{z_2}{\left(1-z_1^2\right)^{\frac{3}{2}}} + G_{z_1 z_1}.$$

Далі, підстановка виразу для контактних перетворень (15) та виразу для похідних функції H у рівняння (12) дає

$$v_{z_1} z_1 + \left(G_{z_1 z_1} - \frac{z_2}{\left(1-z_1^2\right)^{\frac{3}{2}}}\right)\left(G_{z_0} - v_{z_2}\sqrt{1-z_1^2}\right)$$
$$+ v_{z_1}\sqrt{1-z_1^2})\left(-\frac{z_1}{\sqrt{1-z_1^2}}\right)$$
$$= \left(G_{z_1 z_1} - \frac{z_2}{\left(1-z_1^2\right)^{\frac{3}{2}}}\right)\left(G_{z_0} - v_{z_2}\sqrt{1-z_1^2}\right) = 0.$$

Враховуючи, що

$$G_{z_1 z_1} - \frac{z_2}{\left(1-z_1^2\right)^{\frac{3}{2}}} \neq 0,$$

отримуємо

$$G_{z_0} - v_{z_2}\sqrt{1-z_1^2} = 0,$$

що дає нам вираз для функції v:

$$v = \frac{G_{z_0} z_2}{\sqrt{1-z_1^2}} + P(z_0, z_1), \tag{16}$$

де $P(z_0, z_1)$ — це деяка функція від своїх аргументів, яка має бути знайдена нижче.

З (16) ми обчислюємо вирази для похідних функції v:

$$v_{z_0} = \frac{G_{z_0 z_0} z_2}{\sqrt{1-z_1^2}} + P_{z_0}, \quad v_{z_1} = \frac{G_{z_0 z_1} z_2}{\sqrt{1-z_1^2}} + \frac{G_{z_0} z_1 z_2}{\left(1-z_1^2\right)^{\frac{3}{2}}} + P_{z_1},$$
$$v_{z_2} = \frac{G_{z_0}}{\sqrt{1-z_1^2}}. \tag{17}$$

Підстановка (15), (16), (17) у (12) дає

$$v_{z_1}^2 + (v_{z_2} H_{z_1 z_1} - v_{z_1} H_{z_1 z_2})^2 = 2H_{z_1 z_1}(v_{z_0} H_{z_1 z_1} - v_{z_1} H_{z_0 z_1}),$$

$$\left(\frac{G_{z_0z_1}z_2}{\sqrt{1-z_1^2}}+\frac{G_{z_0}z_1z_2}{\left(1-z_1^2\right)^{\frac{3}{2}}}P_{z_1}\right)^2+\left(\frac{G_{z_0}}{\sqrt{1-z_1^2}}\left(G_{z_1z_1}-\frac{z_2}{\left(1-z_1^2\right)^{\frac{3}{2}}}\right)\right.$$
$$\left.+\frac{z_1}{\sqrt{1-z_1^2}}\left(P_{z_1}+\frac{G_{z_0z_1}z_2}{\sqrt{1-z_1^2}}+\frac{G_{z_0}z_1z_2}{\left(1-z_1^2\right)^{\frac{3}{2}}}\right)\right)^2$$
$$=2\left(G_{z_1z_1}-\frac{z_2}{\left(1-z_1^2\right)^{\frac{3}{2}}}\right)$$
$$\times\left(\left(G_{z_1z_1}-\frac{z_2}{\left(1-z_1^2\right)^{\frac{3}{2}}}\right)\left(\frac{G_{z_0z_0}z_2}{\sqrt{1-z_1^2}}+P_{z_0}\right)\right.$$
$$\left.-G_{z_0z_1}\left(\frac{G_{z_0z_1}z_2}{\sqrt{1-z_1^2}}+\frac{G_{z_0}z_1z_2}{\left(1-z_1^2\right)^{\frac{3}{2}}}+P_{z_1}\right)\right). \qquad (18)$$

Далі ми можемо розкласти ці вирази за степенями змінної z_2. З вимоги рівності нулю суми коефіцієнтів при z_2^3 ми отримуємо умову

$$-2\frac{G_{z_0z_0}}{\left(1-z_1^2\right)^{\frac{7}{2}}}=0, \qquad (19)$$

звідки можна зробити висновок, що $G_{z_0z_0}=0$. З вимоги рівності нулю суми коефіцієнтів при z_2^2 ми отримуємо умови

$$\left(\frac{G_{z_0z_1}}{\sqrt{1-z_1^2}}+\frac{G_{z_0}z_1}{\left(1-z_1^2\right)^{\frac{3}{2}}}\right)^2$$
$$+\left(-\frac{G_{z_0}}{\left(1-z_1^2\right)^2}+\frac{G_{z_0z_1}z_1}{1-z_1^2}+\frac{G_{z_0}z_1^2}{\left(1-z_1^2\right)^2}\right)^2$$
$$=-\frac{2}{\left(1-z_1^2\right)^{\frac{3}{2}}}\left(-\frac{P_{z_0}}{\left(1-z_1^2\right)^{\frac{3}{2}}}-G_{z_0z_1}\left(\frac{G_{z_0z_1}}{\sqrt{1-z_1^2}}+\frac{G_{z_0}z_1^2}{\left(1-z_1^2\right)^{\frac{3}{2}}}\right)\right),$$

$$G_{z_0z_1}^2\left(1-z_1^2\right)^2+2G_{z_0z_1}G_{z_0}z_1\left(1-z_1^2\right)+G_{z_0}^2z_1^2$$
$$+\left(1-z_1^2\right)\left(z_1^2G_{z_0z_1}^2-2z_1G_{z_0z_1}G_{z_0}+G_{z_0}^2\right)$$
$$=2\left(P_{z_0}+G_{z_0z_1}^2\left(1-z_1^2\right)+G_{z_0z_1}G_{z_0}z_1\right),$$

$$(G_{z_0} - z_1 G_{z_0 z_1})^2 = 2P_{z_0} + G_{z_0 z_1}^2. \tag{20}$$

З вимоги рівності нулю суми коефіцієнтів при z_2 ми отримуємо умови

$$\begin{aligned}
&P_{z_1}\left(\frac{G_{z_0 z_1}}{\sqrt{1-z_1^2}} + \frac{G_{z_0} z_1}{\left(1-z_1^2\right)^{\frac{3}{2}}}\right) + \left(\frac{G_{z_0} G_{z_1 z_1} + z_1 P_{z_1}}{\sqrt{1-z_1^2}}\right)\\
&\qquad\times\left(-\frac{G_{z_0}}{\left(1-z_1^2\right)^2} + \frac{G_{z_0 z_1} z_1}{1-z_1^2} + \frac{G_{z_0} z_1^2}{\left(1-z_1^2\right)^2}\right)\\
&\quad= G_{z_1 z_1}\left(-\frac{P_{z_0} + G_{z_0 z_1} G_{z_0} z_1}{\left(1-z_1^2\right)^{\frac{3}{2}}} - \frac{G_{z_0 z_1}^2}{\sqrt{1-z_1^2}}\right)\\
&\qquad- \frac{P_{z_0} G_{z_1 z_1} - G_{z_0 z_1} P_{z_1}}{\left(1-z_1^2\right)^{\frac{3}{2}}},\\
&P_{z_1}\left(G_{z_0 z_1}\left(1-z_1^2\right) + G_{z_0} z_1\right) + (G_{z_0} G_{z_1 z_1} + z_1 P_{z_1})(G_{z_0 z_1} z_1 - G_{z_0})\\
&\qquad+ G_{z_1 z_1} G_{z_0 z_1}^2\left(1-z_1^2\right) + G_{z_1 z_1}(P_{z_0} + G_{z_0 z_1} G_{z_0} z_1) + P_{z_0} G_{z_1 z_1}\\
&\qquad- G_{z_0 z_1} P_{z_1} = 0,\\
&G_{z_1 z_1}(G_{z_0}(G_{z_0 z_1} z_1 - G_{z_0}) + G_{z_0 z_1}^2\left(1-z_1^2\right)\\
&\qquad+ 2P_{z_0} + G_{z_0 z_1} G_{z_0} z_1) = 0.
\end{aligned} \tag{21}$$

З умови (21) знаходимо, що

$$G_{z_1 z_1}\left[2P_{z_0} + G_{z_0 z_1}^2 - (G_{z_0} - G_{z_0 z_1} z_1)^2\right] = 0. \tag{22}$$

Рівність нулю виразу у квадратних дужках рівняння (22) еквівалентна умові, яка була визначена в результаті збирання коефіцієнтів при z_2^2 (20). Таким чином, з коефіцієнтів при z_2 ми не отримали ніяких нових умов.

З вимоги рівності нулю суми коефіцієнтів при z_2^0 отримуємо

$$\begin{aligned}
&P_{z_1}^2 + \frac{(G_{z_0} G_{z_1 z_1} + z_1 P_{z_1})^2}{1-z_1^2} = 2G_{z_1 z_1}(P_{z_0} G_{z_1 z_1} - G_{z_0 z_1} P_{z_1}),\\
&P_{z_1}^2\left(1-z_1^2\right) + \left(G_{z_0}^2 G_{z_1 z_1}^2 + 2z_1 P_{z_1} G_{z_0} G_{z_1 z_1} + z_1^2 P_{z_1}^2\right)\\
&\quad= 2G_{z_1 z_1}(P_{z_0} G_{z_1 z_1} - G_{z_0 z_1} P_{z_1})\\
&\qquad- 2z_1^2\left(P_{z_0} G_{z_1 z_1}^2 - G_{z_0 z_1} G_{z_1 z_1} P_{z_1}\right),\\
&P_{z_1}^2 + G_{z_0}^2 G_{z_1 z_1}^2 + 2z_1 P_{z_1} G_{z_0} G_{z_1 z_1}
\end{aligned} \tag{23}$$

$$= 2\bigl(1 - z_1^2\bigr)G_{z_1z_1}(P_{z_0}G_{z_1z_1} - P_{z_1}G_{z_0z_1}). \tag{24}$$

З (18) випливає, що $G_{z_0z_0} = 0$, з (20) робимо висновок, що $P_{z_0z_0} = 0$.

Таким чином, шукані функції G та P повинні мати наступну форму

$$G = g(z_1)z_0 + k(z_1), \quad P = p(z_1)z_0 + r(z_1), \tag{25}$$

де g, k, p, r — певні функції від змінної z_1, умови на які будуть знайдені далі.

Після підстановки (25) у (20) ми отримали

$$2p + g'^2 = (g - z_1g')^2. \tag{26}$$

Після підстановки (25) у (24) ми приходимо до рівняння

$$\begin{aligned}(p'z_0 + r')^2 &+ g^2(g''z_0 + k'')^2 + 2z_1(p'z_0 + r')g(g''z_0 + k'') \\ &= 2\bigl(1 - z_1^2\bigr)(g''z_0 + k'')(p(g''z_0 + k'') - g'(p'z_0 + r')).\end{aligned} \tag{27}$$

Далі ми групуємо коефіцієнти біля степенів z_0. При z_0^2 отримуємо

$$p'^2 + g^2g''^2 + 2z_1p'gg'' = 2\bigl(1 - z_1^2\bigr)\bigl(g''^2p - g''g'p'\bigr). \tag{28}$$

З рівняння (26) ми можемо знайти вираз для функції p через функцію g:

$$p = \tfrac{1}{2}\bigl(g^2 - 2z_1gg' + \bigl(z_1^2 - 1\bigr)g'^2\bigr). \tag{29}$$

Звідси

$$p' = g''\bigl(\bigl(z_1^2 - 1\bigr)g' - z_1g\bigr). \tag{30}$$

Після підстановки виразів для p та p' до (28) ми отримуємо, що

$$\begin{aligned}g''^2\Bigl[\bigl(\bigl(z_1^2 - 1\bigr)g' - z_1g\bigr)^2 &+ g^2g''^2 + 2z_1g\bigl(\bigl(z_1^2 - 1\bigr)g' - z_1g\bigr) \\ &- \bigl(1 - z_1^2\bigr)\bigl(\bigl(g^2 - 2z_1gg' + \bigl(z_1^2 - 1\bigr)g'^2\bigr) \\ &- g'\bigl(\bigl(z_1^2 - 1\bigr)g' - z_1g\bigr)\bigr)\Bigr] = 0.\end{aligned} \tag{31}$$

У квадратних дужках рівняння (31) ми маємо тотожний нуль, тому нові умови на функції G та P можна знайти лише з коефіцієнтів при z_0^2.

Групування коефіцієнтів при z_0 дає наступну умову:

$$2p'r' + 2g^2g''k'' + 2z_1g(p'k'' + r'g'') \\ = 2\big(1 - z_1^2\big)(g''(pk'' - r'g') + k''(pg'' - p'g')).$$

Підстановка виразів (29) та (30) для p та p' призводить до виразів

$$2g''\big[(r' + z_1gk'')\big(\big(z_1^2 - 1\big)g' - z_1g\big) + \big(r'z_1g + k''g^2\big)\big] \\ = 2\big(1 - z_1^2\big)g''\big[k''\big(g^2 - 2z_1gg' + \big(z_1^2 - 1\big)g'^2\big) - r'g' \\ - k''g'\big(\big(z_1^2 - 1\big)g' - z_1g\big)\big],$$

які дають тотожну рівність, тобто ми знову не отримуємо нових умов порівняно зі знайденими раніше.

З вимоги рівності нулю суми коефіцієнтів при z_0^0 ми маємо, що

$$r'^2 + g^2k''^2 + 2z_1r'gk'' - 2\big(1 - z_1^2\big)k''(pk'' - g'r') \\ = \Big(r' - k''\big(\big(z_1^2 - 1\big)g' - z_1g\big)\Big)^2 = 0, \tag{32}$$

та з (32) випливає, що

$$r' = k''\big(\big(z_1^2 - 1\big)g' - z_1g\big),$$

і тоді, якщо $g'' \neq 0$, це буде еквівалентне умові

$$r'g'' - p'k'' = 0.$$

Таким чином, ми знайшли явний вигляд функцій G та P:

$$G = g(z_1)z_0 + k(z_1), \quad P = p(z_1)z_0 + r(z_1), \tag{33}$$

де

$$p = \tfrac{1}{2}\big(g^2 - 2z_1gg' + \big(z_1^2 - 1\big)g'^2\big), \\ r' = k''\big(\big(z_1^2 - 1\big)g' - z_1g\big). \tag{34}$$

3. Застосування обернених контактних перетворень та перетворень годографа. Функція H має вигляд

$$H = z_2\sqrt{1 - z_1^2} + G(z_0, z_1),$$

де $G(z_0, z_1)$ має вигляд (33) з довільними g та k, та

$$v = \frac{G_{z_0} z_2}{\sqrt{1-z_1^2}} + P(z_0, z_1),$$

де $P(z_0, z_1)$ має вигляд (33), де функції p та r знаходяться у відповідності до (34).

Функція w може бути визначена з виразу для H шляхом використання перетворень, обернених до (5) та (6):

$$w = z_1 H_{z_1} - H.$$

Далі ми можемо знову перепозначити z_1 як z, таким чином, ми отримуємо параметричний загальний розв'язок для системи (8), $n = 2$:

$$\begin{aligned} v &= \frac{G_{z_0} z_2}{\sqrt{1-z^2}} + p(z) z_0 + r(z), \\ w &= y_1 z - y_2 \sqrt{1-z^2} - g(z) y_0 - k(z), \\ 0 &= y_1 + \frac{y_2 z}{\sqrt{1-z^2}} - g'(z) y_0 - k'(z). \end{aligned}$$

Застосування перетворень, обернених до (5) та (6), дає можливість знайти параметричний загальний розв'язок для системи пов'язаних рівнянь ейконалу (1) для початкових функцій u та v та $n = 2$:

$$\begin{aligned} u &= \frac{x_1 + \frac{x_2 z}{\sqrt{1-z^2}} - k'(z)}{g'(z)}, \\ v &= \frac{g x_2}{\sqrt{1-z^2}} + \frac{p(z)}{g'(z)} \left[x_1 + \frac{x_2 z}{\sqrt{1-z^2}} - k'(z) \right] + r(z), \\ 0 &= x_0 - x_1 z + x_2 \sqrt{1-z^2} \\ &\quad + \frac{g(z)}{g'(z)} \left\{ x_1 + \frac{x_2 z}{\sqrt{1-z^2}} - k'(z) \right\} - k(z), \end{aligned}$$

де

$$r' = -k''\big(zg + \big(1-z^2\big)g'\big), \quad p = \tfrac{1}{2}\big(-g'^2 + (g - zg')^2\big),$$

де g та k — довільні функції.

4. Особливі випадки. Ми розглядали знаходження загального параметричного розв'язку системи (1) для $n = 2$ з припущеннями, що $u_0 \neq 0$, $v_0 \neq 0$, $w_{y_0} \neq 0$. Остання умова у відповідності до визначення змінних (5) буде виконуватися завжди і не становитиме спеціального випадку для початкової системи.

У випадку, якщо $u_0 = 0$, перше рівняння системи (1) матиме вигляд $-u_1^2 - u_2^2 = 0$, звідки $u_1 = u_2 = 0$, тобто тоді функція u буде сталою, а відповідний розв'язок системи (1) буде тривіальним.

Ще один особливий випадок

$$z_1 = w_{y_1} = -\frac{u_{x_1}}{u_{x_0}} = \pm 1,$$

тобто $u_{x_1} \pm 1 u_{x_0} = 0$, $u = u(x_0 \pm x_1, x_2)$, і початкова система зводиться до системи з двома незалежними змінними та тривіальними розв'язками.

5. Висновки. У цій статті ми навели огляд процедури, яка дозволяє побудувати загальні розв'язки системи (1) для загального та особливих випадків. Ці результати дозволять, зокрема, описати всі анзаци, які редукують багатовимірне рівняння ейконалу до рівнянь з меншим числом просторових змінних, що дасть можливість узагальнити результати, отримані в роботах [1] та [2].

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Збірник праць Інституту математики НАН України 2019, т. 16, № 1, 78–83

УДК 517.9

Точно-розв'язні моделі гідродинамічної стійкості

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Метод відокремлення змінних застосовано до задачі визначення точно-розв'язних моделей гідродинамічної стійкості. З математичної точки зору проблема визначення стійкості даної течії представляє собою розв'язування системи рівнянь, що отримані з рівнянь Нав'є–Стокса лінеарізацією за основними течіями та знаходження множини всіх її можливих розв'язків, які дозволяють розщеплення збурень на нормальні моди. Повністю розглянуто випадок циліндричних координат.

The method of separation of variables is applied to the problem of determining exactly solvable models of hydrodynamic stability. From a mathematical point of view, the problem of determining the stability of a flow is the solving of a system of equations derived from Navier–Stokes equations by linearization along the main flows and finding a set of all possible solutions that allow splitting of perturbations into normal modes. The case of cylindrical coordinates is completely considered.

Класична теорія лінійної стійкості в'язких нестисливих потоків пов'язана з розвитком у просторі та часі нескінченно малих збурень навколо заданого основного потоку [1, 2, 3]. Сформулюємо задачу гідродинамічної стійкості, базуючись на рівнянні Нав'є–Стокса в циліндричних координатах (r, φ, z). Як це звичайно роблять у теорії стійкості, розщепимо поля швидкості і тиску $(\hat{v}_r, \hat{v}_\varphi, \hat{v}_z, \hat{p})$ на 2 складові: основної течії (V_r, V_φ, V_z, P) і збуреної (v_r, v_φ, v_z, p),

$$\hat{v}_r = V_r + v_r, \quad \hat{v}_\varphi = V_\varphi + v_\varphi, \quad \hat{v}_z = V_z + v_z, \quad \hat{p} = P + p. \tag{1}$$

Підставляючи (1) в рівняння Нав'є–Стокса, записане в термінах змінних $(\hat{v}_r, \hat{v}_\varphi, \hat{v}_z, \hat{p})$, і ігноруючи всі доданки, що містять квадрат збуреної амплітуди, а також накладаючи умову, щоб змінні основної течії (V_r, V_φ, V_z, P) самі задовольняли рівняння Нав'є–Стокса, ми отримуємо наступну систему лінеаризованих рівнянь гідродинамічної стійкості в циліндричних координатах:

$$
\begin{aligned}
&\frac{\partial v_r}{\partial t} + V_r\frac{\partial v_r}{\partial r} + v_r\frac{\partial V_r}{\partial r} + \frac{V_\varphi}{r}\frac{\partial v_r}{\partial \varphi} + \frac{v_\varphi}{r}\frac{\partial V_r}{\partial \varphi}\\
&\quad + V_z\frac{\partial v_r}{\partial z} + v_z\frac{\partial V_r}{\partial z} - 2\frac{V_\varphi v_\varphi}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}\\
&\quad + \nu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \varphi^2} + \frac{\partial^2 v_r}{\partial z^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{2}{r^2}\frac{\partial v_\varphi}{\partial \varphi} - \frac{v_r}{r^2}\right),\\
&\frac{\partial v_\varphi}{\partial t} + V_r\frac{\partial v_\varphi}{\partial r} + v_r\frac{\partial V_\varphi}{\partial r} + \frac{V_\varphi}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{v_\varphi}{r}\frac{\partial V_\varphi}{\partial \varphi}\\
&\quad + V_z\frac{\partial v_\varphi}{\partial z} + v_z\frac{\partial V_\varphi}{\partial z} + \frac{V_r v_\varphi}{r} + \frac{v_r V_\varphi}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \varphi}\\
&\quad + \nu\left(\frac{\partial^2 v_\varphi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_\varphi}{\partial \varphi^2} + \frac{\partial^2 v_\varphi}{\partial z^2} + \frac{1}{r}\frac{\partial v_\varphi}{\partial r} + \frac{2}{r^2}\frac{\partial v_r}{\partial \varphi} - \frac{v_\varphi}{r^2}\right),\\
&\frac{\partial v_z}{\partial t} + V_r\frac{\partial v_z}{\partial r} + v_r\frac{\partial V_z}{\partial r} + \frac{V_\varphi}{r}\frac{\partial v_z}{\partial \varphi} + \frac{v_\varphi}{r}\frac{\partial V_z}{\partial \varphi} + V_z\frac{\partial v_z}{\partial z} + v_z\frac{\partial V_z}{\partial z}\\
&\quad = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \varphi^2} + \frac{\partial^2 v_z}{\partial z^2} + \frac{1}{r}\frac{\partial v_z}{\partial r}\right),\\
&\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{\partial v_z}{\partial z} + \frac{v_r}{r} = 0. \qquad (2)
\end{aligned}
$$

Введемо нову систему координат t, $\xi = \xi(t, r)$, $\gamma = \gamma(t, \varphi)$, $\eta = \eta(t, z)$.

Казатимемо, що система (2) допускає відокремлення змінних в нестаціонарній циліндричній системі координат ξ, γ, η, якщо анзац

$$
\begin{aligned}
&v_r = T(t)\exp(a\eta + m\gamma + sS(t))f(\xi),\\
&v_\varphi = T(t)\exp(a\eta + m\gamma + sS(t))g(\xi),\\
&v_z = T(t)\exp(a\eta + m\gamma + sS(t))h(\xi),\\
&p = T_1(t)\exp(a\eta + m\gamma + sS(t))\pi(\xi) \qquad (3)
\end{aligned}
$$

зводить систему рівнянь з частинними похідними (2) до системи 3-х звичайних диференціальних рівнянь другого порядку й одного звичайного диференціального рівняння першого порядку для 4-х функцій $f(\xi)$, $g(\xi)$, $h(\xi)$, $\pi(\xi)$ наступного вигляду:

$$h''(\xi) = U_{11}g'(\xi) + U_{12}h'(\xi) + U_{13}\pi'(\xi)$$

$$
\begin{aligned}
&+U_{14}f(\xi)+U_{15}g(\xi)+U_{16}h(\xi)+U_{17}\pi(\xi),\\
f''(\xi)&=U_{21}g'(\xi)+U_{22}h'(\xi)+U_{23}\pi'(\xi)\\
&+U_{24}f(\xi)+U_{25}g(\xi)+U_{26}h(\xi)+U_{27}\pi(\xi),\\
g''(\xi)&=U_{31}g'(\xi)+U_{32}h'(\xi)+U_{33}\pi'(\xi)\\
&+U_{34}f(\xi)+U_{35}g(\xi)+U_{36}h(\xi)+U_{37}\pi(\xi),\\
f'(\xi)&=U_{41}f(\xi)+U_{42}g(\xi)+U_{43}h(\xi)+U_{44}\pi(\xi). \qquad (4)
\end{aligned}
$$

Тут U_{ij} — це поліноми другого порядку відносно спектральних параметрів a, s, m з коефіцієнтами, які самі є гладкими функціями від ξ.

Основні кроки процедури відокремлення змінних в системі (2) є наступними:

1. Підставляємо анзац (3) в рівняння (2) і записуємо похідні $f''(\xi)$, $g''(\xi)$, $h''(\xi), f'(\xi)$ в термінах функцій $g'(\xi)$, $h'(\xi)$, $\pi'(\xi)$, $f(\xi)$, $g(\xi)$, $h(\xi)$, $\pi(\xi)$, використовуючи рівняння (4).
2. Далі розглядаємо $g'(\xi)$, $h'(\xi)$, $\pi'(\xi)$, $f(\xi)$, $g(\xi)$, $h(\xi)$, $\pi(\xi)$ як нові незалежні змінні. Оскільки функції $\xi(t,r)$, $\gamma(t,\varphi)$, $\eta(t,z)$, $T(t)$, $T_1(t)$, $S(t)$, основні течії V_r, V_φ, V_z і коефіцієнти U_{ij} (які самі є гладкими функціями від ξ) є незалежними відносно цих змінних, ми вимагатимемо, щоб отримана рівність перетворювалась у тотожність при довільних $g'(\xi)$, $h'(\xi)$, $\pi'(\xi)$, $f(\xi)$, $g(\xi)$, $h(\xi)$, $\pi(\xi)$. Іншими словами, ми маємо розщепити цю рівність відносно цих змінних. Після розщеплення ми отримаємо перевизначену систему нелінійних рівнянь в частинних похідних для невідомих функцій $\xi(t,r)$, $\gamma(t,\varphi)$, $\eta(t,z)$, $T(t)$, $T_1(t)$, $S(t)$, основних течій V_r, V_φ, V_z і коефіцієнтів поліномів U_{ij}.
3. Після розв'язання вищеотриманої системи ми отримаємо вичерпний опис координатних систем, в яких система рівнянь (2) допускає відокремлення змінних в рамках нашого означення.

Отже, проблема відокремлення змінних в системі рівнянь (2) зводиться до інтегрування перевизначеної системи рівнянь з частинними похідними для невідомих функцій $\xi(t,r)$, $\gamma(t,\varphi)$, $\eta(t,z)$, $T(t)$, $T_1(t)$, $S(t)$, основних течій V_r, V_φ, V_z і коефіцієнтів поліномів U_{ij}.

Нижче наводимо результати. Для наявності фізичного змісту ми наклали додаткову умову, щоб основні течії самі точно задовольняли рівняння Нав'є–Стокса.

Тривимірні збурення. Загальна форма збурень v_r, v_φ, v_z і p є такою:

$$v_r = T(t)\exp\left(a\eta + m\varphi + s\int T(t)^2\,\mathrm{d}t\right) f(\xi),$$

$$v_\varphi = T(t)\exp\left(a\eta + m\varphi + s\int T(t)^2\,\mathrm{d}t\right) g(\xi),$$

$$v_z = T(t)\exp\left(a\eta + m\varphi + s\int T(t)^2\,\mathrm{d}t\right) h(\xi),$$

$$p = \rho T(t)^2\exp\left(a\eta + m\varphi + s\int T(t)^2\,\mathrm{d}t\right) \pi(\xi), \qquad (5)$$

де

$$\xi = T(t)r, \quad \eta = T(t)z + c(t).$$

Цим збуренням відповідають два класи основних течій, що задовольняють рівняння Нав'є–Стокса. Поля швидкостей для обох класів визначено наступним чином:

$$V_z = A(\xi)T(t) - \frac{zT'(t)}{T(t)} - \beta(t), \quad \beta(t) = \frac{c'(t)}{T(t)},$$

$$V_r = B(\xi)T(t) - r\frac{T'(t)}{T(t)}, \quad V_\varphi = C(\xi)T(t),$$

де функції $T(t)$ і $B(\xi)$ визначаються різним чином для кожного з цих двох класів:

$$\text{Class I:} \quad T(t) = \frac{1}{\sqrt{t}}, \quad B(\xi) = -\frac{3\xi}{4} + \frac{k}{\xi},$$

де функції $A(\xi)$ і $C(\xi)$ задовольняють рівняння

$$(4k + 3\xi^2 - 4\nu)A'(\xi) + \xi(-4k + 3\xi^2 + 4\nu)A''(\xi) + 4\xi^2\nu A'''(\xi) = 0,$$

$$-4\nu k_0\xi + (-4k + 3\xi^2 - 4\nu)C(\xi)$$
$$\quad + \xi(-4k + 3\xi^2 + 4\nu)C'(\xi) + 4\nu\xi^2 C''(\xi) = 0 \qquad (6)$$

і розподіл тисків дано наступним чином:

$$\frac{P}{\rho} = \frac{\nu k_0\varphi}{t} + \frac{x^2}{8t^2}$$

$$+ x\left[\beta'(t) + \frac{\beta(t)}{2t} + t^{-3/2}\left(\nu A''(\xi) - \frac{4k - 3\xi^2 - 4\nu}{4\xi}A'(\xi)\right)\right]$$
$$+ \frac{1}{t}\int \frac{16k^2 - 5\xi^2 + 16\xi^2 C^2(\xi)}{16\xi^3}\,\mathrm{d}\xi + p_0(t).$$

ЗДР (6) можуть бути явним чином розв'язані в термінах неповних гамма-функцій.

$$\text{Class II:}\quad T(t) = 1,\quad B(\xi) = \frac{k}{\xi},$$

де $A(\xi)$ і $C(\xi)$ задовольняють рівняння

$$(k-\nu)A'(\xi) + \xi(\nu - k)A''(\xi) + \xi^2\nu A'''(\xi) = 0,$$
$$\nu k_0\xi + (k+\nu)C(\xi) + \xi(k-\nu)C'(\xi) - \xi^2\nu C''(\xi) = 0 \tag{7}$$

і відповідний розподіл тисків дано наступним чином:

$$\frac{P}{\rho} = \nu k_0\varphi + x\left(\beta'(t) + \nu A''(\xi) + \frac{\nu - k}{\xi}A'(\xi)\right)$$
$$+ \int \frac{k^2 + \xi^2 C^2(\xi)}{\xi^3}\,\mathrm{d}\xi + p_0(t).$$

ЗДР (7) можуть бути явним чином розв'язані в елементарних функціях.

Рівняння з відокремленими змінними можуть бути записані для обох класів таким чином:

$$f(\xi)\big(\xi^2 s + \nu - m^2\nu - a^2\xi^2\nu + a\xi^2 A(\xi) + m\xi C(\xi) + \xi^2 B'(\xi)\big)$$
$$+ 2(m\nu - \xi C(\xi))g(\xi) + \xi((-\nu + \xi B(\xi))f'(\xi)$$
$$+ \xi(\pi'(\xi) - \nu f''(\xi))) = 0,$$
$$\big(\xi^2 s + \nu - m^2\nu - a^2\xi^2\nu + a\xi^2 A(\xi) + \xi B(\xi) + m\xi C(\xi)\big)g(\xi)$$
$$+ f(\xi)\big(-2m\nu + \xi C(\xi) + \xi^2 C'(\xi)\big)$$
$$+ \xi(m\pi(\xi) + (-\nu + \xi B(\xi))g'(\xi) - \xi\nu g''(\xi)) = 0,$$
$$\big(\xi^2 s - m^2\nu - a^2\xi^2\nu + a\xi^2 A(\xi) + m\xi C(\xi)\big)h(\xi)$$
$$+ \xi(a\xi\pi(\xi) + \xi f(\xi)A'(\xi) - \nu h'(\xi) + \xi B(\xi)h'(\xi) - \xi\nu h''(\xi)) = 0,$$
$$f(\xi) + mg(\xi) + \xi(ah(\xi) + f'(\xi)) = 0.$$

Двовимірні збурення. Загальна форма збурень є такою:

$$v_r = T(t) \exp\left(m\varphi + s\int T(t)^2\,\mathrm{d}t\right) f(\xi),$$

$$v_\varphi = T(t) \exp\left(m\varphi + s\int T(t)^2\,\mathrm{d}t\right) g(\xi), \quad v_z = 0,$$

$$p = \rho T(t)^2 \exp\left(m\varphi + s\int T(t)^2\,\mathrm{d}t\right) \pi(\xi), \quad \xi = T(t)r,$$

яка є частинним випадком (5) для $a = 0$.

Цим збуренням відповідає, зокрема, така основна течія:

$$V_z = -kz + \beta(t), \quad V_r = kr/2 + q/r, \quad V_\varphi = \nu B(\xi)T(t),$$

$$\frac{P}{\rho} = -\frac{1}{2}k^2x^2 + x\left(k\beta(t) - \beta'(t)\right) - \frac{4q^2 + k^2r^4}{8r^2} + T^2(t)\left(\nu^2\int\frac{B^2(\xi)}{\xi}\,\mathrm{d}\xi - \frac{1}{2}\nu k_0\varphi\right) + p_0(t),$$

де функції $T(t)$ і $B(\xi)$ задовольняють систему рівнянь

$$T'(t) - \frac{1}{2}\big(QT^3(t) - kT(t)\big) = 0,$$

$$k_0\xi - \big(2q + 2\nu + Q\xi^2\big)B(\xi) - \xi\big(2q - 2\nu + Q\xi^2\big)B'(\xi) + 2\nu\xi^2B''(\xi) = 0,$$

яка приводить до наступних випадків

$$T(t) = \frac{1}{\sqrt{\mathrm{e}^{kt} + 1}} \quad \left(\frac{Q}{k} = 1\right), \quad T(t) = \frac{1}{\sqrt{\mathrm{e}^{kt} - 1}} \quad \left(\frac{Q}{k} = -1\right),$$

$$T(t) = 1 \quad \left(\frac{Q}{k} = 1\right), \quad T(t) = \mathrm{e}^{-kt/2} \quad (Q = 0)\,.$$

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Збірник праць Інституту математики НАН України 2019, т. 16, № 1, 84–99

УДК 517.912:512.816

Describing certain Lie algebra orbits via polynomial equations

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Розглядаються алгебра Гейзенберга $\mathfrak{h}_3$ та тривимірна алгебра Лі $\mathfrak{g}$ з ненульовими комутаційними співвідношеннями $[e_1, e_2] = e_1$ $(= -[e_2, e_1])$. Описано алгебраїчні множини, що є замиканням орбіт векторів структурних сталих, що відповідають $\mathfrak{h}_3$ і $\mathfrak{g}$, а саме: у кожному з випадків побудовано набір поліномів, таких, що множина їх спільних нулів є замиканням орбіти вектора структурних сталих. Такий опис дозволяє надати альтернативний підхід до знаходження всіх можливих вироджень $\mathfrak{h}_3$ та $\mathfrak{g}$ у довільному нескінченному полі за допомогою означення незвідної алгебраїчної множини.

Let $\mathfrak{h}_3$ be the Heisenberg algebra and let $\mathfrak{g}$ be the 3-dimensional Lie algebra having $[e_1, e_2] = e_1$ $(= -[e_2, e_1])$ as its only non-zero commutation relations. We describe the closure of the orbit of a vector of structure constants corresponding to $\mathfrak{h}_3$ and $\mathfrak{g}$ respectively as an algebraic set giving in each case a set of polynomials for which the orbit closure is the set of common zeros. Working over an arbitrary infinite field, this description enables us to give an alternative way, using the definition of an irreducible algebraic set, of obtaining all degenerations of $\mathfrak{h}_3$ and $\mathfrak{g}$ (the degeneration from $\mathfrak{g}$ to $\mathfrak{h}_3$ being one of them).

1. Introduction. In the second half of the twentieth century a lot of works appeared on the study of different types of limit processes between various physical or geometrical theories. Such limit processes naturally lead to the notion of contraction (or degeneration). Possibly the first work in this direction was Segal [11] who studied a limit process of a family of some physically important isomorphic Lie groups. The claim is that if two physical theories are related by a limit process, then the associated invariance groups (and invariance algebras) should also be related by some limit process. This led to a wide investigation of contractions of Lie algebras from the physical point of view. Possibly, the three most famous physical examples of contractions are the following.

- Contraction of relativistic mechanics to classical mechanics was studied in works by Inönü and Wigner [6, 7]. Considering the physical limit process $c \to \infty$ in special relativity theory they showed how the symmetry group of relativistic mechanics (the Poincaré group) contracts to the Galilean group which is the symmetry group of classical mechanics.
- The relation between classical and quantum mechanics can also be expressed in terms of a limit process or, in other words, a contraction [5]. Thus, one can consider classical mechanics as the limit of quantum mechanics under the contraction $\mathfrak{h} \to \mathfrak{a}$, where $\mathfrak{h}$ is the Weyl–Heisenberg algebra and $\mathfrak{a}$ is the abelian Lie algebra of the same dimension. Under this contraction the quantum mechanical commutator $[x, p] = i\hbar$ (corresponding to the Heisenberg uncertainty principle) maps to the Abelian case (that is, the classical mechanics limit) under $\hbar \to 0$.
- The porous medium equation $u_t = m^{-1}\Delta(u^m - 1)$ can be contracted [13] (as $m \to 0$) to the equation $u_t = \Delta \ln u$, which is equivalent to the equation defining the Ricci flow on $\mathbb{R}^2$.

In these (and many other publications) it is shown, in particular, how some basic properties of the "contracted theories" can be reconstructed from the corresponding properties of the "original" theories. In an attempt to unify such observations, Zaitsev [14], independently of Inönü and Wigner, suggested constructing "the theory of physical theories" based on group limits of physical theories. This amounts to including in a uniform system several physical theories being connected together via certain relations. Recently, different types of contractions have been widely used in elementary particle theory, analysis of differential equations and other areas of mathematical and theoretical physics.

Working over $\mathbb{C}$ or $\mathbb{R}$, the statement "Lie algebra $\mathfrak{h}_1$ is a contraction of Lie algebra $\mathfrak{h}_2$" can be rephrased as "$\mathfrak{h}_1$ lies in the closure, in the metric topology, of the orbit of $\mathfrak{h}_2$ under the 'change of basis' action of the group of invertible linear transformations". In [4] the authors show that over $\mathbb{C}$ the orbit closure in the metric topology coincides with the orbit closure in the Zariski topology. Orbit closures with respect to the

Zariski topology are called degenerations. The notion of degeneration is well-defined not only over the fields $\mathbb{C}$ and $\mathbb{R}$ but also over an arbitrary ground field. In fact, this concept of orbit closure under the action of various groups arises naturally in many areas of mathematics (see, for example, [10]).

In [8] we explored the possibility of investigating degenerations over an arbitrary field using elementary algebraic techniques. For this we needed to extend or modify techniques already used over the fields $\mathbb{C}$, $\mathbb{R}$ (for example contractions obtained as limit points resulting from the action of diagonal matrices, also known as generalized Inönü–Wigner contractions) in a way so that they can be applied to the case of degenerations over an arbitrary field. In this paper, although we continue our study of degenerations via an elementary algebraic approach, we take a slightly different path and consider the possibility of obtaining all degenerations (for certain examples of Lie algebras) 'from first principles' by direct application of the definition of an algebraic (Zariski-closed) set. This involves obtaining explicit descriptions of the orbit closures under consideration using polynomial equations.

The paper is organized as follows. In Section 2 we give some necessary background, the setup being over an arbitrary infinite field $\mathbb{F}$. In particular, in Section 2.1 we recall some basic definitions and results on irreducible algebraic sets and regular maps while in Section 2.2 we recall the definition of degeneration together with some basic facts on Lie algebra structure vectors and their orbits under the 'change of basis' action of the general linear group. In Section 3 we perform some explicit computations concerning the orbits (and their closure in the Zariski topology) of certain given Lie algebra structure vectors corresponding to $\mathfrak{h}_3$ and $\mathfrak{g}_2 \oplus \mathfrak{a}_1$ respectively, where $\mathfrak{h}_3$ denotes the Heisenberg algebra, $\mathfrak{g}_2$ denotes the 2-dimensional non-Abelian Lie algebra and $\mathfrak{a}_1$ denotes the 1-dimensional Abelian Lie algebra. This enables us to give a description of the orbit closures of these structure vectors as algebraic sets via polynomial equations and, as a consequence, determine in an alternative way all degenerations of $\mathfrak{h}_3$ and $\mathfrak{g}_2 \oplus \mathfrak{a}_1$ over $\mathbb{F}$. We also obtain descriptions of the particular orbits described above as the intersection of a Zariski-closed set with a Zariski-open set.

2. Preliminaries and generalities. We begin this section by recalling some basic facts on irreducible algebraic sets. We refer the reader to Geck [2] for more details and for proofs of the main results from the theory we will be using.

2.1. Algebraic sets. Fix $\mathbb{F}$ to be an arbitrary infinite field and let m be a positive integer. We consider the ring $F[\boldsymbol{X}] = \mathbb{F}[X_1, \ldots, X_m]$ of polynomials in the indeterminates $X_1, \ldots, X_m$ over $\mathbb{F}$. For each $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{F}^m$ there exists a unique $\mathbb{F}$-algebra homomorphism $\mathbf{ev}_{\boldsymbol{\alpha}} \colon \mathbb{F}[X_1, \ldots, X_m] \to \mathbb{F}$ such that $\mathbf{ev}_{\boldsymbol{\alpha}}(X_i) = \alpha_i$ for all i. Given $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{F}^m$ and $f \in \mathbb{F}[X_1, \ldots, X_m]$ we will be writing more simply $f(\boldsymbol{\alpha}) = f(\alpha_1, \ldots, \alpha_m) = \mathbf{ev}_{\boldsymbol{\alpha}}(f)$.

Definition. Let S be any subset of $\mathbb{F}[X_1, \ldots, X_m]$. The algebraic set $\mathbf{V}(S)$ determined by S is defined by

$$\mathbf{V}(S) = \{\boldsymbol{\alpha} \in \mathbb{F}^m \colon f(\boldsymbol{\alpha}) = 0 \text{ for all } f \in S\}.$$

A subset of $\mathbb{F}^m$ is called *algebraic* if it is of the form $\mathbf{V}(S)$ for some subset $S \subseteq \mathbb{F}[X_1, \ldots, X_m]$. For any subset $V \subseteq \mathbb{F}^m$, the vanishing ideal $\mathbf{I}(V)$ of V is defined by

$$\mathbf{I}(V) = \{f \in \mathbb{F}[X_1, \ldots, X_m] \colon f(\boldsymbol{\alpha}) = 0 \text{ for all } \boldsymbol{\alpha} \in V\}.$$

It is immediate from the above definition that if S_1, S_2 are subsets of $\mathbb{F}[X_1, \ldots, X_m]$ with $S_1 \subseteq S_2$, then $\mathbf{V}(S_2) \subseteq \mathbf{V}(S_1)$ (see [2, Remark 1.1.4]).

It can be shown (see, for example, [2, Remark 1.1.4 and Lemma 1.1.5]) that arbitrary intersections and finite unions of algebraic sets in $\mathbb{F}^m$ are again algebraic. The empty set $\varnothing$ and $\mathbb{F}^m$ itself are clearly algebraic. Thus, the algebraic sets in $\mathbb{F}^m$ form the closed sets of a topology in $\mathbb{F}^m$, which is called the *Zariski topology*. A subset $X \subseteq \mathbb{F}^m$ is open if its complement $\mathbb{F}^m \setminus X$ is algebraic (closed).

We will denote by $\overline{V}$ the closure of a subset V of $\mathbb{F}^m$ in the Zariski topology.

An essential role in our investigation is played by the notion of irreducibility of algebraic sets.

Definition. Let $Z \subseteq \mathbb{F}^m$ be a nonempty algebraic set. We say that Z is *reducible* if we can write $Z = Z_1 \cup Z_2$, where $Z_1, Z_2 \subseteq Z$ are nonempty algebraic subsets with $Z_1 \neq Z$ and $Z_2 \neq Z$. Otherwise, we say that Z is *irreducible*.

Remark 1 (see [2, Example 1.1.13])**.** Our assumption that $\mathbb{F}$ is infinite ensures that $\mathbb{F}^m$ is irreducible.

Definition. Let s, r be positive integers and let $V \subseteq \mathbb{F}^s$ and $W \subseteq \mathbb{F}^r$ be nonempty algebraic sets. We say that $\Phi\colon V \to W$ is a regular map if there exist $f_1, \dots, f_r \in \mathbb{F}[X_1, \dots, X_s]$ such that $\Phi(\boldsymbol{\alpha}) = (f_1(\boldsymbol{\alpha}), \dots, f_r(\boldsymbol{\alpha}))$ for all $\boldsymbol{\alpha} \in V$.

One can then observe (see [2, p. 23]) that regular maps are continuous in the Zariski topology.

Remark 2 (see [2, Remark 1.3.2])**.** Let V, W be as in the definition above and let $\Phi\colon V \to W$ be a regular map. Assume that V is irreducible. Then the Zariski closure $\overline{\Phi(V)} \subseteq W$ is also irreducible.

2.2. Degenerations of Lie algebras. We keep the setup of the previous subsection. In particular $\mathbb{F}$ denotes an arbitrary infinite field but now we assume further that $m = n^3$ for some integer $n \geq 2$ we have fixed. Also let G be the general linear group $\mathrm{GL}(n, \mathbb{F})$.

Now let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in \mathbb{F}^m$ be given. For the rest of our discussion, it will be convenient to relabel the components of $\boldsymbol{\alpha}$ as follows. For $1 \leq r \leq m$ relabel α_r as $\alpha_{i(r),j(r),k(r)}$ where $i(r)$, $j(r)$, $k(r)$ are the unique integers with $1 \leq i(r), j(r), k(r) \leq n$ satisfying $r - 1 = (i(r)-1)n^2 + (j(r)-1)n + (k(r)-1)$. We will write $\boldsymbol{\alpha} = (\alpha_{i,j,k})$ or $\boldsymbol{\alpha} = (\alpha_{ijk})$ for short. For example, in the case $n = 2$ ($m = 8$) we have for $\boldsymbol{\alpha} \in \mathbb{F}^m$,

$$\begin{aligned}\boldsymbol{\alpha} &= (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8)\\ &= (\alpha_{111}, \alpha_{112}, \alpha_{121}, \alpha_{122}, \alpha_{211}, \alpha_{212}, \alpha_{221}, \alpha_{222}).\end{aligned}$$

(The above ordering in fact amounts to writing $\boldsymbol{\alpha} = (\alpha_{ijk}) \in \mathbb{F}^{n^3}$ where the triples (i, j, k) are placed in lexicographic order.)

In a similar manner we relabel the indeterminates $X_1, \dots, X_m$ in $\mathbb{F}[X_1, \dots, X_m]$ and we write $\mathbb{F}[\boldsymbol{X}]$ $(= \mathbb{F}[X_1, \dots, X_m]) = \mathbb{F}[X_{ijk}\colon 1 \leq i, j, k \leq n]$.

Definition. An element $\boldsymbol{\lambda} = (\lambda_{ijk}) \in \mathbb{F}^m$ is called a Lie algebra structure vector if there exists an n-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{F}$ and an ordered $\mathbb{F}$-basis $\hat{b} = (b_1, \dots, b_n)$ of $\mathfrak{g}$ such that $[b_i, b_j] = \sum_{k=1}^{n} \lambda_{ijk} b_k$ for $1 \leq i, j \leq n$. In such a case we call $\boldsymbol{\lambda} = (\lambda_{ijk})$ the structure vector of $\mathfrak{g}$ relative to $\hat{b}$. We denote by $\mathcal{L}_n(\mathbb{F})$ the subset of $\mathbb{F}^m$ consisting of precisely those elements of $\mathbb{F}^m$ which are Lie algebra structure vectors.

We refer the reader to [9] for the basic definitions and properties of Lie algebras.

The properties of the Lie bracket ensure that $\mathcal{L}_n(\mathbb{F})$ is an algebraic subset of $\mathbb{F}^m$. This is because $\mathcal{L}_n(\mathbb{F}) = \mathbf{V}(S)$ where S is the union of the following three subsets of $\mathbb{F}[X_{ijk} \colon 1 \leq i,j,k \leq n]$ (see, for example, [9, pp. 4–5] for a proof of this fact):

$$\{X_{iik} \colon 1 \leq i,k \leq n\}, \quad \{X_{ijk} + X_{jik} \colon 1 \leq i,j,k \leq n\},$$
$$\Big\{ \sum_k (X_{ijk}X_{klr} + X_{jlk}X_{kir} + X_{lik}X_{kjr}) \colon 1 \leq i,j,l,r \leq n \Big\}.$$

Remark 3. We have the following natural action of $G = \mathrm{GL}(n,\mathbb{F})$ on $\mathcal{L}_n(\mathbb{F})$ by 'change of basis'. Let $g = (g_{ij}) \in G$ and let $\boldsymbol{\lambda} = (\lambda_{ijk}) \in \mathcal{L}_n(\mathbb{F})$. Also let $\mathfrak{g}$ be an n-dimensional Lie algebra over $\mathbb{F}$ and $\hat{b} = (b_1, \dots, b_n)$ be an ordered $\mathbb{F}$-basis of $\mathfrak{g}$ such that $\boldsymbol{\lambda} = (\lambda_{ijk})$ is the structure vector of $\mathfrak{g}$ relative to $\hat{b}$. Now let $\hat{b}' = (b'_1, \dots, b'_n)$ be the basis of $\mathfrak{g}$ defined by $b'_j = \sum_{i=1}^n g_{ij} b_i$ for $1 \leq j \leq n$. Also let $\boldsymbol{\lambda}' = (\lambda'_{ijk}) \in \mathbb{F}^m$ be the structure vector of $\mathfrak{g}$ relative to $\hat{b}'$ (so we have $[b'_i, b'_j] = \sum_{k=1}^n \lambda'_{ijk} b'_k$ for $1 \leq i,j \leq n$). We will write $\boldsymbol{\lambda}' = \boldsymbol{\lambda} g$ (clearly, $\boldsymbol{\lambda}' \in \mathcal{L}_n(\mathbb{F})$). We call g the transition matrix from basis $\hat{b}$ to basis $\hat{b}'$ of $\mathfrak{g}$.

It is well known and easy to check, that the above process describes a well-defined (right) action of G on $\mathcal{L}_n(\mathbb{F})$. (See, for example, [8, Remark 2.6] where some details of such a check are given.)

Observe that the orbits relative to the action defined in the preceding remark correspond precisely to the isomorphism classes of n-dimensional Lie algebras over $\mathbb{F}$. We denote by $O(\boldsymbol{\mu})$ the orbit of the Lie algebra structure vector $\boldsymbol{\mu} \in \mathcal{L}_n(\mathbb{F})$ under the action of $\mathrm{GL}(n,\mathbb{F})$ described above.

Example. It is immediate that the zero vector $\mathbf{0} = (0_{\mathbb{F}}, \dots, 0_{\mathbb{F}})$ of $\mathbb{F}^{n^3}$ belongs to $\mathcal{L}_n(\mathbb{F})$ as it corresponds to the n-dimensional Abelian Lie algebra over $\mathbb{F}$ (under any choice of basis). Its orbit consists of precisely one point and hence it is Zariski-closed.

Remark 4. (i) For each $g \in \mathrm{GL}(n,\mathbb{F})$, making use of the action described in Remark 3, we define a function $\Phi_g \colon \mathcal{L}_n(\mathbb{F}) \to \mathcal{L}_n(\mathbb{F}) \colon \boldsymbol{\mu} \mapsto \boldsymbol{\mu} g$, ($\boldsymbol{\mu} \in \mathcal{L}_n(\mathbb{F})$). Then Φ_g is a regular map and hence continuous in the Zariski topology. (To see this we fix $g \in \mathrm{GL}(n,\mathbb{F})$. It follows from the change of basis process that for each $\boldsymbol{\mu} \in \mathcal{L}_n(\mathbb{F})$ we get $\Phi_g(\boldsymbol{\mu}) = (\mathbf{ev}_{\boldsymbol{\mu}}(f_1), \dots, \mathbf{ev}_{\boldsymbol{\mu}}(f_{n^3}))$ where, for $1 \leq i \leq n^3$, f_i is polynomial in $\mathbb{F}[X]$ which only depends on g.)

(ii) In view of item (i), one can give an elementary proof of the fact that the closure of an orbit in $\mathcal{L}_n(\mathbb{F})$ is a union of orbits (see, for example, [8, Lemma 3.1]).

Definition. Let $\mathfrak{g}_1$, $\mathfrak{g}_2$ be n-dimensional Lie algebras over $\mathbb{F}$. We say that $\mathfrak{g}_1$ degenerates to $\mathfrak{g}_2$ (respectively, $\mathfrak{g}_1$ properly degenerates to $\mathfrak{g}_2$) if there exist structure vectors $\boldsymbol{\lambda}_1$ of $\mathfrak{g}_1$ and $\boldsymbol{\lambda}_2$ of $\mathfrak{g}_2$, relative to some bases of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, such that $\boldsymbol{\lambda}_2 \in \overline{O(\boldsymbol{\lambda}_1)}$ (respectively, $\boldsymbol{\lambda}_2 \in \overline{O(\boldsymbol{\lambda}_1)} \setminus O(\boldsymbol{\lambda}_1)$).

It is immediate from Remark 4(ii) that if $\boldsymbol{\lambda} \in \overline{O(\boldsymbol{\mu})}$ and $\boldsymbol{\nu} \in \overline{O(\boldsymbol{\lambda})}$, then $\boldsymbol{\nu} \in \overline{O(\boldsymbol{\mu})}$, $(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{L}_n(\mathbb{F}))$. In other words, the transitivity property holds in the case of degenerations.

Finally for this subsection we remark that there are no proper degenerations over finite fields as finite subsets of $\mathbb{F}^m$ are closed in the Zariski topology.

3. Lie algebra orbit closures via polynomial equations. We continue with our assumption that $\mathbb{F}$ is an arbitrary infinite field.

Below, $\mathfrak{h}_3$ will denote the Heisenberg (Lie) algebra, $\mathfrak{g}_2$ will denote the 2-dimensional non-Abelian Lie algebra and $\mathfrak{a}_k$, for $k \geq 1$, the Abelian Lie algebra of dimension k.

We will make use of the action of $G = \mathrm{GL}(n, \mathbb{F})$ on $\mathcal{L}_n(\mathbb{F})$ described in Remark 3 in order to perform some explicit computations concerning the orbits (and their closure in the Zariski topology) of certain given Lie algebra structure vectors corresponding to $\mathfrak{h}_3$ and $\mathfrak{g}_2 \oplus \mathfrak{a}_1$ respectively. This will allow us to give descriptions of the orbit closures of these structure vectors as algebraic sets (via polynomial equations) and, in addition, obtain descriptions of the particular orbits we investigate here as intersections of a Zariski-closed with a Zariski-open set.

We will also show how these explicit descriptions of the orbits enable us to provide an alternative way of obtaining all degenerations of $\mathfrak{h}_3$ and $\mathfrak{g}_2 \oplus \mathfrak{a}_1$ over $\mathbb{F}$.

3.1. The Heisenberg algebra. We consider the Heisenberg algebra $\mathfrak{h}_3$. This (3-dimensional) algebra has an $\mathbb{F}$-basis $\hat{e} = (e_1, e_2, e_3)$ relative to which the only non-zero products (commutation relations) are $[e_2, e_3] = e_1 = -[e_3, e_2]$. The structure vector of $\mathfrak{h}_3$ relative to $\hat{e}$ is $\boldsymbol{\eta} = (\eta_{ijk}) \in \mathbb{F}^{27}$ where η_{231} and η_{321} (with $\eta_{231} = 1$, $\eta_{321} = -1$) are the only nonzero components of $\boldsymbol{\eta}$. First we determine $O(\boldsymbol{\eta})$ as a subset of $\mathbb{F}^{27}$. For this, let $g = (g_{ij}) \in \mathrm{GL}(3, \mathbb{F})$ and suppose that M_{ij} $(i, j = 1, 2, 3)$ is the determinant of the matrix obtained from g by deleting its i-th row and j-th column. Assume further that g is the transition matrix from basis (e_1, e_2, e_3) to the basis (e_1', e_2', e_3') of $\mathfrak{h}_3$. (So (e_1', e_2', e_3') is the basis of $\mathfrak{h}_3$ given by $e_j' = \sum_{i=1}^{3} g_{ij} e_i$ for $1 \leq j \leq 3$.) An easy computation then shows that, relative to this new basis, the multiplication in $\mathfrak{h}_3$ is given by

$$\begin{aligned}
[e_1', e_2'] &= (\det g)^{-1} M_{13}(M_{11}e_1' - M_{12}e_2' + M_{13}e_3'),\\
[e_1', e_3'] &= (\det g)^{-1} M_{12}(M_{11}e_1' - M_{12}e_2' + M_{13}e_3'),\\
[e_2', e_3'] &= (\det g)^{-1} M_{11}(M_{11}e_1' - M_{12}e_2' + M_{13}e_3').
\end{aligned}$$

It follows that there exist $\alpha, \beta, \gamma, \delta \in \mathbb{F}$ such that

$$\begin{aligned}
[e_1', e_2'] &= \gamma\delta(\alpha e_1' - \beta e_2' + \gamma e_3'),\\
[e_1', e_3'] &= \beta\delta(\alpha e_1' - \beta e_2' + \gamma e_3'),\\
[e_2', e_3'] &= \alpha\delta(\alpha e_1' - \beta e_2' + \gamma e_3').
\end{aligned}$$

The above relations motivate the following definition. For $\alpha, \beta, \gamma, \delta \in \mathbb{F}$, let $\boldsymbol{\eta}'(\alpha, \beta, \gamma, \delta) \in \mathbb{F}^{27}$ be defined by $\boldsymbol{\eta}'(\alpha, \beta, \gamma, \delta) = (0, 0, 0, \alpha\gamma\delta, -\beta\gamma\delta, \gamma^2\delta, \alpha\beta\delta, -\beta^2\delta, \beta\gamma\delta, -\alpha\gamma\delta, \beta\gamma\delta, -\gamma^2\delta, 0, 0, 0, \alpha^2\delta, -\alpha\beta\delta, \alpha\gamma\delta, -\alpha\beta\delta, \beta^2\delta, -\beta\gamma\delta, -\alpha^2\delta, \alpha\beta\delta, -\alpha\gamma\delta, 0, 0, 0)$.

We aim to show that the subset V of $\mathbb{F}^{27}$ defined by

$$V = \left\{\boldsymbol{\eta}'(\alpha, \beta, \gamma, \delta) \in \mathbb{F}^{27} \colon \alpha, \beta, \gamma, \delta \in \mathbb{F}\right\}$$

is in fact the (disjoint) union of $O(\boldsymbol{\eta})$ and $O(\mathbf{0})$ (recall that $\mathbf{0}$, the zero vector of $\mathbb{F}^{27}$, is the unique structure vector corresponding to the 3-dimensional Abelian Lie algebra). It is clear from the above discussion that $O(\boldsymbol{\eta}) \subseteq V$, hence it suffices to show that any nonzero vector $\boldsymbol{v} \in V$ belongs to $O(\boldsymbol{\eta})$. For this, it will be convenient to consider the decomposition $V = V_1 \cup V_2 \cup V_3$ where the subsets V_1, V_2, V_3 of V are defined as follows: First, for $\mu, \nu, \lambda, \sigma, \tau, \kappa \in \mathbb{F}$, define the elements $\boldsymbol{\eta}_1(\mu, \nu, \lambda)$, $\boldsymbol{\eta}_2(\tau, \sigma)$ and $\boldsymbol{\eta}_3(\kappa)$ of $\mathbb{F}^{27}$ by

$$\begin{aligned}
\boldsymbol{\eta}_1(\mu, \nu, \lambda) = (&0, 0, 0, \nu\lambda, -\mu\nu\lambda, \nu^2\lambda, \mu\lambda, -\mu^2\lambda, \mu\nu\lambda, -\nu\lambda, \mu\nu\lambda,\\
&-\nu^2\lambda, 0, 0, 0, \lambda, -\mu\lambda, \nu\lambda, -\mu\lambda, \mu^2\lambda, -\mu\nu\lambda, -\lambda, \mu\lambda,\\
&-\nu\lambda, 0, 0, 0),\\
\boldsymbol{\eta}_2(\tau, \sigma) = (&0, 0, 0, 0, \sigma\tau, -\sigma\tau^2, 0, \sigma, -\sigma\tau, 0, -\sigma\tau, \sigma\tau^2, 0, 0, 0, 0,\\
&0, 0, 0, -\sigma, \sigma\tau, 0, 0, 0, 0, 0, 0),\\
\boldsymbol{\eta}_3(\kappa) = (&0, 0, 0, 0, 0, \kappa, 0, 0, 0, 0, 0, -\kappa, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,\\
&0, 0, 0, 0).
\end{aligned}$$

We then let $V_1 = \{\boldsymbol{\eta}_1(\mu,\nu,\lambda)\colon \mu,\nu,\lambda \in \mathbb{F}\}$, $V_2 = \{\boldsymbol{\eta}_2(\tau,\sigma)\colon \tau,\sigma \in \mathbb{F}\}$ and $V_3 = \{\boldsymbol{\eta}_3(\kappa)\colon \kappa \in \mathbb{F}\}$.

In order to establish that V is indeed the union of the three sets above, it suffices to verify that $V_1 = \{\boldsymbol{\eta}'(\alpha,\beta,\gamma,\delta) \in V\colon \alpha \neq 0\}$, $V_2 = \{\boldsymbol{\eta}'(\alpha,\beta,\gamma,\delta) \in V\colon \alpha = 0 \text{ and } \beta \neq 0\}$ and $V_3 = \{\boldsymbol{\eta}'(\alpha,\beta,\gamma,\delta) \in V\colon \alpha = 0 \text{ and } \beta = 0\}$. That the above equalities of sets in fact hold is immediate from the relations $\boldsymbol{\eta}'(1,\mu,\nu,\lambda) = \boldsymbol{\eta}_1(\mu,\nu,\lambda)$, $\boldsymbol{\eta}_1(\beta\alpha^{-1},\gamma\alpha^{-1},\delta\alpha^2) = \boldsymbol{\eta}'(\alpha,\beta,\gamma,\delta)$ (for $\alpha \neq 0$), $\boldsymbol{\eta}'(0,1,\tau,-\sigma) = \boldsymbol{\eta}_2(\tau,\sigma)$, $\boldsymbol{\eta}_2(\gamma\beta^{-1},-\delta\beta^2) = \boldsymbol{\eta}'(0,\beta,\gamma,\delta)$ (for $\beta \neq 0$) and $\boldsymbol{\eta}'(0,0,1,\kappa) = \boldsymbol{\eta}_3(\kappa)$, $\boldsymbol{\eta}_3(\delta\gamma^2) = \boldsymbol{\eta}'(0,0,\gamma,\delta)$.

Since $V = V_1 \cup V_2 \cup V_3$, we can see that any nonzero element of V has one of the following forms: $\boldsymbol{\eta}_1(\mu,\nu,\lambda)$ (with $\lambda \neq 0$), $\boldsymbol{\eta}_2(\tau,\sigma)$ (with $\sigma \neq 0$) or $\boldsymbol{\eta}_3(\kappa)$ (with $\kappa \neq 0$).

Moreover, for $\lambda \neq 0$ we have $\boldsymbol{\eta} g_1(\mu,\nu,\lambda) = \boldsymbol{\eta}_1(\mu,\nu,\lambda)$, for $\sigma \neq 0$ we have $\boldsymbol{\eta} g_2(\tau,\sigma) = \boldsymbol{\eta}_2(\tau,\sigma)$ and finally for $\kappa \neq 0$ we have $\boldsymbol{\eta} g_3(\kappa) = \boldsymbol{\eta}_3(\kappa)$ where, for $\lambda \neq 0$, $\sigma \neq 0$, $\kappa \neq 0$ respectively, the matrices

$$g_1(\mu,\nu,\lambda) = \begin{bmatrix} \lambda^{-1} & 0 & 0 \\ \mu & 1 & 0 \\ -\nu & 0 & 1 \end{bmatrix}, \quad g_2(\tau,\sigma) = \begin{bmatrix} 0 & \sigma^{-1} & 0 \\ 1 & 0 & 0 \\ 0 & \tau & 1 \end{bmatrix},$$

$$g_3(\kappa) = \begin{bmatrix} 0 & 0 & \kappa^{-1} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

all belong to $G = \mathrm{GL}(3,\mathbb{F})$. This establishes that $V \setminus \{\mathbf{0}\} = O(\boldsymbol{\eta})$.

Our next aim is to show that V is an irreducible algebraic set. For this, let $S = S_1 \cup S_2 \cup S_3$ where S_1, S_2, S_3 are the following subsets of $\mathbb{F}[X_{ijk}\colon 1 \leq i,j,k \leq 3]$:

$$\begin{aligned} &S_1 = \{X_{iik}\colon 1 \leq i,k \leq 3\}, \quad S_2 = \{X_{ijk} + X_{jik}\colon 1 \leq i,j,k \leq 3\}, \\ &S_3 = \{X_{121} - X_{233}, X_{131} + X_{232}, X_{122} + X_{133}, X_{122}^2 + X_{123}X_{132}, \\ &\qquad X_{121}^2 - X_{123}X_{231}, X_{131}^2 + X_{132}X_{231}, X_{121}X_{131} + X_{122}X_{231}\}. \end{aligned}$$

Observe that $S \subseteq \mathbf{I}(V)$. We claim that $V = \mathbf{V}(S)$. It is clear that $V \subseteq \mathbf{V}(S)$. To establish the reverse inclusion $\mathbf{V}(S) \subseteq V$, let $\boldsymbol{\gamma} = (\gamma_{ijk}) \in \mathbb{F}^{27}$ be a common zero of the elements of S. Since $\boldsymbol{\gamma}$ is a common zero of the elements of $S_1 \cup S_2$, we see that the shape of $\boldsymbol{\gamma}$ is determined once we determine the shape of the auxiliary vector $\hat{\boldsymbol{\gamma}} = (\gamma_{121},\gamma_{122},\gamma_{123},\gamma_{131},\gamma_{132},\gamma_{133},\gamma_{231},\gamma_{232},\gamma_{233}) \in \mathbb{F}^9$. Invoking now the fact that $\boldsymbol{\gamma}$ is a common zero of the polynomials of degree 1 in S_3 we see that in fact $\hat{\boldsymbol{\gamma}}$ has shape $(\gamma_{121}, \gamma_{122}, \gamma_{123}, \gamma_{131}, \gamma_{132}, -\gamma_{122}, \gamma_{231}, -\gamma_{131}, \gamma_{121})$. We will consider the cases $\gamma_{231} \neq 0$ and $\gamma_{231} = 0$ separately. If $\lambda = \gamma_{231} \neq 0$ we can set $\mu = \gamma_{131}\lambda^{-1}$ and $\nu = \gamma_{121}\lambda^{-1}$ from which we can deduce that $\gamma_{123} = \nu^2\lambda$ (since $\gamma_{121}^2 - \gamma_{123}\gamma_{231} = 0$), $\gamma_{132} = -\mu^2\lambda$ (since $\gamma_{131}^2 + \gamma_{132}\gamma_{231} = 0$) and $\gamma_{122} = -\mu\nu\lambda$ (since $\gamma_{121}\gamma_{131} + \gamma_{122}\gamma_{231} = 0$). Hence $\boldsymbol{\gamma} \in V_1$ whenever $\gamma_{231} \neq 0$.

For the case $\gamma_{231} = 0$, by similar argument, one can show that if $\gamma_{132} \neq 0$, then $\boldsymbol{\gamma} \in V_2$ and if $\gamma_{132} = 0$ then $\boldsymbol{\gamma} \in V_3$. We conclude that $V = \mathbf{V}(S)$ and hence V is an algebraic set.

Next, we consider the map $\Phi\colon \mathbb{F}^4 \to \mathbb{F}^{27}\colon (\alpha, \beta, \gamma, \delta) \mapsto \boldsymbol{\eta}'(\alpha, \beta, \gamma, \delta)$. Clearly Φ is a regular map having V as its image. Thus, $\overline{\Phi(\mathbb{F}^4)} = \overline{V} = V$. Invoking Remarks 1 and 2, we see that V is irreducible. It follows that $O(\boldsymbol{\eta})$ is not closed in the Zariski topology. (Note that if $O(\boldsymbol{\eta})$ were Zariski-closed this would imply that $V = O(\boldsymbol{\eta}) \cup \{\mathbf{0}\}$ is reducible, being the union of two nonempty closed sets.) Hence, $O(\boldsymbol{\eta})$ is properly contained in $\overline{O(\boldsymbol{\eta})}$. Also $\overline{O(\boldsymbol{\eta})} \subseteq V$ since $O(\boldsymbol{\eta}) \subseteq V$ and V is an algebraic set. We conclude that $\overline{O(\boldsymbol{\eta})} = V = O(\boldsymbol{\eta}) \cup \{\mathbf{0}\}$. In other words, over an arbitrary infinite field, the only proper degeneration of $\mathfrak{h}_3$ is to the Abelian Lie algebra $\mathfrak{a}_3$. We remark here that this is a well-known fact and has been proved using different methods over various fields, see for example [1, 3, 8, 12]. In the discussion above we presented an alternative way of obtaining it, based on the definition of an irreducible algebraic set.

3.2. The algebra $\mathfrak{g}_2 \oplus \mathfrak{a}_1$. In this subsection we perform a similar investigation for the algebra $\mathfrak{g} = \mathfrak{g}_2 \oplus \mathfrak{a}_1$. Note that this algebra has an $\mathbb{F}$-basis $\hat{b} = (b_1, b_2, b_3)$ relative to which the only non-zero commutation relations are given by $[b_1, b_2] = b_1 = -[b_2, b_1]$. Let $\boldsymbol{\rho} = (\rho_{ijk}) \in \mathbb{F}^{27}$ be the structure vector of $\mathfrak{g}$ relative to the basis $\hat{b}$. Suppose now that $g \in G = \mathrm{GL}(3, \mathbb{F})$ is the transition matrix from $\hat{b}$ to the basis $\hat{b}' = (b_1', b_2', b_3')$ of $\mathfrak{g}$. It is then easy to show that

$$
\begin{aligned}
[b_1', b_2'] &= (\det g)^{-1} M_{33}(M_{11}b_1' - M_{12}b_2' + M_{13}b_3'),\\
[b_1', b_3'] &= (\det g)^{-1} M_{32}(M_{11}b_1' - M_{12}b_2' + M_{13}b_3'),\\
[b_2', b_3'] &= (\det g)^{-1} M_{31}(M_{11}b_1' - M_{12}b_2' + M_{13}b_3'),
\end{aligned}
$$

where, as before, M_{ij} denotes the determinant of the matrix obtained from g by deleting its i-th row and j-th column (in particular, the M_{ij} are elements of our field $\mathbb{F}$). It follows that there exist $\chi_1, \psi_1, \omega_1, \chi_2, \psi_2, \omega_2, \delta$

$\in \mathbb{F}$ such that

$$[b'_1, b'_2] = \delta\chi_2(\chi_1 b'_1 - \psi_1 b'_2 + \omega_1 b'_3),$$
$$[b'_1, b'_3] = \delta\psi_2(\chi_1 b'_1 - \psi_1 b'_2 + \omega_1 b'_3),$$
$$[b'_2, b'_3] = \delta\omega_2(\chi_1 b'_1 - \psi_1 b'_2 + \omega_1 b'_3).$$

This prompts us to define $\boldsymbol{\rho}'(\chi_1, \psi_1, \omega_1, \chi_2, \psi_2, \omega_2, \delta) \in \mathbb{F}^{27}$ by

$$\boldsymbol{\rho}'(\chi_1, \psi_1, \omega_1, \chi_2, \psi_2, \omega_2, \delta) = (0, 0, 0, \chi_1\chi_2\delta, -\psi_1\chi_2\delta, \omega_1\chi_2\delta, \chi_1\psi_2\delta, -\psi_1\psi_2\delta, \omega_1\psi_2\delta, -\chi_1\chi_2\delta, \psi_1\chi_2\delta, -\omega_1\chi_2\delta, 0, 0, 0, \chi_1\omega_2\delta, -\psi_1\omega_2\delta, \omega_1\omega_2\delta, -\chi_1\psi_2\delta, \psi_1\psi_2\delta, -\omega_1\psi_2\delta, -\chi_1\omega_2\delta, \psi_1\omega_2\delta, -\omega_1\omega_2\delta, 0, 0, 0),$$

and the subset U of $\mathbb{F}^{27}$ by $U = \{\boldsymbol{\rho}'(\chi_1, \psi_1, \omega_1, \chi_2, \psi_2, \omega_2, \delta)\colon \chi_1, \psi_1, \omega_1, \chi_2, \psi_2, \omega_2, \delta \in \mathbb{F}\}$.

It is then clear that $O(\boldsymbol{\rho}) \subseteq U$. We want to show that U is an algebraic set containing $V = O(\boldsymbol{\eta}) \cup \{\mathbf{0}\}$ (we keep the notation for V, $\boldsymbol{\eta}$, $\boldsymbol{\eta}'$ and also for S, S_1, S_2, S_3 introduced in the previous subsection). The inclusion $V \subseteq U$ is immediate from the fact that $\boldsymbol{\eta}'(\alpha, \beta, \gamma, \delta) = \boldsymbol{\rho}'(\alpha, \beta, \gamma, \gamma, \beta, \alpha, \delta)$.

Next, we define the subset T of $\mathbf{I}(U)$ by $T = S_1 \cup S_2 \cup T_3$ where

$$\begin{aligned} T_3 = \{&X_{121}X_{132} - X_{122}X_{131}, X_{121}X_{232} - X_{122}X_{231}, \\ &X_{131}X_{232} - X_{132}X_{231}, X_{121}X_{133} - X_{123}X_{131}, \\ &X_{121}X_{233} - X_{123}X_{231}, X_{232}X_{123} - X_{122}X_{233}, \\ &X_{122}X_{133} - X_{123}X_{132}, X_{132}X_{233} - X_{133}X_{232}, \\ &X_{233}X_{131} - X_{133}X_{231}\} \end{aligned}$$

(recall the definition of S_1 and S_2 in Section 3.1).

Now let $S' = T \cup \{X_{121} - X_{233},\ X_{131} + X_{232},\ X_{122} + X_{133}\} \subseteq T \cup S_3$. It is easy to check that $V \subseteq \mathbf{V}(S')$. We also have $\mathbf{V}(S') = \mathbf{V}(T \cup S_3)$. To see this last equality of sets, note first that $\mathbf{V}(T \cup S_3) \subseteq \mathbf{V}(S')$ since $S' \subseteq T \cup S_3$. On the other hand, any $\boldsymbol{\nu} \in \mathbf{V}(S')$ is a common zero of every polynomial in $T \cup S_3$. Hence, we also have $\mathbf{V}(S') \subseteq \mathbf{V}(T \cup S_3)$. Since $V \subseteq \mathbf{V}(S')$, we get $V \subseteq \mathbf{V}(T \cup S_3)$. But $T \cup S_3 \supseteq S$, so $\mathbf{V}(T \cup S_3) \subseteq \mathbf{V}(S) = V$. We conclude that V $(= \mathbf{V}(S)) = \mathbf{V}(T \cup S_3) = \mathbf{V}(S')$.

We aim to show that $U = \mathbf{V}(T)$. This would imply that U is an algebraic set (and also provide an alternative way of seeing that $V \subseteq U$ in view of the observation above).

Clearly, $U \subseteq \mathbf{V}(T)$. In order to establish the reverse inclusion, it will be convenient to decompose U as a union of three subsets which contain among them all elements of $\mathbf{V}(T)$. With $\alpha, \beta, \gamma, \mu, \nu, \phi, \rho, \sigma, \tau, \zeta, \theta, \xi, \kappa \in \mathbb{F}$, define the elements $\boldsymbol{\rho}_1(\alpha, \beta, \gamma, \mu, \nu, \phi)$, $\boldsymbol{\rho}_2(\sigma, \tau, \rho, \zeta)$ and $\boldsymbol{\rho}_3(\theta, \xi, \kappa) \in \mathbb{F}^{27}$ by

$$\begin{aligned}\boldsymbol{\rho}_1(\alpha, \beta, \gamma, \mu, \nu, \phi) = (&0, 0, 0, \mu\alpha, -\mu\beta, \mu\gamma, \nu\alpha, -\nu\beta, \nu\gamma, -\mu\alpha, \mu\beta,\\ &- \mu\gamma, 0, 0, 0, \phi\alpha, -\phi\beta, \phi\gamma, -\nu\alpha, \nu\beta, -\nu\gamma,\\ &- \phi\alpha, \phi\beta, -\phi\gamma, 0, 0, 0),\\ \boldsymbol{\rho}_2(\sigma, \tau, \rho, \zeta) = (&0, 0, 0, 0, \sigma, -\sigma\zeta, 0, \tau, -\tau\zeta, 0, -\sigma, \sigma\zeta, 0, 0, 0, 0, \rho,\\ &- \rho\zeta, 0, -\tau, \tau\zeta, 0, -\rho, \rho\zeta, 0, 0, 0),\\ \boldsymbol{\rho}_3(\theta, \xi, \kappa) = (&0, 0, 0, 0, 0, \theta, 0, 0, \xi, 0, 0, -\theta, 0, 0, 0, 0, 0, \kappa, 0, 0, -\xi,\\ &0, 0, -\kappa, 0, 0, 0).\end{aligned}$$

Also define the subsets U_1, U_2 and U_3 of $\mathbb{F}^{27}$ by $U_1 = \{\boldsymbol{\rho}_1(\alpha, \beta, \gamma, \mu, \nu, \phi)\colon \alpha, \beta, \gamma, \mu, \nu, \phi \in \mathbb{F} \text{ and } \alpha \neq 0\}$, $U_2 = \{\boldsymbol{\rho}_2(\sigma, \tau, \rho, \zeta)\colon \sigma, \tau, \rho, \zeta \in \mathbb{F}\}$ and $U_3 = \{\boldsymbol{\rho}_3(\theta, \xi, \kappa)\colon \theta, \xi, \kappa \in \mathbb{F}\}$.

It is then immediate from the relations $\boldsymbol{\rho}_1(\alpha, \beta, \gamma, \mu, \nu, \phi) = \boldsymbol{\rho}'(\chi_1 = \alpha,\ \psi_1 = \beta,\ \omega_1 = \gamma,\ \chi_2 = \mu,\ \psi_2 = \nu,\ \omega_2 = \phi,\ \delta = 1)$, $\boldsymbol{\rho}_2(\sigma, \tau, \rho, \zeta) = \boldsymbol{\rho}'(\chi_1 = 0,\ \psi_1 = -1,\ \omega_1 = -\zeta,\ \chi_2 = \sigma,\ \psi_2 = \tau,\ \omega_2 = \rho,\ \delta = 1)$ and $\boldsymbol{\rho}_3(\theta, \xi, \kappa) = \boldsymbol{\rho}'(\chi_1 = 0,\ \psi_1 = 0,\ \omega_1 = 1,\ \chi_2 = \theta,\ \psi_2 = \xi,\ \omega_2 = \kappa,\ \delta = 1)$ that $U_i \subseteq U$ for $i = 1, 2, 3$.

We now show that $\mathbf{V}(T) \subseteq U_1 \cup U_2 \cup U_3$. Let $\boldsymbol{\gamma} = (\gamma_{ijk}) \in \mathbb{F}^{27}$ be a common zero of all polynomials in T. As $T \supseteq S_1 \cup S_2$, similarly to the Heisenberg algebra case, we will work with the auxiliary vector $\hat{\boldsymbol{\gamma}} = (\gamma_{121}, \gamma_{122}, \gamma_{123}, \gamma_{131}, \gamma_{132}, \gamma_{133}, \gamma_{231}, \gamma_{232}, \gamma_{233}) \in \mathbb{F}^9$. Again, we will need to consider different subcases. We begin by considering the case $\gamma_{121} \neq 0$. Since $\boldsymbol{\gamma} \in \mathbf{V}(T)$, we get $\hat{\boldsymbol{\gamma}} = (\gamma_{121}, \gamma_{122}, \gamma_{123}, \gamma_{131}, \gamma_{122}\gamma_{131}\gamma_{121}^{-1}, \gamma_{123}\gamma_{131}\gamma_{121}^{-1}, \gamma_{231}, \gamma_{122}\gamma_{231}\gamma_{121}^{-1}, \gamma_{123}\gamma_{231}\gamma_{121}^{-1})$. For example, to see that $\gamma_{132} = \gamma_{122}\gamma_{131}\gamma_{121}^{-1}$, note that $\boldsymbol{\gamma}$ is a zero of the polynomial $X_{121}X_{132} - X_{122}X_{131}$ which belongs to T. On setting $\mu = 1$, $\nu = \gamma_{131}\gamma_{121}^{-1}$, $\phi = \gamma_{231}\gamma_{121}^{-1}$, $\alpha = \gamma_{121}$ $(\neq 0)$, $\beta = -\gamma_{122}$, $\gamma = \gamma_{123}$, we see that $\boldsymbol{\gamma} = \boldsymbol{\rho}_1(\alpha, \beta, \gamma, \mu, \nu, \phi)$ where $\alpha \neq 0$, so $\boldsymbol{\gamma} \in U_1$. Next we consider the case $\gamma_{121} = 0$. We split this case into the subcases $\gamma_{122} \neq 0$ (where, by similar argument as above, we can show that $\boldsymbol{\gamma} \in U_2$) and $\gamma_{122} = 0$. It remains to consider the case when both γ_{121} and γ_{122} are equal to zero and the next step is to split this case into subcases according to whether $\gamma_{123} \neq 0$ (we can show then that $\boldsymbol{\gamma} \in U_3$) or $\gamma_{123} = 0$. Continuing in a similar fashion, we finally deduce that $\mathbf{V}(T)$ is indeed a subset of $U_1 \cup U_2 \cup U_3$. Summing up the above discussion, we see that $U \subseteq \mathbf{V}(T) \subseteq U_1 \cup U_2 \cup U_3 \subseteq U$. This forces $U = U_1 \cup U_2 \cup U_3 = \mathbf{V}(T)$. Recall now that $V = O(\boldsymbol{\eta}) \cup \{\mathbf{0}\} \subseteq U$. In order to show that $U = O(\boldsymbol{\rho}) \cup O(\boldsymbol{\eta}) \cup \{\mathbf{0}\}$, we find, for each $\boldsymbol{\delta} \in U \setminus V$, an invertible matrix $g(\boldsymbol{\delta}) \in G$ such that $\boldsymbol{\delta} = \boldsymbol{\rho} g(\boldsymbol{\delta})$.

In the table below we summarize the results of this computation, listing also the corresponding matrices $g = g(\boldsymbol{\delta})$. We first split into subcases according to whether $\boldsymbol{\delta} \in U \setminus V$ is of the form $\boldsymbol{\rho}_1$ (with $\alpha \neq 0$), $\boldsymbol{\rho}_2$ or $\boldsymbol{\rho}_3$ and as it turns out, depending on the values of the elements of $\mathbb{F}$ involved, we need to split into further subcases.

It is now useful to recall that $V = \mathbf{V}(S')$ where $S' = T \cup \{X_{121} - X_{233}, X_{131} + X_{232}, X_{122} + X_{133}\} \subseteq T \cup S_3$. Let $\boldsymbol{\rho}' = (\rho'_{ijk}) \in U$. It follows that $\boldsymbol{\rho}' \in V$ if, and only if all three conditions $\rho'_{121} - \rho'_{233} = 0$, $\rho'_{131} + \rho'_{232} = 0$ and $\rho'_{122} + \rho'_{133} = 0$ are satisfied. In particular, in the case $\boldsymbol{\rho}' = \boldsymbol{\rho}_1(\alpha, \beta, \gamma, \mu, \nu, \phi)$, we have $\boldsymbol{\rho}' \in V$ if, and only if, all of the conditions $\mu\alpha - \phi\gamma = 0$, $\nu\alpha - \phi\beta = 0$ and $-\mu\beta + \nu\gamma = 0$ are satisfied. For $\boldsymbol{\rho}_1$ to be an element of U_1 we have the restriction $\alpha \neq 0$, so in this case, the third of the last three conditions follows from the other two (this is because the conditions $\mu\alpha - \phi\gamma = 0$ and $\nu\alpha - \phi\beta = 0$ are equivalent to the conditions $\mu = \phi\gamma\alpha^{-1}$ and $\nu = \phi\beta\alpha^{-1}$ if $\alpha \neq 0$). For simplicity, in the table below we will write $A_1 = \mu\alpha - \phi\gamma$, $A_2 = \nu\alpha - \phi\beta$. Similar observations can be made in the cases $\boldsymbol{\rho}'$ has form $\boldsymbol{\rho}_2$ or $\boldsymbol{\rho}_3$ (as it can also be seen from the table). Moreover, in the table below, vector $\boldsymbol{\rho}_1 = \boldsymbol{\rho}_1(\alpha, \beta, \gamma, \mu, \nu, \phi)$ will always be considered under the restriction $\alpha \neq 0$, compare with the definition of set U_1.

$\boldsymbol{\rho}_i$	conditions	transition matrix g	$\det g$
$\boldsymbol{\rho}_1$	$A_1 \neq 0,\ A_2 \neq 0$	$\begin{bmatrix} \nu & \phi & 0 \\ \mu\beta - \nu\gamma & A_1 & A_2 \\ -\gamma & 0 & \alpha \end{bmatrix}$	$A_1 A_2$
$\boldsymbol{\rho}_1$	$A_1 = 0,\ A_2 \neq 0,$ $\phi\gamma \neq 0$	$\begin{bmatrix} \beta & \alpha & \alpha\phi^{-1}\gamma^{-1}A_2 \\ -\gamma\alpha^{-1}A_2 & 0 & A_2 \\ \phi\beta\alpha^{-1} & \phi & 0 \end{bmatrix}$	$-A_2^2$
$\boldsymbol{\rho}_1$	$A_1 = 0,\ A_2 \neq 0,$ $\phi = 0$ $(\Rightarrow \mu = 0,\ \nu \neq 0)$	$\begin{bmatrix} 1 & 0 & 0 \\ -\gamma\nu & 0 & \alpha\nu \\ -\gamma + \beta & \alpha & \alpha \end{bmatrix}$	$-\alpha^2\nu$
$\boldsymbol{\rho}_1$	$A_1 = 0,\ A_2 \neq 0,$ $\phi \neq 0,\ \gamma = 0$ $(\Rightarrow \mu = 0)$	$\begin{bmatrix} \nu & \phi & 0 \\ 0 & 0 & A_2 \\ \beta & \alpha & 0 \end{bmatrix}$	$-A_2^2$

$\boldsymbol{\rho}_1$	$A_1 \neq 0,\ A_2 = 0,$ $\beta \neq 0,\ \gamma \neq 0$	$\begin{bmatrix} 0 & \alpha^2\mu\gamma & \alpha\gamma\phi\beta \\ \alpha^{-1}\beta A_1 & A_1 & 0 \\ -\alpha^{-1}A_1 & 0 & \gamma^{-1}A_1 \end{bmatrix}$	$-\beta A_1^3$
$\boldsymbol{\rho}_1$	$A_1 \neq 0,\ A_2 = 0,$ $\gamma = 0\ (\Rightarrow \mu \neq 0)$	$\begin{bmatrix} -\mu & 0 & \phi \\ \beta\mu & \mu\alpha & 0 \\ \beta & \alpha & 1 \end{bmatrix}$	$-\mu^2\alpha$
$\boldsymbol{\rho}_1$	$A_1 \neq 0,\ A_2 = 0,$ $\beta = 0\ (\Rightarrow \nu = 0)$	$\begin{bmatrix} \mu & 0 & -\phi \\ 0 & A_1 & 0 \\ -\gamma & 0 & \alpha \end{bmatrix}$	A_1^2
$\boldsymbol{\rho}_1$	$A_1 = 0,\ A_2 = 0$	$\boldsymbol{\rho}_1 \in \overline{O(\boldsymbol{\eta})}$	—
$\boldsymbol{\rho}_2$	$\rho \neq 0,\ \tau\zeta - \sigma \neq 0$	$\begin{bmatrix} 0 & -\sigma & -\tau \\ \tau\zeta - \sigma & \rho\zeta & \rho \\ 1 & 0 & 0 \end{bmatrix}$	$\rho(\tau\zeta - \sigma)$
$\boldsymbol{\rho}_2$	$\rho \neq 0,\ \tau\zeta - \sigma = 0$	$\begin{bmatrix} \tau & \rho & 0 \\ 0 & \rho\zeta & \rho \\ 1 & 0 & 0 \end{bmatrix}$	ρ^2
$\boldsymbol{\rho}_2$	$\rho = 0,\ \tau\zeta - \sigma \neq 0$	$\begin{bmatrix} 0 & \sigma & \tau \\ \tau\zeta - \sigma & 0 & 0 \\ 0 & \zeta & 1 \end{bmatrix}$	$(\tau\zeta - \sigma)^2$
$\boldsymbol{\rho}_2$	$\rho = 0,\ \tau\zeta - \sigma = 0$	$\boldsymbol{\rho}_2 \in \overline{O(\boldsymbol{\eta})}$	—
$\boldsymbol{\rho}_3$	$\kappa \neq 0,\ \xi \neq 0$	$\begin{bmatrix} 1 & \xi^{-1}(\kappa + \theta) & 1 \\ -\xi & -\kappa & 0 \\ 1 & 0 & 0 \end{bmatrix}$	κ
$\boldsymbol{\rho}_3$	$\kappa \neq 0,\ \xi = 0$	$\begin{bmatrix} -\theta\kappa^{-1} & 0 & 1 \\ 0 & -\kappa & 0 \\ 1 & 0 & 0 \end{bmatrix}$	κ
$\boldsymbol{\rho}_3$	$\kappa = 0,\ \xi \neq 0$	$\begin{bmatrix} 0 & \theta\xi^{-1} & 1 \\ -\xi & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	$-\xi$
$\boldsymbol{\rho}_3$	$\kappa = 0,\ \xi = 0$	$\boldsymbol{\rho}_3 \in \overline{O(\boldsymbol{\eta})}$	—

The computation above establishes that $U = O(\boldsymbol{\rho}) \cup O(\boldsymbol{\eta}) \cup \{\mathbf{0}\}$. Now recall that U $(= \mathbf{V}(T))$ is Zariski-closed. In fact, by similar argument as in the case of the set V, we can show that U is irreducible, considering now the regular map $\Phi\colon \mathbb{F}^7 \to U = \bar{U} \subseteq F^{27}$: $(\chi_1, \psi_1, \omega_1, \chi_2, \psi_2, \omega_2, \delta) \mapsto (0,0,0,\ \chi_1\chi_2\delta,\ -\psi_1\chi_2\delta,\ \omega_1\chi_2\delta,\ \chi_1\psi_2\delta,\ -\psi_1\psi_2\delta, \omega_1\psi_2\delta,\ -\chi_1\chi_2\delta,\ \psi_1\chi_2\delta, -\omega_1\chi_2\delta, 0, 0, 0,\ \chi_1\omega_2\delta, -\psi_1\omega_2\delta, \omega_1\omega_2\delta, -\chi_1\psi_2\delta, \psi_1\psi_2\delta, -\omega_1\psi_2\delta, -\chi_1\omega_2\delta, \psi_1\omega_2\delta,\ -\omega_1\omega_2\delta,\ 0,0,0) = \boldsymbol{\rho}'(\chi_1, \psi_1, \omega_1, \chi_2, \psi_2, \omega_2, \delta)$.

Since U is Zariski-closed and $O(\boldsymbol{\rho}) \subseteq U$, we get $\overline{O(\boldsymbol{\rho})} \subseteq U$. Invoking the fact that $U = O(\boldsymbol{\rho}) \cup O(\boldsymbol{\eta}) \cup \{\mathbf{0}\}$ we can deduce that $\mathfrak{h}_3$ and $\mathfrak{a}_3$

are the only possible Lie algebras which $\mathfrak{g}_2 \oplus \mathfrak{a}_1$ can properly degenerate to. In order to establish that $\mathfrak{g}_2 \oplus \mathfrak{a}_1$ in fact degenerates to both $\mathfrak{h}_3$ and $\mathfrak{a}_3$ it suffices to show that $\overline{O(\boldsymbol{\rho})} = U$. Since U is irreducible and $\overline{O(\boldsymbol{\eta})} = O(\boldsymbol{\eta}) \cup \{\mathbf{0}\}$ we get that $O(\boldsymbol{\rho})$ is not Zariski-closed. It follows that $O(\boldsymbol{\rho})$ is properly contained in $\overline{O(\boldsymbol{\rho})}$. If $\boldsymbol{\eta} \notin \overline{O(\boldsymbol{\rho})}$, then $O(\boldsymbol{\eta}) \cap \overline{O(\boldsymbol{\rho})} = \varnothing$ since $\overline{O(\boldsymbol{\rho})}$ is a union of orbits (see Remark 4(ii)). It would then follow that $\overline{O(\boldsymbol{\rho})} = O(\boldsymbol{\rho}) \cup \{\mathbf{0}\}$, contradicting the fact that U is irreducible. We conclude that $\boldsymbol{\eta} \in \overline{O(\boldsymbol{\rho})}$. It follows that $O(\boldsymbol{\eta}) \subseteq \overline{O(\boldsymbol{\rho})}$ and hence $\overline{O(\boldsymbol{\eta})} \subseteq \overline{O(\boldsymbol{\rho})}$. Since $\mathbf{0} \in \overline{O(\boldsymbol{\eta})}$, we get that $\mathbf{0} \in \overline{O(\boldsymbol{\rho})}$. Summing up, we have shown $\overline{O(\boldsymbol{\rho})} \subseteq U = O(\boldsymbol{\rho}) \cup O(\boldsymbol{\eta}) \cup \{\mathbf{0}\} \subseteq \overline{O(\boldsymbol{\rho})}$. Hence, $U = \overline{O(\boldsymbol{\rho})}$ as required.

We remark here that it is well-known that, over an infinite field, any Lie algebra degenerates to the abelian Lie algebra of the same dimension. Also note that already in [1] it is shown that $\mathfrak{g}$ degenerates to $\mathfrak{h}_3$ in the case the ground field is $\mathbb{R}$. In view of [8, Lemma 3.9] the technique used in [1] can be extended to obtain a degeneration from $\mathfrak{g}$ to $\mathfrak{h}_3$ now over an arbitrary infinite field. In the discussion above we provided an alternative way of obtaining this particular degeneration using the notion of an irreducible algebraic set.

We close this subsection with some general comments regarding our sets above. First, we can observe that $O(\boldsymbol{\rho}) = U \setminus \overline{O(\boldsymbol{\eta})} = \overline{O(\boldsymbol{\rho})} \setminus \overline{O(\boldsymbol{\eta})}$ so $O(\boldsymbol{\rho})$ is open in its closure (compare [2, Proposition 2.5.2] for the case of an algebraically closed field). Now let W be the union of the three principal open sets $\{\boldsymbol{\alpha} \in \mathbb{F}^{n^3} : f_i(\boldsymbol{\alpha}) \neq 0\}$ for $i = 1, 2, 3$ where $f_1 = X_{121} - X_{233}$, $f_2 = X_{131} + X_{232}$ and $f_3 = X_{122} + X_{133}$. Since $\overline{O(\boldsymbol{\rho})} = \mathbf{V}(T)$ and $\overline{O(\boldsymbol{\eta})} = \mathbf{V}(S')$ where $S' = T \cup \{f_1, f_2, f_3\}$, we see that $O(\boldsymbol{\rho}) = \mathbf{V}(T) \cap W$. This in fact verifies that $O(\boldsymbol{\rho})$ consists of precisely those points in U $(= \overline{O(\boldsymbol{\rho})})$ which do not correspond to unimodular Lie algebras (compare, for example, with [8, Remark 4.12]).

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Збірник праць Інституту математики НАН України 2019, т. 16, № 1, 100–112

УДК 517.912:512.816

Generic realizations of conformal and de Sitter algebras

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Отримано нові породжуючі реалізації конформної алгебри Лі та двох алгебр де Сіттера. Побудовано деформацію алгебри Пуанкаре до алгебр де Сіттера.

New generic realizations of conformal Lie algebra and two de Sitter algebras are obtained. Deformation of the Poincaré algebra to the de Sitter ones is constructed.

1. Introduction. Each well-established physical theory has its own certain fundamental invariance group and, therefore, realizations (representations by first-order differential operators) of their Lie algebras are effectively used for reduction, integration, differential invariants, etc., see e.g. [1, 2, 3, 5, 8].

In this work we consider three types of conformal groups: standard conformal group $\mathrm{C}(3,1)$ and two conformal groups of pseudoeuclidian spaces $\mathrm{C}(3,0)$ and $\mathrm{C}(2,1)$. For the respective Lie algebras $\mathfrak{c}(3,1)$, $\mathfrak{c}(3,0)$ and $\mathfrak{c}(2,1)$ we construct the maximal possible (generic) realizations using the algebraic approach proposed in [7]. Some covariant realizations of the conformal and de Sitter algebras are well known, but we first represent realizations in fifteen and ten essential variables respectively. Realizations in smaller number of variables can be obtained from the given ones by means of projection with respect a subalgebra.

The paper is arranged as follows. Fist we outline the algorithm for construction of realizations and define the conformal Lie algebra. Then we obtain it's generic realization and we do the same for the both de Sitter Lie algebras $\mathfrak{so}(4,1)$ and $\mathfrak{so}(3,2)$. And, finally, we include naturally the contraction parameters to de Sitter algebras in such a way, that contraction results are the Poincaré algebra.

2. Definitions and conventions. Let V be an n-dimensional vector space over the field of real numbers. Consider a Lie algebra $\mathfrak{g}$ on V spanned by a basis $\{e_1, e_2, \dots, e_n\}$ with the structure constants $C_{ij}^k \in \mathbb{R}$, here and below $i, j, k = 1, 2, \dots, n$. We denote an open domain of $\mathbb{R}^m$ as M and $\mathrm{Vect}(M)$ is the Lie algebra of smooth vector fields on M with the Lie product defined as commutator (i.e., the Lie algebra of first-order linear differential operators with analytical function coefficients).

A *realization of a Lie algebra* $\mathfrak{g}$ *in vector fields on* M is a homomorphism $R(\mathfrak{g}) = R\colon \mathfrak{g} \to \mathrm{Vect}(M)$. The realization is called *faithful* if $\ker R = \{0\}$ and *unfaithful* otherwise.

In Lie theory realizations are considered locally at some neighborhood $U_x \subset M \subset \mathbb{R}^m$ of a point $x \in M$ and in most of the cases without loss of generality the realization can be considered in a neighborhood of a zero point $x = 0$.

Denote local coordinates of a point $x \in M$ as $(x_1, \dots, x_m)$, then in coordinate form a realization $R(\mathfrak{g})$ is performed by the images $\Xi_i(x)$ of the basis elements e_i of a general form

$$\Xi_i(x) = R(e_i) = \sum_{l=1}^{m} \xi_{il}(x_1, x_2, \dots, x_m)\partial_l, \tag{1}$$

hereafter $\partial_l = \frac{\partial}{\partial x_l}$ and the coefficients $\xi_{il}(x_1, x_2, \dots, x_m)$ are smooth (analytic) functions.

Let us fix a point $x \in M$ and let R_x be a realization of $\mathfrak{g}$ at this point. Consider the linear map $R_x\colon \mathfrak{g} \to \mathrm{Vect}(M)(x)$ that transforms a vector $v \in \mathfrak{g}$ to it's image $R(v(x))$ at x. The matrix that corresponds to this linear map is the n by m matrix ξ formed by the coefficients of the realization (1)

$$\xi(x) = \begin{pmatrix} \xi_{11}(x) & \xi_{12}(x) & \dots & \xi_{1m}(x) \\ \xi_{21}(x) & \xi_{22}(x) & \dots & \xi_{2m}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{n1}(x) & \xi_{n2}(x) & \dots & \xi_{nm}(x) \end{pmatrix}.$$

The rank of the linear map R_x, or, equivalently, the rank of the matrix $\xi(x)$ at a point x is called a *rank of realization* R at point x and is denoted $\operatorname{rank} R_x$. The realization rank value possess the obvious inequality $0 \le \operatorname{rank} R_x \le n$, where n is the dimension of a Lie algebra $\mathfrak{g}$. The second inequality is dictated by the number of rows in matrix ξ, which is equal to the number of basis vector fields of $\mathfrak{g}$.

A realization R of a Lie algebra $\mathfrak{g}$ is called *transitive* if the action of the local Lie group corresponding to R is transitive. Or, equivalently (see [4]), a realization R of a Lie algebra $\mathfrak{g}$ is called *transitive* if rank $R_p = m$ for all $p \in M$.

For many practical applications it is necessary to decide if two given sets of first order differential operators (with the isomorphic commutation relations) can be transformed to each other or not. This task is rather complicated even in the case of small number of operators and variables.

Roughly speaking, two realizations are equivalent, if they can be transformed to the identical form by means of non-singular automorphic basis changes ($e_i \mapsto \tilde{e}_i$) and 1 to 1 changes of variables ($x_l \mapsto y_l = \varphi_l(x)$) with non-zero Jacobi determinant.

Let us have a diffeomorphism of M such that for the corresponding $x, y \in M$ we have $y_1 = \varphi_1(x_1, \dots, x_m)$, $y_2 = \varphi_2(x_1, \dots, x_m)$, $\dots$, $y_m = \varphi_m(x_1, \dots, x_m)$. Then the realization of the form (1) transforms to the following:

$$\tilde{R}(e_i) = \sum_{l=1}^{m} \tilde{\xi}_{il}(y)\partial_{y_l} = \sum_{l=1}^{m} \left(\sum_{l'=1}^{m} \tilde{\xi}_{il'}(x) \frac{\partial \varphi_l(x)}{\partial x_{l'}} \right) \partial_{y_l}.$$

Note, that the coefficients $\tilde{\xi}_{il}(y)$ are written in terms of y using the inverse transformation φ^{-1}.

It is obvious that application of transformations from $\operatorname{Aut}(\mathfrak{g})$ to the realization R does not change the rank of R, and none of diffeomorphisms of M can change the realization rank either. Therefore the equivalent realizations have the same ranks.

Let a realization $R(x)\colon \mathfrak{g} \to \operatorname{Vect}(M)$ has a rank $r = \operatorname{rank} R < m$ at a regular point $x \in M$, where $m = \dim M$. Then there exists a locally equivalent realization $\tilde{R}(y)\colon \mathfrak{g} \to \operatorname{Vect}(M)$ at a regular point $y \in M$ such that the coefficients of basic vector fields $\tilde{\xi}_{il}(y) = 0$ for all $i = 1, \dots, n$, $l = r+1, \dots, m$. To prove this let us construct the desired diffeomorphism. Since the realization rank is equal to r it is known from the theory of invariants [9] that there are $m - r$ functionally independent invariants $J_1(x_1, \dots, x_m), \dots, J_{m-r}(x_1, \dots, x_m)$ of the realization R. The diffeomorphism of the form $y_a = x_a$, $a = 1, \dots, r$; $y_{r+b} = J_b$,

$b = 1, \dots, m-r$ gives the following zero coefficients of the realization $\tilde{R}$: $\tilde{\xi}_{i(r+b)}(y) = R(e_i)(J_b) = 0$ for all $i = 1, \dots, n$, $b = 1, \dots, m-r$.

The above variables $y_1, \dots, y_r$ are called *essential* and the rest of non-zero variables from $y_{r+1}, \dots, y_m$ are called *additional*.

Example 1. Consider two-dimensional abelian Lie algebra $2A_1$. It is well-known that the basis elements of this algebra can be realized by two operators of translations

$$R_1(e_1) = \partial_1, \quad R_1(e_2) = \partial_2.$$

It was shown in [10] that there are exactly two inequivalent realizations of $2A_1$, and the second one is

$$R_2(e_1) = \partial_1, \quad R_2(e_2) = x_2\partial_1.$$

In these cases $\operatorname{rank} R_1 = 2$ and $\operatorname{rank} R_2 = 1$.

Consider the formal sum of these realizations $R_3 = R_1 + R_2$ (R_1 for the variables (x_1, x_2) and R_2 for the variables (x_3, x_4)), namely

$$R_3(e_1) = \partial_1 + \partial_3, \quad R_3(e_2) = \partial_2 + x_4\partial_3.$$

As far as $[\partial_1 + \partial_3, \partial_2 + x_4\partial_3] = 0$, then R_3 do realize the Lie algebra $2A_1$ in the space of four variables (x_1, x_2, x_3, x_4) and $\operatorname{rank} R_3 = 2$, what means that the number of essential variables is equal to 2.

Indeed, the diffeomorphism φ given by the non-singular functions

$$\varphi_1(x_1, \dots, x_4) = x_1, \quad \varphi_2(x_1, \dots, x_4) = x_2,$$
$$\varphi_3(x_1, \dots, x_4) = x_1 - x_3 + x_2x_4, \quad \varphi_4(x_1, \dots, x_4) = x_4$$

transforms the realization R_3 to the equivalent realization R_1 in 2 essential variables.

In case of transitive realizations all variables are essential and, since $\operatorname{rank} R \le n$, any transitive realization of a Lie algebra is realized in not more then n variables.

A recent paper [7] establishes the one-to-one correspondence between inequivalent transitive realizations of a Lie algebra $\mathfrak{g}$ and Int-inequivalent subalgebras of $\mathfrak{g}$. Moreover, this relation was extended to the non-transitive case as well, see [4].

The coefficients $\xi_k^i(x)$ of the generic realization

$$\Xi_i = \sum_{k=1}^{n} \xi_k^i(x) \frac{\partial}{\partial x_k}, \quad i = 1, 2, \dots, n,$$

can be recovered from the left-invariant differential one-forms

$$\Omega^i = \sum_{l=1}^{n} \omega_i^l(x) \mathrm{d}x_l$$

using the duality $\omega_i^l(x)\xi_k^i(x) = \delta_k^l$ and the coefficients $\omega_i^l(x)$ of the differential one-forms are constructed as follows:

$$\omega_i^l(x) = \left(A^{(1)}\left(x^1\right)A^{(2)}\left(x^2\right) \cdots A^{(i-1)}\left(x^{i-1}\right)\right)_i^l,$$

where $i = 2, 3, \dots, n$, $l = 1, 2, \dots, n$, $\omega_1^l = \delta_1^l$, and the matrices $A^{(p)}$, $p = 1, 2, \dots, n$, are the exponential solutions of the system

$$\dot{A}^{(p)}(t) = -\operatorname{ad}_{e_p} A^{(p)}(t), \quad A^{(p)}(0) = I.$$

All the rest of transitive realizations of a fixed Lie algebra are constructed by means of projection of the generic realization using the known set of $\mathrm{Aut}(\mathfrak{g})$-inequivalent subalgebras and the following rule.

Let $\mathfrak{h} = \langle e_{m+1}, \dots, e_n \rangle$ be a subalgebra of $\mathfrak{g} = \langle e_1, \dots, e_n \rangle$ with a complementary space $\{e_1, \dots, e_m\}$, then, using the above approach and the shortcut $\partial_i = \frac{\partial}{\partial x_i}$, we will obtain the realization of basis elements in the form

$$\begin{aligned} R(e_i) = {} & \xi_i^1(x_1, x_2, \dots, x_m)\partial_1 + \dots + \xi_i^m(x_1, x_2, \dots, x_m)\partial_m \\ & + \xi_i^{m+1}(x_1, x_2, \dots, x_n)\partial_{m+1} + \dots + \xi_i^n(x_1, x_2, \dots, x_n)\partial_n. \end{aligned}$$

The realization projected on the coordinates $x_1, x_2, \dots, x_m$ is well defined and has the form

$$\operatorname{pr}_{\mathfrak{h}} R(e_i) = \xi_i^1(x_1, x_2, \dots, x_m)\partial_1 + \dots + \xi_i^m(x_1, x_2, \dots, x_m)\partial_m.$$

The subalgebra that corresponds to the given realization is the kernel of its linear map at the origin of coordinates. In other words at the point $x = 0 \in \mathbb{R}^m$ the realization vectors that form a basis of corresponding subalgebra are identically equal to zero.

Example 2. Consider the realizations

$$R_1\colon\ e_1=\partial_1,\quad e_2=x_2\partial_1,\quad e_3=x_1\partial_1+2x_2\partial_2,$$
$$R_2\colon\ e_1=\partial_1,\quad e_2=x_1\partial_1-x_2\partial_2,\quad e_3=\partial_2.$$

At the origin of coordinates $x=0$ their basis vectors have the form

$$R_1(x=0)\colon\ e_1=\partial_1,\quad e_2=0,\quad e_3=0,$$
$$R_2(x=0)\colon\ e_1=\partial_1,\quad e_2=0,\quad e_3=\partial_2.$$

Therefore the realization R_1 corresponds to the subalgebra $\langle e_2, e_3\rangle$ and R_2 corresponds to $\langle e_2\rangle$.

The structure of realizations constructed by means of the algebraic method reminds a tree diagram, namely: a realization corresponding to a subalgebra $\mathfrak{h}_1$ can be constructed by means of projection from a realization corresponding to a subalgebra $\mathfrak{h}_2$ if $\mathfrak{h}_2\subset\mathfrak{h}_1$.

Note that all inequivalent realizations of a fixed Lie algebra can be obtained by the above method, as far as any realization corresponds to a quotient group G/H that acts effectively on some subspace M, where H is a subgroup that corresponds to some subalgebra $\mathfrak{h}$.

In this paper we use the above method to construct the realizations of three conformal Lie algebras in maximal possible number of essential variables, that is we construct realizations that correspond to zero subalgebras.

3. Conformal Lie algebra. First of all we consider a conformal group and it's 15-dimensional Lie algebra $\mathfrak{c}(3,1)$. The conformal Lie group $\mathrm{C}(3,1)=\mathrm{SO}(4,2)=\mathrm{SU}(2,2)$ of the Minkowski space is the maximal invariance group of the Maxwell equations in the flat space-time. This group in many aspects unite all physical groups. It is generated by 10 Poincaré generators P_μ, $J_{\mu\nu}$, dilatation generator D and generators of special conformal transformations K_μ, hereafter $\mu,\nu=1,2,\dots,4$. The non-zero commutation relations of the Lie algebra are

$$[J_{\mu\nu},J_{\rho\sigma}]=g_{\mu\rho}J_{\nu\sigma}-g_{\nu\rho}J_{\mu\sigma}+g_{\mu\sigma}J_{\rho\nu}-g_{\nu\sigma}J_{\rho\mu},\tag{2}$$
$$[J_{\mu\nu},P_\rho]=g_{\mu\rho}P_\nu-g_{\nu\rho}P_\mu,\tag{3}$$
$$[J_{\mu\nu},K_\rho]=g_{\mu\rho}K_\nu-g_{\nu\rho}K_\mu,\tag{4}$$
$$[P_\mu,K_\nu]=2(g_{\mu\nu}D+J_{\mu\nu}),\tag{5}$$
$$[P_\mu,D]=P_\mu,\tag{6}$$

$$[K_\mu, D] = -K_\mu. \tag{7}$$

Here $g_{\mu\nu}$ is the metric tensor of the Minkowski space $g_{11} = g_{22} = g_{33} = -g_{44} = 1$.

It is possible to consider conformal groups $\mathrm{C}(p,q)$ of the pseudoeuclidian spaces with metric tensors

$$g_{11} = g_{22} = \cdots = g_{pp} = -g_{p+1,p+1} = \cdots = -g_{p+q,p+q} = 1 \tag{8}$$

and $\mu, \nu = 1, \ldots, p+q = n$.

Consider the group $\mathrm{SO}(p+1, q+1) = \mathrm{span}\{I_{ab}\}$, $I_{ab} = -I_{ba}$ with the commutators

$$[I_{ab}, I_{cd}] = g_{ac}I_{bd} - g_{bc}I_{ad} + g_{ad}I_{cb} - g_{bd}I_{ca},$$

where g_{ab} are from (8) and $g_{n+1,n+1} = -g_{n+2,n+2} = 1$. Then matching

$$J_{\mu\nu} = I_{\mu\nu}, \quad P_\mu = I_{\mu,n+1} - I_{\mu,n+2}, \quad K_\mu = I_{\mu,n+1} + I_{\mu,n+2},$$
$$D = I_{n+1,n+2}$$

we get the isomorphism $\mathrm{C}(p,q) \simeq \mathrm{SO}(p+1,q+1)$. Therefore a number of well-known groups (like de Sitter groups) are conformal groups of pseudoeuclidian spaces. Consider the well-known realization of the conformal group

$$P_\mu = \partial_\mu, \quad J_{\mu\nu} = x_\nu\partial_\mu - x_\mu\partial_\nu, \quad D = x_\nu\partial_\nu,$$
$$K_\mu = 2x_\mu x_\nu\partial_\nu - x^2\partial_\mu;$$

hereafter the summation with respect to the repeated indices is implied and $x^2 = x_1^2 + \cdots + x_n^2$.

Let us define the subalgebra that corresponds to the given realization. To do this we study the realization at the point $x = (0,0,0,0)$ and see that the kernel of this linear map coincides with the subalgebra $\mathrm{span}\{J_{\mu\nu}, D, K_\mu\}$. Indeed, this is proven by the construction and projection of the generic realization of $\mathfrak{c}(3,1)$ with the following complementary part $\{P_\mu, J_{\mu\nu}, K_\mu, D\}$ taken in the lexicographical order. To make formula more readable we have introduced the shortcuts:

$$\sin x_i = \mathrm{s}_i, \quad \cos x_i = \mathrm{c}_i, \quad \tan x_i = \mathrm{t}_i, \quad \sinh x_i = \mathrm{sh}_i,$$
$$\cosh x_i = \mathrm{ch}_i, \quad \tanh x_i = \mathrm{th}_i, \quad i = \overline{1,10}.$$

$R_{\mathrm{generic}}(\mathfrak{c}(3,1))$:

$$P_1 = \partial_1, \quad P_2 = \partial_2, \quad P_3 = \partial_3, \quad P_4 = \partial_4,$$

$$J_{12} = x_2\partial_1 - x_1\partial_{x_2} + \partial_5,$$

$$J_{13} = x_3\partial_1 - x_1\partial_3 - \mathrm{th}_6\mathrm{s}_5\partial_5 + \mathrm{c}_5\partial_6 + 2\frac{\mathrm{s}_5}{\mathrm{ch}_6}\partial_8,$$

$$\begin{aligned} J_{14} = {} & -x_4\partial_1 - x_1\partial_4 + 2\frac{\mathrm{t}_7\mathrm{s}_5}{\mathrm{ch}_6}\partial_5 + \mathrm{t}_7\mathrm{s}_6\mathrm{c}_5\partial_6 + \mathrm{c}_5\mathrm{c}_6\partial_7 \\ & + \frac{\mathrm{s}_9\mathrm{s}_6\mathrm{c}_5\mathrm{c}_8 + \mathrm{th}_6\mathrm{s}_7\mathrm{c}_9\mathrm{s}_5 + \mathrm{s}_9\mathrm{s}_8\mathrm{s}_5}{\mathrm{c}_7\mathrm{c}_9}\partial_8 \\ & - \frac{\mathrm{c}_5\mathrm{s}_6\mathrm{s}_8 - \mathrm{s}_5\mathrm{c}_8}{\mathrm{c}_7}\partial_9 + \frac{\mathrm{s}_5\mathrm{s}_8 + \mathrm{c}_5\mathrm{s}_6\mathrm{c}_8}{\mathrm{c}_7\mathrm{c}_9}\partial_{10}, \end{aligned}$$

$$J_{23} = x_3\partial_2 - x_2\partial_3 + -\mathrm{th}_6\mathrm{c}_5\partial_5 - \mathrm{s}_5\partial_6 + \frac{\mathrm{c}_5}{\mathrm{ch}_6}\partial_8,$$

$$\begin{aligned} J_{24} = {} & -x_4\partial_2 - x_2\partial_4 + \frac{\mathrm{t}_7\mathrm{c}_5}{\mathrm{c}_7\mathrm{ch}_6}\partial_5 - \mathrm{t}_7\mathrm{s}_5\mathrm{s}_6\partial_6 - \mathrm{s}_5\mathrm{c}_6\partial_7 \\ & + \frac{\mathrm{th}_6\mathrm{s}_7\mathrm{c}_9\mathrm{c}_5 - \mathrm{s}_9\mathrm{s}_6\mathrm{c}_8\mathrm{s}_5 + \mathrm{s}_9\mathrm{c}_5\mathrm{s}_8}{\mathrm{c}_7\mathrm{c}_9}\partial_8 \\ & + \frac{\mathrm{c}_5\mathrm{c}_8 + \mathrm{s}_5\mathrm{s}_6\mathrm{s}_8}{\mathrm{c}_7}\partial_9 - \frac{\mathrm{s}_5\mathrm{s}_6\mathrm{c}_8 - \mathrm{c}_5\mathrm{s}_8}{\mathrm{c}_7\mathrm{c}_9}\partial_{10}, \end{aligned}$$

$$\begin{aligned} J_{34} = {} & -x_4\partial_3 - x_3\partial_4 + \mathrm{c}_6\mathrm{t}_7\partial_6 - \mathrm{s}_6\partial_7 + \frac{\mathrm{t}_9\mathrm{c}_8\mathrm{c}_6}{\mathrm{c}_7}\partial_8 \\ & - \frac{\mathrm{s}_8\mathrm{c}_6}{\mathrm{c}_7}\partial_9 + \frac{\mathrm{c}_6\mathrm{c}_8}{\mathrm{c}_7\mathrm{c}_9}\partial_{10}, \end{aligned}$$

$$\begin{aligned} K_1 = {} & \left(x_1^2 - x_2^2 - x_3^2 + x_4^2\right)\partial_1 + 2x_1x_2\partial_2 + 2x_1x_3\partial_3 + 2x_1x_4\partial_4 \\ & - 2\left(x_2 + \frac{x_4\mathrm{t}_7\mathrm{s}_5}{\mathrm{ch}_6} - x_3\mathrm{th}_6\mathrm{s}_5\right)\partial_5 \\ & - 2(x_4\mathrm{t}_7\mathrm{s}_6 + x_3)\mathrm{c}_5\partial_6 - 2x_4\mathrm{c}_5\mathrm{c}_6\partial_7 \\ & - 2\left(\frac{x_4\mathrm{t}_9\mathrm{s}_6\mathrm{c}_5\mathrm{c}_8}{\mathrm{c}_7} + x_4\mathrm{t}_7\mathrm{th}_6\mathrm{s}_5 + \frac{x_3\mathrm{s}_5}{\mathrm{ch}_6} + \frac{x_4\mathrm{t}_9\mathrm{s}_5\mathrm{s}_8}{\mathrm{c}_7}\right)\partial_8 \\ & + 2\frac{(\mathrm{c}_5\mathrm{s}_6\mathrm{s}_8 - \mathrm{s}_5\mathrm{c}_8)x_4}{\mathrm{c}_7}\partial_9 \\ & - 2\frac{(\mathrm{s}_5\mathrm{s}_8 + \mathrm{c}_5\mathrm{s}_6\mathrm{c}_8)x_4}{\mathrm{c}_7\mathrm{c}_9}\partial_{10} - (2x_1x_{11} - \mathrm{ch}_7\mathrm{c}_5\mathrm{c}_6)\partial_{11} \\ & + (\mathrm{sh}_7\mathrm{sh}_9\mathrm{c}_5\mathrm{c}_6 + \mathrm{ch}_9(\mathrm{s}_5\mathrm{c}_8 - \mathrm{s}_6\mathrm{s}_8\mathrm{c}_5) - 2x_1x_{12})\partial_{12} \\ & + (\mathrm{sh}_9\mathrm{sh}_{10}(\mathrm{s}_5\mathrm{c}_8 - \mathrm{c}_5\mathrm{s}_6\mathrm{s}_8) + \mathrm{ch}_{10}(\mathrm{s}_6\mathrm{c}_8\mathrm{c}_5 + \mathrm{s}_5\mathrm{s}_8) \end{aligned}$$

$$
\begin{aligned}
&+ \mathrm{sh}_7\mathrm{ch}_9\mathrm{sh}_{10}\mathrm{c}_5\mathrm{c}_6 - 2x_1x_{13})\partial_{13} \\
&+ (\mathrm{sh}_9\mathrm{ch}_{10}(\mathrm{s}_5\mathrm{c}_8 + \mathrm{c}_5\mathrm{s}_6\mathrm{s}_8) + \mathrm{sh}_{10}(\mathrm{s}_6\mathrm{c}_8\mathrm{c}_5 + \mathrm{s}_5\mathrm{s}_8) \\
&+ \mathrm{sh}_7\mathrm{ch}_9\mathrm{ch}_{10}\mathrm{c}_5\mathrm{c}_6 - 2x_1x_{14})\partial_{14} + 2x_1\partial_{15}, \\
K_2 = {}& 2x_1x_2\partial_1 + \big(-{x_1}^2 + {x_2}^2 - {x_3}^2 + {x_4}^2\big)\partial_{x_2} + 2x_2x_3\partial_3 \\
&+ 2x_2x_4\partial_4 + 2\left(\frac{x_4\mathrm{c}_5\mathrm{t}_7}{\mathrm{ch}_6} - x_1 - x_3\mathrm{c}_5\mathrm{th}_6\right)\partial_5 \\
&+ 2(x_3 + x_4\mathrm{t}_7\mathrm{s}_6)\mathrm{s}_5\partial_6 + 2x_4\mathrm{s}_5\mathrm{c}_6\partial_7 \\
&+ 2\left(\frac{x_4\mathrm{t}_9}{\mathrm{c}_7}(\mathrm{s}_6\mathrm{c}_8\mathrm{s}_5 - \mathrm{c}_5\mathrm{s}_8) - x_4\mathrm{th}_7\mathrm{c}_5 - \frac{x_3\mathrm{c}_5}{\mathrm{ch}_6}\right)\partial_8 \\
&- 2\frac{(\mathrm{c}_5\mathrm{c}_8 + \mathrm{s}_5\mathrm{s}_6\mathrm{s}_8)x_4}{\mathrm{c}_7}\partial_9 + 2\frac{(\mathrm{s}_5\mathrm{s}_6\mathrm{c}_8 - \mathrm{c}_5\mathrm{s}_8)x_4}{\mathrm{c}_7\mathrm{c}_9}\partial_{10} \\
&- (2x_2x_{11} + \mathrm{ch}_7\mathrm{s}_5\mathrm{c}_6)\partial_{11} + \big(\mathrm{ch}_9(\mathrm{c}_5\mathrm{c}_8 + \mathrm{s}_5\mathrm{s}_6\mathrm{s}_8) \\
&- \mathrm{sh}_7\mathrm{sh}_9\mathrm{s}_5\mathrm{c}_6 - 2x_2x_{12}\big)\partial_{12} + \big(\mathrm{sh}_9\mathrm{ch}_{10}\mathrm{c}_5\mathrm{c}_8 \\
&+ \mathrm{ch}_{10}\mathrm{c}_5\mathrm{s}_8 - \mathrm{sh}_7\mathrm{sh}_{10}\mathrm{ch}_9\mathrm{s}_5\mathrm{c}_6 - \mathrm{ch}_{10}\mathrm{s}_5\mathrm{s}_6\mathrm{c}_8 \\
&+ \mathrm{sh}_9\mathrm{sh}_{10}\mathrm{s}_5\mathrm{s}_6\mathrm{s}_8 - 2x_2x_{13}\big)\partial_{13} \\
&+ \big(\mathrm{sh}_9\mathrm{ch}_{10}(\mathrm{c}_5\mathrm{c}_8 + \mathrm{s}_5\mathrm{s}_6\mathrm{c}_8) + \mathrm{sh}_{10}(\mathrm{s}_8\mathrm{c}_5 - \mathrm{s}_5\mathrm{s}_6\mathrm{c}_8) \\
&- \mathrm{sh}_7\mathrm{ch}_9\mathrm{ch}_{10}\mathrm{s}_5\mathrm{c}_6 - 2x_2x_{14}\big)\partial_{14} + 2x_2\partial_{15}, \\
K_3 = {}& 2x_1x_3\partial_1 + 2x_2x_3\partial_{x_2} + \big(-x_1^2 - x_2^2 + x_3^2 + x_4^2\big)\partial_3 + 2x_3x_4\partial_4 \\
&- 2\mathrm{th}_6(x_1\mathrm{s}_5 + x_2\mathrm{c}_5)\partial_5 - 2(x_4\mathrm{t}_7\mathrm{c}_6 - x_1\mathrm{c}_5 + x_2\mathrm{s}_5)\partial_6 \\
&+ 2x_4\mathrm{s}_6\partial_7 + \left(\frac{x_1\mathrm{s}_5 + x_2\mathrm{c}_5}{\mathrm{ch}_6} - 2\frac{x_4\mathrm{t}_9\mathrm{c}_6\mathrm{c}_8}{\mathrm{c}_7}\right)\partial_8 \\
&+ 2\frac{\mathrm{s}_8\mathrm{c}_6x_4}{\mathrm{c}_7}\partial_9 - 2\frac{\mathrm{c}_6\mathrm{c}_8x_4}{\mathrm{c}_7\mathrm{c}_9}\partial_{10} - (\mathrm{ch}_7\mathrm{s}_6 + 2x_3x_{11})\partial_{11} \\
&- (\mathrm{ch}_9\mathrm{c}_6\mathrm{s}_8 + 2x_3x_{12} + \mathrm{sh}_7\mathrm{sh}_9\mathrm{s}_6)\partial_{12} \\
&+ \big(\mathrm{ch}_{10}\mathrm{c}_6\mathrm{c}_8 - \mathrm{sh}_9\mathrm{sh}_{10}\mathrm{c}_6\mathrm{s}_8 - \mathrm{sh}_7\mathrm{ch}_9\mathrm{sh}_{10}\mathrm{s}_6 \\
&- 2x_3x_{13}\big)\partial_{13} + \big(\mathrm{ch}_{10}\mathrm{c}_6\mathrm{c}_8 - \mathrm{sh}_9\mathrm{ch}_{10}\mathrm{c}_6\mathrm{s}_8 \\
&- \mathrm{sh}_7\mathrm{ch}_9\mathrm{ch}_{10}\mathrm{s}_6 - 2x_3x_{14}\big)\partial_{14} + 2x_3\partial_{15}, \\
K_4 = {}& -2x_1x_4\partial_1 - 2x_2x_4\partial_{x_2} - 2x_4x_3\partial_3 - \big(x_1^2 + x_2^2 + x_3^2 + x_4^2\big)\partial_4 \\
&+ 2\frac{\mathrm{t}_7(x_1\mathrm{s}_5 + x_2\mathrm{c}_5)}{\mathrm{ch}_6}\partial_5 + 2\mathrm{t}_7(x_1\mathrm{s}_6\mathrm{c}_5 - x_2\mathrm{s}_6\mathrm{s}_5 + x_3\mathrm{c}_6)\partial_6 \\
&- 2(x_2\mathrm{c}_6\mathrm{s}_5 - x_1\mathrm{c}_6\mathrm{c}_5 + x_3\mathrm{s}_6)\partial_7
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\left(\frac{\mathrm{t}_9}{\mathrm{c}_7}(x_2\mathrm{s}_5\mathrm{s}_6\mathrm{c}_8 - x_1\mathrm{s}_6\mathrm{c}_5\mathrm{c}_8 - x_3\mathrm{c}_6\mathrm{c}_8 - x_1\mathrm{s}_5\mathrm{s}_8\right.\\
&\left. - x_2\mathrm{s}_8\mathrm{c}_5) - \mathrm{th}_6\mathrm{t}_7(x_1\mathrm{s}_5 + x_2\mathrm{c}_5)\right)\partial_8\\
&- \frac{2}{\mathrm{c}_7}(x_2 - \mathrm{s}_5\mathrm{s}_6\mathrm{s}_8 + x_1\mathrm{c}_5\mathrm{s}_6\mathrm{s}_8 + x_3\mathrm{c}_6\mathrm{s}_8 - x_1\mathrm{s}_5\mathrm{c}_8\\
&- x_2\mathrm{c}_5\mathrm{c}_8)\partial_9 + \frac{2}{\mathrm{c}_7\mathrm{c}_9}(x_1\mathrm{c}_5\mathrm{s}_6\mathrm{c}_8 - x_2\mathrm{s}_5\mathrm{s}_6\mathrm{c}_8 + x_3\mathrm{c}_6\mathrm{c}_8\\
&+ x_1\mathrm{s}_5\mathrm{s}_8 + x_2\mathrm{c}_5\mathrm{s}_8)\partial_{10} + (2x_4x_{11} + \mathrm{sh}_7)\partial_{11}\\
&+ (2x_4x_{12} + \mathrm{ch}_7\mathrm{sh}_9)\partial_{12} + (2x_4x_{13} + \mathrm{ch}_7\mathrm{ch}_9\mathrm{sh}_{10})\partial_{13}\\
&+ (2x_4x_{14} + \mathrm{ch}_7\mathrm{ch}_9\mathrm{ch}_{10})\partial_{14} - 2x_4\partial_{15},
\end{aligned}
$$

$$
\begin{aligned}
D = {}& x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + x_4\partial_4 - x_{11}\partial_{11} - x_{12}\partial_{12} - x_{13}\partial_{13}\\
&- x_{14}\partial_{14} + \partial_{15}.
\end{aligned}
$$

4. De Sitter Lie algebras. Consider de Sitter groups $SO(4,1)$ and SO(3,2) that are the groups of isometry transformations of pseudoeuclidean spaces with metric forms $x_1^2 + x_2^2 + x_3^2 - x_4^2 + x_5^2$ and $x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2$ respectively. Them are the movement groups of 4-dimensional Riemann spaces of a constant curvature (de Sitter spaces). Both de Sitter spaces describe the expanding Universe, where the radial velocities of galaxies are approximately proportional to distances from any space point. For the de Sitter Lie algebras we can use the isomorphisms $\mathfrak{c}(3,0) \sim \mathfrak{so}(4,1)$ and $\mathfrak{c}(2,1) \sim \mathfrak{so}(3,2)$ with the conformal commutation relations (2)–(7) for the metric tensors $g_{11} = g_{22} = g_{33} = 1$ and $g_{11} = g_{22} = -g_{33} = 1$ respectively. Then, constructing the generic realization by the method given in second section (with the complementary part $\{P_\mu, J_{\mu\nu}, K_\mu, D\}$ taken in the lexicographical order), we have got two following realizations. Note that it is possible to construct one realization for both de Sitter algebras (putting the parameter to the commutation relations that changes the tensor sign), but this essentially complicates calculations and appearance of realizations.

$R_{\text{generic}}(\mathfrak{c}(3,0))$:

$$
\begin{aligned}
&P_1 = \partial_1, \quad P_2 = \partial_2, \quad P_3 = \partial_3, \quad J_{12} = x_2\partial_1 - x_1\partial_2 + \partial_4,\\
&J_{13} = x_3\partial_1 - x_1\partial_3 - \mathrm{th}_5\mathrm{s}_4\partial_4 + \mathrm{c}_4\partial_5 + \frac{\mathrm{s}_4}{\mathrm{ch}_5}\partial_6,\\
&J_{23} = x_3\partial_2 - x_2\partial_3 - \mathrm{th}_5\mathrm{c}_4\partial_4 - \mathrm{s}_4\partial_5 + \frac{\mathrm{c}_4}{\mathrm{ch}_5}\partial_6,
\end{aligned}
$$

$$
\begin{aligned}
K_1 &= \left(x_1^2 - x_2^2 - x_3^2\right)\partial_1 + 2x_1x_2\partial_2 + 2x_1x_3\partial_3 \\
&\quad + 2(x_3\mathrm{s}_4\mathrm{th}_5 - x_2)\partial_4 - 2x_3\mathrm{c}_4\partial_5 - 2\frac{x_3\mathrm{s}_4}{\mathrm{ch}_5}\partial_6 \\
&\quad + (\mathrm{c}_4\mathrm{c}_5 - 2x_1x_7)\partial_7 + (\mathrm{s}_4\mathrm{ch}_6 - \mathrm{c}_4\mathrm{sh}_5\mathrm{sh}_6 - 2x_1x_8)\partial_8 \\
&\quad + (\mathrm{c}_4\mathrm{s}_5\mathrm{c}_6 + \mathrm{s}_4\mathrm{s}_6 - 2x_1x_9)\partial_9 + 2x_1\partial_{10}, \\
K_2 &= 2x_1x_2\partial_1 + \left(x_2^2 - x_1^2 - x_3^2\right)\partial_2 + 2x_2x_3\partial_3 \\
&\quad + 2(x_1 + x_3\mathrm{c}_4\mathrm{th}_5)\partial_4 + 2x_3\mathrm{s}_4\partial_5 - 2\frac{x_3\mathrm{c}_4}{\mathrm{ch}_5}\partial_6 \\
&\quad - (\mathrm{s}_4\mathrm{c}_5 + 2x_2x_7)\partial_7 + (\mathrm{s}_4\mathrm{s}_5\mathrm{s}_6 + \mathrm{c}_4\mathrm{c}_6 - 2x_8x_2)\partial_8 \\
&\quad + (\mathrm{c}_4\mathrm{s}_6 - \mathrm{s}_4\mathrm{s}_5\mathrm{c}_6 - 2x_2x_9)\partial_9 + 2x_2\partial_{10}, \\
K_3 &= 2x_1x_3\partial_1 + 2x_2x_3\partial_2 + \left(x_3^2 - x_1^2 - x_2^2\right)\partial_3 \\
&\quad - 2(x_1\mathrm{s}_4 + x_2\mathrm{c}_4)\mathrm{th}_5\partial_4 + 2(x_1\mathrm{c}_4 - x_2\mathrm{s}_4)\partial_5 \\
&\quad + 2\frac{x_1\mathrm{s}_4 + x_2\mathrm{c}_4}{\mathrm{ch}_5}\partial_6 - (\mathrm{s}_5 + 2x_3x_7)\partial_7 \\
&\quad - (\mathrm{c}_5\mathrm{s}_6 + 2x_8x_3)\partial_8 + (\mathrm{c}_5\mathrm{c}_6 - 2x_9x_3)\partial_9 + 2x_3\partial_{10}, \\
D &= x_1\partial_1 + x_2\partial_2 + x_3\partial_3 - x_7\partial_7 - x_8\partial_8 - x_9\partial_9 + \partial_{10}.
\end{aligned}
$$

$R_{\text{generic}}(\mathfrak{c}(2,1))$:

$$
\begin{aligned}
&P_1 = \partial_1, \quad P_2 = \partial_2, \quad P_3 = \partial_3, \quad J_{12} = x_2\partial_1 - x_1\partial_2 + \partial_4, \\
&J_{13} = -x_3\partial_1 - x_1\partial_3 + \mathrm{s}_4\mathrm{t}_5\partial_4 + \mathrm{c}_4\partial_5 + \frac{\mathrm{s}_4}{\mathrm{c}_5}\partial_6, \\
&J_{23} = -x_3\partial_2 - x_2\partial_3 + \mathrm{c}_4\mathrm{t}_5\partial_4 - \mathrm{s}_4\partial_5 + \frac{\mathrm{c}_4}{\mathrm{c}_5}\partial_6, \\
&K_1 = \left(x_1^2 - x_2^2 + x_3^2\right)\partial_1 + 2x_1x_2\partial_2 + 2x_1x_3\partial_3 \\
&\qquad - 2(x_2 + x_3\mathrm{s}_4\mathrm{t}_5)\partial_4 - 2x_3\mathrm{c}_4\partial_5 - 2\frac{x_3\mathrm{s}_4}{\mathrm{c}_5}\partial_6 \\
&\qquad + (\mathrm{c}_4\mathrm{ch}_5 - 2x_1x_7)\partial_7 + (\mathrm{s}_4\mathrm{ch}_6 + \mathrm{c}_4\mathrm{sh}_5\mathrm{sh}_6 - 2x_1x_8)\partial_8 \\
&\qquad + (\mathrm{s}_4\mathrm{sh}_6 + \mathrm{c}_4\mathrm{sh}_5\mathrm{ch}_6 - 2x_1x_9)\partial_9 + 2x_1\partial_{10}, \\
&K_2 = 2x_1x_2\partial_1 + \left(x_2^2 + x_3^2 - x_1^2\right)\partial_2 + 2x_2x_3\partial_3 \\
&\qquad - 2(x_3\mathrm{c}_4\mathrm{t}_5 - x_1)\partial_4 + 2x_3\mathrm{s}_4\partial_5 - 2\frac{x_3\mathrm{c}_4}{\mathrm{c}_5}\partial_6 \\
&\qquad - (2x_2x_7 + \mathrm{s}_4\mathrm{ch}_5)\partial_7 + (\mathrm{c}_4\mathrm{ch}_6 - \mathrm{s}_4\mathrm{sh}_5\mathrm{sh}_6 - 2x_2x_8)\partial_8 \\
&\qquad + (\mathrm{c}_4\mathrm{sh}_6 - \mathrm{s}_4\mathrm{sh}_5\mathrm{ch}_6 - 2x_2x_9)\partial_9 + 2x_2\partial_{10}, \\
&K_3 = -2x_1x_3\partial_1 - 2x_2x_3\partial_2 - \left(x_1^2 + x_2^2 + x_3^2\right)\partial_3
\end{aligned}
$$

$$
\begin{aligned}
&+ 2(\mathrm{t}_5(x_1\mathrm{s}_4 + x_2\mathrm{c}_4)\partial_4 + 2(x_1\mathrm{c}_4 - x_2\mathrm{s}_4)\partial_5 \\
&+ 2\frac{x_1\mathrm{s}_4 + x_2\mathrm{c}_4}{\mathrm{c}_5}\partial_6 + (2x_3x_7 + \mathrm{sh}_5)\partial_7 \\
&+ (\mathrm{ch}_5\mathrm{sh}_6 + 2x_3x_8)\partial_8 + (\mathrm{ch}_5\mathrm{ch}_6 + 2x_3x_9)\partial_9 - 2x_3\partial_{10}, \\
D &= x_1\partial_1 + x_2\partial_2 + x_3\partial_3 - x_7\partial_7 - x_8\partial_8 - x_9\partial_9 + \partial_{10}.
\end{aligned}
$$

5. Connection to the Poincaré Lie algebra. Classical Poincaré algebra $\mathfrak{p}(1,3)$ is ten-dimensional and formed by the operators $\{P_\mu,\ J_{\mu\nu}\}$ with the commutation relations (2) and (3). Extending this set of commutation relations by the following ones

$$
[P_\nu, P_\mu] = \tau J_{\mu\nu}, \quad \tau \in \mathbb{R} \tag{9}
$$

we get the well-defined 10-dimensional Lie algebra which is the deformation $\mathfrak{p}^\tau(1,3)$ of $\mathfrak{p}(1,3)$ to the both de Sitter algebras at the same time. Indeed, for $\tau = 0$ $\mathfrak{p}^\tau(1,3)$ coincides with the Poincaré algebra, for $\tau \geq 0$ $\mathfrak{p}^\tau(1,3) \sim \mathfrak{so}(4,1)$ and for $\tau \leq 0$ $\mathfrak{p}^\tau(1,3) \sim \mathfrak{so}(3,2)$. So, one can construct uniform realizations for the both de Sitter and Poincaré algebras applying the algebraic method to the structure constants from the deformed relations (2), (3) and (9). The inverse connection between de Sitter and Poincaré algebras is provided by standard Inönü–Wigner contraction [6] with respect to the six-dimensional subalgebra $\mathfrak{so}(3,1)$.

The result of the paper can be used for construction of differential invariants and respective invariant differential equations [9].

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Збірник праць Інституту математики НАН України 2019, т. 16, № 1, 113–130

УДК 517.912:512.816

Exact solvability of PDM systems with extended Lie symmetries

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Показано, що усі рівняння Шрьодінгера зі змінним параметром маси, які допускають алгебри інваріантності розмірності більше п'яти (повний список таких рівнянь наведено у роботі [*J. Math. Phys.* **58** (2017), 083508, 16 pp.], є точно розв'язними. Знайдено у явному вигляді відповідні розв'язки та показано їх суперсиметричну природу.

It is shown that all PDM Schrödinger equations admitting more than five-dimensional Lie symmetry algebras (whose completed list can be found in paper [*J. Math. Phys.* **58** (2017), 083508, 16 pp.] are exactly solvable. The corresponding exact solutions are presented. The supersymmetric aspects of the exactly solvable systems are discussed.

1. Introduction. Group classification of differential equations consists in the specification of non-equivalent classes of such equations which possess the same symmetry groups. It is a rather attractive research field which has both fundamental and application values.

A perfect example of group classification of fundamental equations of mathematical physics was presented by Boyer [3] who had specified all inequivalent Schrödinger equations with time independent potentials admitting symmetries with respect to Lie groups, see also [1, 7, 10], where particular important symmetries were discussed, and [14], where the Boyer results were corrected. These old results have a big impact since include a priori information about all symmetry groups which can be admitted by the fundamental equation of quantum mechanics. Let us mention also that the nonlinear Schrödinger equation as well as the generalized Ginsburg–Landau quasilinear equations have been classified also [11, 15] as well as symmetries of more general systems of reaction-diffusion equations [16, 17].

In contrary, the group classification of Schrödinger equations with position dependent mass (PDM) was waited for a very long time. There were many papers devoted to PDM Schrödinger equations with particular symmetries, see, e.g., [5, 8, 20, 21]. But the complete group classification of these equations appears only recently in [18] and [13, 19] for the stationary and time dependent equations correspondingly. A systematic search for the higher order symmetries if the PDM systems started in [12]. So late making of such important job have to cause the blame for experts in group analysis of differential equations, taking into account the fundamental role played by such equations in modern theoretical physics!

Let us remind that the PDM Schrödinger equations are requested for the description of various condensed-matter systems such as semiconductors, quantum liquids, and metal clusters, quantum wells, wires and dots, super-lattice band structures, etc.

It happens that the number of PDM systems with different Lie symmetries is rather extended. Namely, in [13] seventy classes of such systems are specified. Twenty of them are defined up to arbitrary parameters, the remaining fifty systems include arbitrary functions.

The knowledge of all Lie groups which can be admitted by the PDM Schrödinger equations has both fundamental and application values. In particular, when construct the models with a priory requested symmetries we can use the complete lists of inequivalent PDM systems presented in [19] for $d = 2$ and [13] for $d = 3$. Moreover, in many cases a sufficiently extended symmetry induces integrability or exact solvability of the system, and just this aspect will be discussed in the present paper.

It will be shown that all PDM systems admitting six parametric Lie groups of symmetries or more extended symmetries are exactly solvable. Moreover, the complete sets of solutions of the corresponding stationary PDM Schrödinger equations will be presented explicitly.

There exist a tight connection between the complete solvability and various types of higher symmetries and supersymmetries. We will see that extended Lie symmetries also can cause the exact solvability. Moreover, the systems admitting extended Lie symmetries in many cases are supersymmetric and superintegrable.

2. PDM Schrödinger equations with extended Lie symmetries. In [13] we present the group classification of PDM Schrödinger equations

$$L\psi \equiv \left(\mathrm{i}\frac{\partial}{\partial t} - H\right)\psi = 0, \tag{1}$$

where H is the PDM Hamiltonian of the following generic form

$$H = \tfrac{1}{4}\left(m^\alpha p_a m^\beta p_a m^\gamma + m^\gamma p_a m^\beta p_a m^\alpha\right) + \hat{V}, \quad p_a = -\mathrm{i}\frac{\partial}{\partial x_a}. \quad (2)$$

Here $m = m(\mathbf{x})$ and $\hat{V} = \hat{V}(\mathbf{x})$ are the mass and potential depending on spatial variables $\mathbf{x} = (x_1, x_2, x_3)$, and summation with respect to the repeating indices a is imposed over the values $a = 1, 2, 3$. In addition, α, β and γ are the ambiguity parameters satisfying the condition $\alpha+\beta+\gamma = -1$.

The choice of values of the ambiguity parameters can be motivated by physical reasons, see a short discussion of this point in [13].

Hamiltonian (2) can be rewritten in the following more compact form

$$H = \tfrac{1}{2}p_a f p_a + V, \quad (3)$$

where

$$V = \hat{V} + \tfrac{1}{4}(\alpha+\gamma)f_{aa} + \alpha\gamma\frac{f_a f_a}{2f} \quad (4)$$

with $f = \frac{1}{m}$, $f_a = \frac{\partial f}{\partial x_a}$ and $f_{aa} = \Delta f = \frac{\partial f_a}{\partial x_a}$.

In the following text representation (4) will be used.

In accordance with [13] there is a big variety of Hamiltonians (4) generating non-equivalent continuous point symmetries of equation (2). The corresponding potential and mass terms are defined up to arbitrary parameters or even up to arbitrary functions.

In the present paper we consider the PDM systems defined up to arbitrary parameters. Only such systems admit the most extended Lie symmetries. Using the classification results presented in [13, 18] we enumerate these systems in the following Table 1, where $\varphi = \arctan\frac{x_2}{x_1}$ and the other Greek letters denote arbitrary constants parameters, which are supposed not to be zero simultaneously. Moreover, λ and ω are either real or imaginary, the remaining parameters are real.

The symmetry operators presented in column 4 of the table are given by the following formulae

$$P_i = p_i = -\mathrm{i}\frac{\partial}{\partial x_i}, \quad D = x_n p_n - \tfrac{3\mathrm{i}}{2},$$
$$M_{ij} = x_i p_j - x_j p_j, \quad M_{0i} = \tfrac{1}{2}\left(K^i + P_i\right), \quad M_{4i} = \tfrac{1}{2}\left(K^i + P_i\right),$$

$$B_1^1 = \lambda \sin(\lambda t) M_{12}\big(\lambda^2\varphi + \nu\big)\cos(\lambda t), \quad B_2^1 = \frac{\partial}{\partial t} B_1^1,$$

$$B_1^2 = \sin(\lambda t) D - \cos(\lambda t)\left(\lambda \ln(r) + \tfrac{\nu}{\lambda}\right), \quad B_2^2 = \frac{\partial}{\partial t} B_1^2,$$

$$N_1^1 = \omega \cos(\omega\sigma t) L_3 - \sin(\omega\sigma t)\big(\mathrm{i}\partial_t - \omega^2 \mathrm{e}^{-\sigma\Theta}\big), \quad N_2^1 = \frac{\partial}{\partial t} N_1^1,$$

$$N_1^2 = \omega \cos(\omega\sigma t) D + \sin(\omega\sigma t)\big(\mathrm{i}\partial_t - \omega^2 r^{-\sigma}\big), \quad N_2^2 = \frac{\partial}{\partial t} N_1^1, \qquad (5)$$

where $K_i = x_n x_n p_i - 2x_i D$ and indices i, j, k, n take the values 1, 2, 3.

Rather surprisingly, all systems (except ones given in items 4 and 5) presented in Table 1 are exactly solvable. In the following sections we present their exact solutions. To obtain these solutions we use some nice properties of the considered systems like superintegrability and supersymmetry with shape invariance. Let us remind that the quantum mechanical system is called superintegrable if it admits more integrals of motion than its number of degrees of freedom.

In accordance with Table 1 we can indicate 11 inequivalent PDM systems which are defined up to arbitrary parameters and admit Lie symmetry algebras of dimension five or higher. Notice that the systems fixed in items 4 and 5 admit five dimension symmetry algebras while the remaining systems admit more extended symmetries.

3. Systems with fixed mass and potentials. Firstly we consider those systems whose mass and potential terms are fixed, i.e., do not include arbitrary parameters. These systems are presented in items 1, 2 of Table 1 and others provided the mass does not depends on parameters and parameters of the potential are trivial.

3.1. System invariant with respect to algebra $\mathfrak{so}(4)$. Consider Hamiltonian (3) with functions f and V presented in item 1 of Table 1:

$$H = \tfrac{1}{2} p_a \big(1 + r^2\big)^2 p_a - 3r^2. \qquad (6)$$

The eigenvalue problem for this Hamiltonian can be written in the following form

$$H\psi = 2E\psi, \qquad (7)$$

where E are yet unknown numbers.

Equation (7) admits six integrals of motion M_{AB}, $A, B = 1, 2, 3, 4$, presented in equation (5). Let us write them explicitly

$$M^{ab} = x^a p^b - x^b p^a, \quad M^{4a} = \tfrac{1}{2}\big(r^2 - 1\big) p^a - x^a x^b p^b + \tfrac{3\mathrm{i}}{2} x^a. \qquad (8)$$

Table 4. PDM systems with extended Lie symmetries.

no.	inverse mass f	potential V	symmetries
1	$\left(r^2+1\right)^2$	$-3r^2$	M_{41}, M_{42}, M_{43}, M_{21}, M_{31}, M_{32}
2	$\left(r^2-1\right)^2$	$-3r^2$	M_{01}, M_{02}, M_{03}, M_{21}, M_{31}, M_{32}
3	x_3^2	$\nu\ln(x_3)$	$P_1,\ P_2,\ M_{12},\ D+\nu t$
4	$\tilde{r}^3$	$\kappa x_3+\lambda\tilde{r}$	$P_3+\kappa t,\ D+\mathrm{i}t\partial_t,\ M_{12}$
5	x_1^3	$\lambda x_1+\kappa x_3$	$P_3+\kappa t,\ P_2,\ D+\mathrm{i}t\partial_t$
6	$x_3^{\sigma+2}$	κx_3^σ	$P_1,\ P_2,\ M_{12},\ D+\mathrm{i}\sigma t\partial_t$, $\sigma\neq 0,1,-2$
7	$\tilde{r}^{\sigma+2}\mathrm{e}^{\lambda\varphi}$	$\kappa\tilde{r}^\sigma\mathrm{e}^{\lambda\varphi}$	$M_{12}+\mathrm{i}\lambda t\partial_t,\ P_3$, $D+\mathrm{i}\sigma t\partial_t,\ \sigma\neq 0$
8	$\tilde{r}^2$	$\frac{\lambda^2}{2}\varphi^2+\mu\varphi+\nu\ln(\tilde{r})$	$B_1^1,\ B_2^1,\ D+\nu t,\ P_3$
9	$\tilde{r}^2\mathrm{e}^{\sigma\varphi}$	$\kappa\mathrm{e}^{\sigma\varphi}+\frac{\omega^2}{2}\mathrm{e}^{-\sigma\varphi}$	$N_1^1,\ N_2^1, P_3,\ D,\ K_3$
10	r^2	$\nu\ln(r)+\frac{\lambda^2}{2}\ln(r)^2$	$B_1^2,\ B_2^2,\ L_1,\ L_2,\ L_3$
11	$r^{2+\sigma}$	$\kappa r^\sigma+\frac{\omega^2}{2}r^{-\sigma}$	$N_1^2,\ N_2^2,\ L_1,\ L_2,\ L_3$

Operators (8) form a basis of algebra $\mathfrak{so}(4)$. Moreover, the first Casimir operator of this algebra is proportional to Hamiltonian (6) up to the constant shift

$$C_1=\tfrac{1}{2}M_{AB}M_{AB}=\tfrac{1}{2}(H-9),$$

while the second Casimir operator $C_2=\varepsilon_{ABCD}M_{AB}M_{CD}$ appears to be zero.

Thus like the Hydrogen atom system (7) admits six integrals of motion belonging to algebra $\mathfrak{so}(4)$ and is maximally superintegrable.

Using our knowledge of unitary representations of algebra $\mathfrak{so}(4)$ is possible to find eigenvalues E algebraically

$$E=4n^2+5, \tag{9}$$

where $n=0,1,2,\dots$ are natural numbers.

To find the eigenvectors of Hamiltonian (6) corresponding to eigenvalues (9) we use the rotation invariance of (7) and separate variables. Introducing spherical variables and expanding solutions via spherical functions

$$\psi = \frac{1}{r}\sum_{l,m}\phi_{lm}(r)Y^l_m, \tag{10}$$

we come to the following equations for radial functions

$$\left(-\left(r^2+1\right)^2\left(\frac{\partial^2}{\partial r^2}-\frac{l(l+1)}{r^2}\right)-4r\left(r^2+1\right)\frac{\partial}{\partial r}-2r^2\right)\varphi_{lm}$$
$$=\left(4n^2+1\right)\varphi_{lm},$$

where $l=0,1,2,\dots$ are parameters numerating eigenvalues of the squared orbital momentum. The square integrable solutions of these equations are

$$\varphi_{lm} = C^n_{lm}\left(r^2+1\right)^{-n-\frac{1}{2}}r^{l+1}\mathcal{F}\left([A,B],[C]-r^2\right), \tag{11}$$

where

$$A=-n+l+1,\quad B=-n+\tfrac{1}{2},\quad C=l+\tfrac{3}{2}.$$

$\mathcal{F}(\cdots)$ is the hypergeometric function and C^n_{lm} are integration constants. Solutions (11) tend to zero at infinity provided n is a natural number and $l\le n-1$.

Thus the system (7) is maximally superintegrable and exactly solvable.

3.2. System invariant with respect to algebra $\mathfrak{so}(1,3)$. The next Hamiltonian we consider corresponds to functions f and V presented in item 2 of Table 1. The related eigenvalue problem includes the following equation

$$H\psi \equiv -\tfrac{1}{2}\left(\partial_a\left(1-r^2\right)^2\partial_a+6r^2\right)\psi = E\psi. \tag{12}$$

Equation (12) admits six integrals of motion $M_{\mu\nu}$, $\mu,\nu=0,1,2,3$, given by equation (5), which can be written explicitly in the following form

$$M_{ab} = x^ap^b - x^bp^a,$$
$$M_{0a} = \tfrac{1}{2}\left(r^2+1\right)p^a - x^ax^bp^b + \tfrac{3\mathrm{i}}{2}x^a,\quad a,b=1,2,3. \tag{13}$$

These operators form a basis of algebra $\mathfrak{so}(1,3)$, i.e., the Lie algebra of Lorentz group.

As in the previous section, the corresponding first Casimir operator is expressed via the Hamiltonian, namely

$$C_1 = \tfrac{1}{2}M^{ab}M^{ab} - M^{0a}M^{0a} = \tfrac{1}{2}(H+9), \tag{14}$$

while the second one appears to be zero.

Using our knowledge of irreducible unitary representations of Lorentz group we find eigenvalues of C_1 and C_2 in the form [2, 9]:

$$c_1 = 1 - j_0^2 - j_1^2, \quad c_2 = 2\mathrm{i} j_0 j_1,$$

where j_0 and j_1 are quantum numbers labeling irreducible representations. Since the second Casimir operator C_2 is trivial, we have $c_1 = j_0 = 0$. So there are two possibilities [9]: either j_1 is an arbitrary imaginary number, and the corresponding representation belongs to the principal series, or j_1 is a real number satisfying $|j_1| \le 1$, and we come to the subsidiary series of IRs. So

$$j_1 = \mathrm{i}\lambda, \quad c_1 = 1 - j_1^2 = \lambda^2 + 1, \tag{15}$$

where λ is an arbitrary real number, or, alternatively,

$$0 \le j_1 \le 1, \quad c_1 = 1 - j_1^2. \tag{16}$$

In accordance with (14) the related eigenvalues E in (12) are

$$E = -5 - j_1^2. \tag{17}$$

In view of the rotational invariance of equation (12) it is convenient to represent solutions in form (10). As a result we obtain the following radial equations

$$\begin{aligned}\left(-\left(r^2-1\right)^2\left(\frac{\partial^2}{\partial r^2} - \frac{l(l+1)}{r^2}\right) - 4r\left(r^2-1\right)\frac{\partial}{\partial r} - 2r^2\right)\varphi_{lm} \\ = (\tilde{E}+4)\varphi_{lm}.\end{aligned} \tag{18}$$

The general solution of (18) is

$$\varphi_{lm} = \left(1-r^2\right)^{-\frac{1}{2}-k}\left(C_{lm}^k r^{l+1}\mathcal{F}\left([A,B],[C],r^2\right)\right.$$

$$+\tilde{C}^k_{lm} r^{-l}\mathcal{F}\big([\tilde{A},\tilde{B}],[\tilde{C}],r^2\big)\big), \tag{19}$$

where

$$A=-k+l+1,\quad B=-k+\tfrac{1}{2},\quad C=l+\tfrac{3}{2},$$
$$\tilde{A}=-k-l,\quad \tilde{B}=-k+\tfrac{1}{2},\quad \tilde{C}=\tfrac{1}{2}-l,\quad k=\tfrac{1}{2}\sqrt{-\tilde{E}-5}$$

and is singular at $r=1$. However, for $\tilde{C}^k_{lm}=0$ and $k=j_1$ the solutions are normalizable in some specific metric [18].

Thus the system presented in item 7 of Table 1 is exactly solvable too. The corresponding eigenvalues and eigenvectors are given by equations (15), (16), (17) and (19), respectively.

3.3. Scale invariant systems. Consider one more PDM system which is presented in item 3 of the table and includes the following Hamiltonian: Let us note that the free fall effective potential appears also one more system specified in Table 1. Thus, considering the inverse mass and potential specified in item 3 we come to the following Hamiltonian

$$H=-\frac{1}{2}\left(x_3\frac{\partial}{\partial x_3}x_3\frac{\partial}{\partial x_3}+x_3\frac{\partial}{\partial x_3}+x_3^2\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)\right)$$
$$+\nu\ln(x_3). \tag{20}$$

Equation (12) with Hamiltonian given in (20) can be easily solved by separation of variables in Cartesian coordinates. Expanding the wave function ψ via eigenfunctions of integrals of motion P_1 and P_2:

$$\psi=\exp(-\mathrm{i}(k_1x_1+k_2x_2))\Phi(k_1,k_2,x_3) \tag{21}$$

and introducing new variable $y=\ln(x_3)$ we come to the following equation for $\Phi=\Phi(k_1,k_2,x_3)$:

$$-\frac{\partial^2\Phi}{\partial y^2}+\left(\left(k_1^2+k_2^2\right)\exp(2y)+2\nu y\right)\Phi=\tilde{E}\Phi, \tag{22}$$

where $\tilde{E}=2E-\frac{1}{4}$.

Here we consider the simplest version of equation (22) when parameter ν is trivial

$$-\frac{\partial^2\Phi}{\partial y^2}+\left(k_1^2+k_2^2\right)\exp(2y)\Phi=\tilde{E}\Phi. \tag{23}$$

This equation is scale invariant and can be easily solved. Its square integrable solutions are given by Bessel functions

$$\Psi = C^E_{k_1k_2} K_{\mathrm{i}\sqrt{\tilde{E}}}\Big(\sqrt{k_1^2+k_2^2}\,\ln(x_3)\Big),$$

where $C^E_{k_1k_2}$ are integration constants and $\tilde{E}$ are arbitrary real parameters.

It is interesting to note that there are rather non-trivial relations between the results given in the present and previous sections. Equation (23) admits six integrals of motion which are nothing but the following operators

$$P_1,\ P_2,\ K_1,\ K_2,\ M_{12},\ D, \tag{24}$$

which are presented in equations (5).

Like operators (13) integrals of motion (24) form a basis of the Lie algebra of Lorentz group, and we again can find the eigenvalues of Hamiltonian (23) algebraically by direct analogy with the above. We will not present this routine procedure since there exist strong equivalence relations between Hamiltonians (23) with zero ν and (6). To find them we note that basis (24) is equivalent to the following linear combinations of the basis elements

$$M_{01},\ M_{02}, M_{04},\ M_{41}\ M_{42},\ M_{12}, \tag{25}$$

whose expressions via operators (24) are given by equation (5). To reduce (25) to the set (13) it is sufficient to change subindices 4 to 3, i.e., to make the rotation in the plane 43. The infinitesimal operator for such rotation is given by the following operator

$$M_{43} = \tfrac{1}{2}(K_3+P_3) = \tfrac{1}{2}\big(r^2-1\big)p_3 - x_3x_bp_b + \tfrac{3\mathrm{i}}{2}x_3,$$

which belongs to the equivalence group of equations. Solving the corresponding Lie equations and choosing the group parameter be equal $\frac{\pi}{2}$ we easily find the requested equivalence transformations.

One more scale invariant system is presented in item 8 where all parameters of potential are zero. The relation Hamiltonian looks as follows

$$H = -\tilde{r}\frac{\partial}{\partial x_\alpha}\tilde{r}\frac{\partial}{\partial x_\alpha} - x_\alpha\frac{\partial}{\partial x_\alpha} - \tilde{r}^2\frac{\partial^2}{\partial x_3^2}, \quad \alpha = 1, 2. \tag{26}$$

Considering the eigenvalue problem for (26) it is convenient to use the cylindrical variables

$$\tilde{r} = \sqrt{x_1^2 + x_2^2}, \quad \varphi = \arctan\frac{x_2}{x_1}, \quad x_3 = z \tag{27}$$

and expand solutions via eigenfunctions of M^{12} and $P_3 = -\mathrm{i}\frac{\partial}{\partial z}$:

$$\Psi = \exp[\mathrm{i}(\kappa\varphi + \omega z)]\Phi_{\kappa\omega}(\tilde{r}), \quad \kappa = 0, \pm 1, \pm 2, \ldots, \quad -\infty < \omega < \infty.$$

As a result we come to the following equations for radial functions $\Phi = \Phi_{\kappa\omega}(\tilde{r})$:

$$-\left(\tilde{r}\frac{\partial}{\partial\tilde{r}}\tilde{r}\frac{\partial}{\partial\tilde{r}} + \tilde{r}\frac{\partial}{\partial\tilde{r}} + \omega^2\right)\Phi = \left(\tilde{E} - \kappa^2\right)\Phi.$$

Square integrable (with the weight $\tilde{r}$) solutions of this equation are

$$\Phi_{\kappa\omega} = \frac{1}{\tilde{r}}J_\alpha(\omega\tilde{r}), \quad \alpha = \kappa^2 + 1 - \tilde{E}, \tag{28}$$

where $J_\alpha(\omega\tilde{r})$ is Bessel function of the first kind. Functions (28) are normalizable and disappear at $\tilde{r} = 0$ provided $\alpha \leq 0$. The rescaled energies $\tilde{E}$ continuously take the values $\kappa^2 \leq \tilde{E} \leq \infty$.

The last scale invariant system which we have to consider is fixed in item 10 where $\nu = \lambda = 0$. We will do it later in the end of the following section.

4. Systems defined up to arbitrary parameters. In previous section we present exact solutions for systems with fixed potential and mass terms. In the following we deal with the systems defined up to arbitrary parameters.

4.1. The system with oscillator effective potential. Let us consider equation (1) with f and V are functions fixed in item 10 of Table 1, i.e.,

$$\mathrm{i}\frac{\partial\psi}{\partial t} = \left(-\frac{1}{2}\frac{\partial}{\partial x_a}r^2\frac{\partial}{\partial x_a} + \nu\ln(r) + \frac{\lambda^2}{2}\ln(r)^2\right)\psi.$$

These equations admit extended Lie symmetries (whose generators are indicated in the table) being invariant with respect to six-parametrical Lie group. Let us show that they also admit hidden supersymmetries.

In view of the rotational invariance and symmetry of the considered equations with respect to shifts of time variable, it is reasonable to search for their solutions in spherical variables, i.e., in the following form

$$\Psi = \mathrm{e}^{-iEt} R_{lm}(r) Y_{lm}(\varphi, \theta), \tag{29}$$

where φ and θ are angular variables and $Y_{lm}(\varphi, \varphi)$ are spherical functions, i.e., eigenvectors of $L^2 = L_1^2 + L_2^2 + M_{12}^2$ and M_{12}. As a result we come to the following radial equations

$$\left(-r\frac{\partial R_{lm}}{\partial r} r \frac{\partial R_{lm}}{\partial r} - r\frac{\partial R_{lm}}{\partial r} + l(l+1) + \nu \ln(r) + \frac{\lambda^2}{2} \ln(r)^2\right) R_{lm} = 2ER_{lm}. \tag{30}$$

Introducing new variable $y = \sqrt{2}\ln(r)$ we can rewrite equation (30) in the following form

$$\left(-\frac{\partial^2}{\partial y^2} + l(l+1) + \nu y + \frac{\lambda^2}{2} y^2\right) R_{lm}(y) = \tilde{E} R_{lm}(y), \tag{31}$$

where $\tilde{E} = E - \frac{1}{4}$.

Let $\lambda \neq 0$ then equation (31) is reduced to the 1D harmonic oscillator up to the additional term $l(l+1)$. The admissible eigenvalues $\tilde{E}$ are given by the following formula

$$\tilde{E} = n + l(l+1),$$

where n is a natural number. The corresponding eigenfunctions are well known and we will not presented them here. The same is true for supersymmetric aspects of the considered system.

If parameter λ is equal to zero then (31) reduces to equation with free fall potential slightly modified by the term $l(l+1)$. The corresponding solutions can be found in textbooks devoted to quantum mechanics. If both parameters ν and λ are zero, equation (31) is solved by trigonometric or hyperbolic functions. The corresponding PDM Schrödinger equation is scale invariant, i.e., belongs to the class considered in the previous section.

4.2. The systems with potentials equivalent to 3d oscillator. Consider now the system represented in item 11 of the table. The corresponding equation (1) takes the following form

$$\mathrm{i}\frac{\partial\psi}{\partial t}=\left(-\frac{1}{2}\partial_a r^{\sigma+2}\partial_a+\kappa r^{2\sigma}+\frac{\omega^2}{r^{2\sigma}}\right)\psi. \tag{32}$$

Like in previous section we represent the wave function in the form given in (29) and came to the following radial equation

$$-r^{2\sigma+2}\frac{\partial^2 R_{lm}}{\partial r^2}-(2\sigma+4)r^{2\sigma+1}\frac{\partial R_{lm}}{\partial r} + \left(r^{2\sigma}(l(l+1)+\kappa)+\omega^2 r^{-2\sigma}\right)R_{lm}=2ER_{lm}. \tag{33}$$

Using the Liouville transform

$$r\to z=r^{-\sigma},\quad R_{lm}\to \tilde{R}_{lm}=z^{\frac{\sigma+3}{2\sigma}}R_{lm},$$

we reduce (33) to the following form

$$-\sigma^2\frac{\partial^2\tilde{R}_{lm}}{\partial z^2}+\left(\frac{l(l+1)+\delta}{z^2}+\omega^2 z^2\right)\tilde{R}_{lm}=2E\tilde{R}_{lm}, \tag{34}$$

where $\delta=\frac{3}{4}(\sigma+1)(\sigma+3)+2\kappa$.

Equation (34) describes a deformed 3d harmonic oscillator including two deformation parameters, namely, σ and κ.

Let

$$2\kappa=-\sigma^2-3\sigma-2,$$

then equation (34) is reduced to the following form

$$H_l\tilde{R}_{lm}\equiv\left(-\sigma^2\frac{\partial^2}{\partial z^2}+\frac{(2l+1)^2-\sigma^2}{4z^2}+\omega^2 z^2\right)\tilde{R}_{lm}=2E\tilde{R}_{lm}. \tag{35}$$

Equation (35) is shape invariant. Hamiltonian H_r can be factorized

$$H_l=a_l^+a_l-C_l, \tag{36}$$

where

$$a=-\sigma\frac{\partial}{\partial z}+W,\quad a^+=\sigma\frac{\partial}{\partial z}+W,$$
$$W=\frac{2l+1+\sigma}{2z}+\omega z,\quad C_l=\omega(2l+2\sigma+1).$$

The superpartner $\hat{H}_l$ of Hamiltonian (36) has the following property

$$\hat{H}_l \equiv a_l a_l^+ + C_l = H_{l+\sigma} + C_l.$$

Thus our Hamiltonian is shape invariant.

Thus to solve equation (35) we can use the standard tools of SUSY quantum mechanics and find the admissible eigenvalues in the following form

$$E_n = \omega\left(2n\sigma + l + \sigma + \tfrac{1}{2}\right) = \omega\left(2n + l + \tfrac{3}{2}\right) + \delta\omega(2n+1), \qquad (37)$$

where $\delta = \sigma - 1$.

Equation (37) represents the spectrum of 3d isotropic harmonic oscillator deformed by the term proportional to δ.

For equation (34) we obtain in the analogous way

$$E_n = \frac{\omega}{2}\left(\sigma(2n+1) + \sqrt{(2l+1)^2 + \tilde{\kappa}}\right), \qquad (38)$$

where $\tilde{\kappa} = 8(\kappa+1) + \sigma(\sigma+3)$. The related eigenvectors are expressed via the confluent hypergeometric functions $\mathcal{F}$:

$$R_n = \mathrm{e}^{-\frac{\omega r^\sigma}{2\sigma}} r^{\sigma n - \frac{E_n}{\omega}} \mathcal{F}\left(-n, \frac{E_n}{\sigma\omega} - n, \frac{\omega}{\sigma} r^{-\sigma}\right),$$

where n is integer and E_n is eigenvalue (38).

4.3. System with angular oscillator potential. The next system which we consider is specified by the inverse mass and potential presented in item 8 of the table. The corresponding Hamiltonian is

$$H = p_a r^2 p_a + \frac{\lambda^2}{2}\varphi^2 + \sigma\varphi + \nu\ln(\tilde{r}).$$

The corresponding eigenvalue equation is separable in cylindrical variables, thus it is reasonable to represent the wave function as follows

$$\psi = \Psi(\tilde{r})\Phi(\varphi)\exp(-\mathrm{i}kx_3). \qquad (39)$$

As a result we obtain the following equations for radial and angular variables

$$\left(-\tilde{r}\partial_{\tilde{r}}\tilde{r}\partial_{\tilde{r}} - \tilde{r}\partial_{\tilde{r}} + \nu\ln(\tilde{r}) + k^2\tilde{r}^2 - \mu\right)\Psi(\tilde{r}) = 0 \qquad (40)$$

and

$$\left(-\frac{\partial^2}{\partial\varphi^2}+\frac{\lambda^2}{2}\varphi^2+\sigma\varphi-\mu\right)\Phi(\varphi)=0, \tag{41}$$

where μ is a separation constant.

For λ nonzero equation (41) is equivalent to the Harmonic oscillator. The specificity of this system is that, in contrast with (31), it includes angular variable φ whose origin is

$$0\le\varphi\le 2\pi. \tag{42}$$

For trivial λ our equation (41) is reduced to equation with free fall potential, but again for the angular variable satisfying (42).

The radial equation (40) is simple solvable too. In the case $k=0$ we again come to the free fall potential.

4.4. Systems with Morse effective potential. The next system we consider is specified by the inverse mass and potentials represented in item 9 of Table 1. The corresponding Hamiltonian is

$$H=-\frac{\partial}{\partial x_a}\tilde{r}^2\mathrm{e}^{\sigma\varphi}\frac{\partial}{\partial x_a}+\kappa\mathrm{e}^{\sigma\varphi}+\frac{\omega^2}{2}\mathrm{e}^{-\sigma\varphi}.$$

Introducing again the cylindric variables and representing the wave function in the form (39) we come to the following equations for the radial and angular variables

$$\left(-\left(\frac{\partial^2}{\partial y^2}+\frac{\partial}{\partial y}\right)+\mu+k^2\mathrm{e}^{2y}\right)\Psi(\tilde{r})=\mu\Psi(\tilde{r})$$

and

$$\left(-\mathrm{e}^{\sigma\varphi}\left(\frac{\partial^2}{\partial\varphi^2}+\kappa-\mu\right)+\frac{\omega^2}{2}\mathrm{e}^{-\sigma\varphi}\right)\Phi(\varphi)=\tilde{E}\Phi(\varphi). \tag{43}$$

Dividing all terms in (43) by $\exp(\sigma\varphi)$ we obtain the following equation

$$\left(-\left(\frac{\partial^2}{\partial\varphi^2}+\kappa-\mu\right)+\frac{\omega^2}{2}\mathrm{e}^{-2\sigma\varphi}\right)\Phi(\varphi)=\mathrm{e}^{-\sigma\varphi}\tilde{E}\Phi(\varphi).$$

or

$$\left(-\left(\frac{\partial^2}{\partial\varphi^2}\right)+\frac{\omega^2}{2}\mathrm{e}^{-2\sigma\varphi}-\tilde{E}\mathrm{e}^{-\sigma\varphi}\right)\Phi(\varphi)=\hat{E}\Phi(\varphi), \tag{44}$$

where we denote $\hat{E}=\mu-\kappa$.

Formula (44) represents the Schrödinger equation with Morse potential. This equation is shape invariant and also can be solved using tools of SUSY quantum mechanics. We demonstrate this procedure using another system.

Considering the mass and potential presented in item 6 of Table 1 we come to the following Hamiltonian

$$H = \tfrac{1}{2} p_a x_3^{\sigma+2} p_a + \kappa x_3^{\sigma}.$$

Equation (12) with Hamiltonian (20) can be solved by separation of variables in Cartesian coordinates. Expanding the wave function ψ via eigenfunctions of integrals of motion P_1 and P_2 in the form (21) and introducing new variable $y = \ln(x_3)$ we reduce the problem to the following equation for $\Phi(k_1, k_2, x_3)$:

$$\left(-\frac{\partial}{\partial x_3} x_3^{\sigma+2} \frac{\partial}{\partial x_3} + x_3^{\sigma+2} k^2 + 2\kappa x_3^{\sigma}\right) \Phi = 2E\Phi, \tag{45}$$

where $k^2 = k_1^2 + k_2^2$.

Dividing all terms in (45) by x_3^{σ} we can rewrite it in the following form

$$\left(-\frac{\partial^2}{\partial y^2} - (\sigma+1)\frac{\partial}{\partial y} - 2E\exp(-\sigma y) + k^2 \exp(2y) + 2\kappa\right) \Phi = 0.$$

In the particular case $\sigma = 2$ we again come to the equation with Morse effective potential.

One more system which can be related to Morse potential is represented in item 7 and include the following Hamiltonian

$$H = \tfrac{1}{2} p_a \exp(\lambda\varphi) \tilde{r}^{\sigma+2} p_a + \nu \exp(\lambda\varphi) \tilde{r}^{\sigma}.$$

The corresponding equation (12) is separable in the cylindrical variables (27) provided $\sigma \cdot \lambda = 0$ and again includes the Morse effective potential.

Let us return to equation (33) and solve it using approach analogous to the presented above. In other words, we will change the roles of eigenvalues and coupling constants.

First we divide all terms in (33) by $r^{2\sigma}$ and obtain

$$-r^2 \frac{\partial^2 R_{lm}}{\partial r^2} - (2\sigma+4) r \frac{\partial R_{lm}}{\partial r}$$

$$+\left(\omega^2 r^{-4\sigma}+\mu r^{-2\sigma}\right)R_{lm}=\varepsilon R_{lm}, \tag{46}$$

where

$$\varepsilon=-l(l+1)-2\kappa, \quad \mu=-2E. \tag{47}$$

Applying the Liouville transform

$$r\to\rho=\ln(r), \quad R_{lm}\to\tilde{R}_{lm}=\mathrm{e}^{-\frac{\sigma+3}{2}}R_{lm}$$

we reduce (46) to a more compact form

$$H_\nu\tilde{R}_{lm}\equiv\left(-\frac{\partial^2}{\partial\rho^2}+\omega^2\mathrm{e}^{-2\sigma\rho}+(2\omega\nu+\omega\sigma)\mathrm{e}^{-\sigma\rho}\right)\tilde{R}_{lm}=\hat{\varepsilon}\tilde{R}_{lm}, \tag{48}$$

where

$$\hat{\varepsilon}=\varepsilon-\left(\frac{\sigma+3}{2}\right)^2, \quad \nu=\frac{\mu}{2\omega}-\frac{\sigma}{2}. \tag{49}$$

Like (44) equation (48) includes the familiar Morse potential and so is shape invariant. Indeed, denoting $\mu=2\omega(\nu+\frac{\sigma}{2})$ we can factorize Hamiltonian H_ν like it was done in (36) where index l should be changed to ν and

$$W=\nu-\omega\mathrm{e}^{-a\rho}, \quad C_\nu=\nu^2$$

and the shape invariance is easy recognized.

To find the admissible eigenvalues ε and the corresponding eigenvectors we can directly use the results presented in [4], see item 4 of Table 4.1 there

$$\hat{\varepsilon}=\hat{\varepsilon}_n=-(\nu-n\sigma)^2, \quad \left(\tilde{R}_{lm}\right)_n=y^{\frac{\nu}{\sigma}-n}\mathrm{e}^{-\frac{y}{2}}L_n^{2(\frac{\nu}{\sigma}-n)}(y),$$

where $y=\frac{2\omega}{\sigma}r^{-\sigma}$.

Thus we find the admissible values of $\hat{\varepsilon}_n$. Using definitions (47) and (49) we can find the corresponding values of E which are in perfect accordance with (38).

Discussion. The results presented above in Section 2 include the complete list of continuous symmetries which can be admitted by PDM Schrödinger equations, provided these equations are defined up to arbitrary parameters. All such systems appear to be exactly solvable.

It is important to note that the list of symmetries presented in the fourth column of the table is valid only for the case of nonzero parameters defined the potential and mass terms. If some (or all) of these parameters are trivial, the corresponding PDM Schrödinger equation can have more extended set of symmetries. For example, it is the case for the potential and PDM presented in item 3 of the table, compare the list of symmetries presented in column 4 with (24). The completed list of non-equivalent symmetries can be found in [13] which generalizes the Boyer results [3] to the case of PDM Schrödinger equations. As other extensions of results of [3] we can mention the group classification of the nonlinear Schrödinger equations [15] and the analysis of its conditional symmetries [6].

Thanks to their extended symmetries the majority of the presented systems is exactly solvable. In Sections 3 and 4 we present the corresponding solutions explicitly and discuss supersymmetric aspects of some of them. However, two of the presented systems (whose mass and potential are presented in items 4 and 5 of Table 1) are not separable, if both arbitrary parameters κ and λ are nonzero. And just these systems have the most small symmetry. On the other hand, all systems admitting six- or higher-dimensional Lie symmetry algebras are separable and exactly solvable.

In addition to the symmetry under the six parameter Lie group, equation (32) (which we call deformed 3d isotropic harmonic oscillator) possesses a hidden dynamical symmetry with respect to group $\mathrm{SO}(1,2)$. The effective radial Hamiltonian is shape invariant, and its eigenvalues can be found algebraically. In spite on the qualitative difference of its spectra (37) and (38) of the standard 3d oscillator, it keeps the main supersymmetric properties of the latter. We show that the shape invariance of PDM problems usually attends their extended symmetries.

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Збірник праць Інституту математики НАН України 2019, т. 16, № 1, 131–154

УДК 517.912:512.816

Equivalence groupoid of a class of general Burgers–Korteweg–de Vries equations with space-dependent coefficients

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Описано групоїд еквівалентності класу загальних рівнянь Бюргерса–Кортевега–де Фріза з просторовими коефіцієнтами. Показано, що цей клас зводиться сім'єю перетворень еквівалентності до свого підкласу з чотиривимірною звичайною групою еквівалентності. Прокласифіковано допустимі перетворення цього підкласу та виокремлені підкласи, що допускають максимальні нетривіальні умовні групи еквівалентності. Виявляється, що всі вони мають розмірність більшу за чотири. Зокрема, знайдено декілька нових класів диференціальних рівнянь, нормалізованих в узагальненому сенсі. Жоден з них не допускає єдину ефективну узагальнену групу еквівалентності.

We describe the equivalence groupoid of the class of general Burgers–Korteweg–de Vries equations with space-dependent coefficients. This class is shown to reduce by a family of equivalence transformations to a subclass with a four-dimensional usual equivalence group. Classified are admissible transformations of this subclass and singled out its subclasses admitting maximal nontrivial conditional equivalence groups. All of them turn out to have dimension higher than four. In particular, few new examples of nontrivial cases of normalization in the generalized sense of classes of differential equations appeared this way. Neither of classes discussed possesses a unique effective generalized equivalence group.

1. Introduction. A number of evolution equations that are important in mathematical physics are of the general form

$$u_t + C(t,x)uu_x = \sum_{k=0}^{r} A^k(t,x)u_k + B(t,x). \tag{1}$$

In particular, this includes Burgers, Korteweg–de Vries (KdV), Kuramoto–Sivashinsky, Kawahara, and generalized Burgers–KdV equations.

Here and in the following the integer parameter r is fixed, and $r \geqslant 2$. We require the condition $CA^r \neq 0$ guaranteeing that equations from the class (1) are nonlinear and of genuine order r. Throughout the paper we use the standard index derivative notation $u_t = \partial u/\partial t$, $u_k = \partial^k u/\partial x^k$.

The class (1) and its various subclasses were subject to studying from the symmetry analysis point of view, see [6] for an extensive list of references. Recently, the class (1) became a source of examples of nontrivial equivalence groups [6]. In fact, the first examples of classes with generalized and extended generalized equivalence groups are of the form (1) (with some additional restrictions). Moreover, detailed studying thereof allowed the authors to introduce the concept of an effective generalized equivalence group of a class of differential equations. Furthermore, the structure of this class is so flexible, that a “reasonable” singled out subclass thereof is likely to possess normalization properties in some sense. Nonetheless, it is not the case for a subclass $\bar{\mathcal{F}}$ of equations with the arbitrary elements being time-independent,

$$u_t + C(x)uu_x = \sum_{k=0}^{r} A^k(x)u_k + B(x), \quad \text{where} \quad A^rC \neq 0. \tag{2}$$

The aim of this paper is to thoroughly study admissible transformations of the class $\bar{\mathcal{F}}$. In a nutshell, the results of this paper comprise the following four facts. Any equation in $\bar{\mathcal{F}}$ is mapped by an equivalence transformation of $\bar{\mathcal{F}}$ to an equation in the subclass $\mathcal{F}$ of reduced general Burgers–Korteweg–de Vries equations with space-dependent coefficients, singled out by conditions $C = 1$ and $A^1 = 0$. The subclass $\mathcal{F}$ is not normalized in any sense, and its usual equivalence group is four-dimensional. Classified are admissible transformations of the class $\mathcal{F}$ and singled out are its subclasses admitting maximal nontrivial conditional equivalence subgroups of the equivalence group of $\mathcal{F}$,

$$\begin{aligned}\hat{\mathcal{F}}_{\mathrm{I},1}\colon \quad u_t + uu_x &= \left(\frac{\alpha+2}{a_{01}}b_1 + a_{01}|x+\beta|^\alpha\right)u \\ &\quad + (x+\beta)\left(b_2|x+\beta|^{2\alpha} + b_1|x+\beta|^\alpha - \frac{b_1^2(\alpha+1)}{a_{01}^2}\right) \\ &\quad + \sum_{j=2}^{r} a_j(x+\beta)^j|x+\beta|^\alpha u_j \quad \text{with} \quad \alpha a_r a_{01} \neq 0,\end{aligned}$$

$$\hat{\mathcal F}_{\mathrm{I},01}\colon\quad u_t+uu_x=\sum_{j=2}^{r}a_j(x+\beta)^j|x+\beta|^\alpha u_j+a_{00}u$$
$$+(x+\beta)\left(b_2|x+\beta|^{2\alpha}-\frac{\alpha+1}{(\alpha+2)^2}a_{00}^2\right)$$
$$\text{with}\quad (\alpha+2)a_r\neq 0,$$

$$\hat{\mathcal F}_{\mathrm{I},00}\colon\quad u_t+uu_x=\sum_{j=2}^{r}a_j(x+\beta)^{j-2}u_j+b_0(x+\beta)$$
$$+b_2(x+\beta)^{-5}\quad\text{with}\quad a_r\neq 0,$$

$$\hat{\mathcal F}_{\mathrm{II},0}\colon\quad u_t+uu_x=\sum_{j=2}^{r}a_j(x+\beta)^j u_j+a_{00}u+b_0\ \text{with}\ \ a_r\neq 0,$$

$$\hat{\mathcal F}_{\mathrm{II},1}\colon\quad u_t+uu_x=\sum_{j=2}^{r}a_j(x+\beta)^j u_j+(a_{01}\ln|x+\beta|+a_{00})u$$
$$+(x+\beta)\left(-\frac{a_{01}^2}{4}\ln^2|x+\beta|+\left(\frac{a_{01}^2}{4}-\frac{a_{00}a_{01}}{2}\right)\ln|x+\beta|+b_0\right)$$
$$\text{with}\quad a_ra_{01}\neq 0,$$

$$\mathcal F_{\mathrm{III}}\colon\quad u_t+uu_x=\sum_{j=2}^{r}a_je^{\alpha x}u_j+(a_{01}e^{\alpha x}+a_{00})u+b_2e^{2\alpha x}$$
$$-\frac{a_{00}a_{01}}{\alpha}e^{\alpha x}-\frac{a_{00}^2+a_{00}}{2\alpha}\ \text{with}\ \ \alpha a_r\neq 0,$$

$$\mathcal F_{\mathrm{IV},1}\colon\quad u_t+uu_x=\sum_{j=2}^{r}a_ju_j+a_0u+b_1x+b_0$$
$$\text{with}\quad \alpha a_r\sum_{j=2}^{r-1}|a_j|\neq 0,$$

$$\mathcal F_{\mathrm{IV},0}^{r>2}\colon\quad u_t+uu_x=a_ru_r+a_0u+\frac{r-1}{(r-2)^2}a_0^2x+b_0$$
$$\text{with}\quad \alpha a_r\neq 0,\ r>2,$$

$$\mathcal F_{\mathrm{IV},0}^{r=2}\colon\quad u_t+uu_x=a_2u_2+b_1x+b_0\quad\text{with}\quad \alpha a_r\neq 0.$$

All these subclasses but $\hat{\mathcal F}_{\mathrm{II},0}$ are normalized in the generalized sense. The class $\hat{\mathcal F}_{\mathrm{II},0}$ is normalized in the usual sense.

The main result of the paper is described in the following theorem.

Theorem 1. *The usual equivalence group of the class $\mathcal{F}$ of reduced general Burgers–Korteweg–de Vries equations with space-dependent coefficients is four-dimensional. The list of maximal nontrivial conditional equivalence subgroups is exhausted by the generalized equivalence groups of the normalized subclasses $\hat{\mathcal{F}}_{\mathrm{I},1}$, $\hat{\mathcal{F}}_{\mathrm{I},01}$, $\hat{\mathcal{F}}_{\mathrm{I},00}$, $\hat{\mathcal{F}}_{\mathrm{II},1}$, $\mathcal{F}_{\mathrm{III}}$, $\mathcal{F}_{\mathrm{IV},1}$, $\mathcal{F}^{r>2}_{\mathrm{IV},0}$, $\mathcal{F}^{r=2}_{\mathrm{IV},0}$ and the usual equivalence group of the normalized subclass $\hat{\mathcal{F}}_{\mathrm{II},0}$. The equivalence groupoid of the class $\mathcal{F}$ is generated by its usual equivalence group and the equivalence groups of the above subclasses.*

For all classes normalized in the generalized sense, we can take their effective generalized equivalence subgroups as maximal conditional equivalence groups. Denote by $\mathcal{F}_0$ the complement to the union of the above subclasses in the class $\mathcal{F}$. It is a normalized class in the usual sense, and its equivalence group coincides with that of $\mathcal{F}$.

Corollary 2. *The class $\mathcal{F}$ is a union of the normalized (in either the generalized or the usual sense) classes $\hat{\mathcal{F}}_{\mathrm{I},1}$, $\hat{\mathcal{F}}_{\mathrm{I},01}$, $\hat{\mathcal{F}}_{\mathrm{I},00}$, $\hat{\mathcal{F}}_{\mathrm{II},1}$, $\hat{\mathcal{F}}_{\mathrm{II},0}$, $\mathcal{F}_{\mathrm{III}}$, $\mathcal{F}_{\mathrm{IV},1}$, $\mathcal{F}^{r>2}_{\mathrm{IV},0}$, $\mathcal{F}^{r=2}_{\mathrm{IV},0}$ and $\mathcal{F}_0$.*

The structure of this paper is as follows. Firstly, we remind in Section 2 theoretical foundations related to equivalence within classes of differential equations. Following [6] in Section 3 we recall the structure of the equivalence groupoids of the superclass of general Burgers–Korteweg–de Vries equations, its subclass of equations with time-independent coefficients and gauging of these classes to the corresponding subclasses of reduced equations. In Section 4 we give the complete classification of admissible transformations of the class $\mathcal{F}$ of reduced general Burgers–KdV equations with space-dependent coefficients. In [6] there were found subclasses of the class $\mathcal{F}$ possessing admissible transformations that are not generated by the equivalence transformations of $\mathcal{F}$. But the question of a structure of equivalence groupoids of these subgroups was not addressed there. Here we fill this gap by comprehensive description of all these subclasses and their equivalence groups (for subclasses normalized in the generalized sense we present either the entire generalized equivalence group, or its effective generalized equivalence group or both of them). By partitioning if necessary these subclasses we achieve a normalization of "subsubclasses" in either usual or generalized sense. Thus we present the superclass $\mathcal{F}$ as a union of normalized classes of differential equations described in Theorem 1. For the two normalized subclasses to be able to have a closed form of group transformations we apply a non-standard approach, the technical crux of which is as follows. First we gauge the class under consideration by a family of equivalence transformations thereof to a nice normalized subclass. Then every equivalence transformation in the class under consideration would be a composition of the gauging mapping, an equivalence transformation within the nice subclass and the inverse of a (not the same as before because we consider not symmetry but equivalence transformations of the superclass) gauging mapping. This procedure may explain an appearance of generalized equivalence groups for most of the considered subclasses. In fact, the determining systems of ODEs are exactly solvable for all but the two equivalence groups and this procedure is only lurking in the background, but we could use it almost everywhere. In this case, even if a nice underlying subclass is normalized in the usual sense, we compose its equivalence transformations with transformations from the families parameterized by arbitrary elements of the superclass, and thus parameterize the equivalence transformations thereof by arbitrary elements, making them generalized.

2. Equivalence of classes of differential equations. We recall the essential notions for the present paper only. See [6, 8, 9] for more details. Let $\mathcal{L}_\theta$ denote a system of differential equations of the form

$$L\big(x, u^{(r)}, \theta\big(x, u^{(r)}\big)\big) = 0,$$

where $x = (x_1, \dots, x_n)$ is the n independent variables, $u = (u^1, \dots, u^m)$ is the m dependent variables, and L is a tuple of differential functions in u. We use the standard short-hand notation $u^{(r)}$ to denote the tuple of derivatives of u with respect to x up to order r, which also includes u as the derivatives of order zero. The system $\mathcal{L}_\theta$ is parameterized by the tuple of functions $\theta = (\theta^1(x, u^{(r)}), \dots, \theta^k(x, u^{(r)}))$, called the arbitrary elements running through the solution set $\mathcal{S}$ of an auxiliary system of differential relations in θ. Thus, the *class of (systems of) differential equations* $\mathcal{L}|_\mathcal{S}$ is the parameterized family of systems $\mathcal{L}_\theta$, such that θ lies in $\mathcal{S}$.

Equivalence of classes of differential equations is based on studying how equations from a given class are mapped to each other. The notion of *admissible transformations*, which constitute the *equivalence groupoid* of the class $\mathcal{L}|_\mathcal{S}$, formalizes this study. An admissible transformation is a triple $(\theta, \tilde\theta, \varphi)$, where $\theta, \tilde\theta \in \mathcal{S}$ are arbitrary-element tuples associated with equations $\mathcal{L}_\theta$ and $\mathcal{L}_{\tilde\theta}$ from the class $\mathcal{L}_\mathcal{S}$ that are similar, and φ is a point transformation in the space of (x,u) that maps $\mathcal{L}_\theta$ to $\mathcal{L}_{\tilde\theta}$.

A related notion of relevance in the group classification of differential equations is that of *equivalence transformations.* Usual equivalence transformations are point transformations in the joint space of independent variables, derivatives of u up to order r and arbitrary elements that are projectable to the space of $(x,u^{(r')})$ for each $r'=0,\dots,r$, with respect the contact structure of the rth order jet space coordinatized by the r-jets $(x,u^{(r)})$ and map every system from the class $\mathcal{L}|_\mathcal{S}$ to a system from the same class. The Lie (pseudo)group constituted by the equivalence transformations of $\mathcal{L}|_\mathcal{S}$ is called the *usual equivalence group* of this class and denoted by $G^\sim$.

Each equivalence transformation $\mathcal{T}\in G^\sim$ generates a family of admissible transformations parameterized by θ,

$$G^\sim\ni\mathcal{T}\to\big\{(\theta,\mathcal{T}\theta,\pi_*\mathcal{T})\,|\,\theta\in\mathcal{S}\big\}\subset\mathcal{G}^\sim,$$

and therefore the usual equivalence group $G^\sim$ gives rise to a subgroupoid of the equivalence groupoid $\mathcal{G}^\sim$. The function π is the projection of the space of $(x,u^{(r)},\theta)$ to the space of equation variables only, $\pi(x,u^{(r)},\theta)=(x,u)$. The pushforward $\pi_*\mathcal{T}$ of $\mathcal{T}$ by π is then just the restriction of $\mathcal{T}$ to the space of (x,u).

The projectability property for equivalence transformations can be neglected. Then these equivalence transformations constitute a Lie pseudogroup $\bar G^\sim$ called the *generalized equivalence group* of the class. See the first discussion of this notion in [3, 4] and the further development in [8, 9]. When the generalized equivalence group coincides with the usual one the situation is considered to be trivial. Similarly to usual equivalence transformations, each element of $\bar G^\sim$ generates a family of admissible transformations parameterized by θ,

$$\bar G^\sim\ni\mathcal{T}\to\big\{(\theta',\mathcal{T}\theta',\pi_*(\mathcal{T}|_{\theta=\theta'(x,u)}))\,|\,\theta'\in\mathcal{S}\big\}\subset\mathcal{G}^\sim,$$

and thus the generalized equivalence group $\bar G^\sim$ also generates a subgroupoid $\bar{\mathcal{H}}$ of the equivalence groupoid $\mathcal{G}^\sim$.

Definition 3. Any minimal subgroup of $\bar G^\sim$ that generates the same subgroupoid of $\mathcal{G}^\sim$ as the entire group $\bar G^\sim$ does is called an *effective generalized equivalence group* of the class $\mathcal{L}|_\mathcal{S}$.

If the entire group $\bar G^\sim$ is effective itself, then its uniqueness is evident. At the same time, there exist classes of differential equations, where effective generalized equivalence groups are proper subgroups of the corresponding generalized equivalence groups that are even not normal. Hence each of these effective generalized equivalence groups is not unique since it differs from some of subgroups non-identically similar to it, and all of these subgroups are also effective generalized equivalence groups of the same class.

The class of differential equations $\mathcal{L}|_\mathcal{S}$ is *normalized* in the usual (resp. generalized) sense if the subgroupoid induced by its usual (resp. generalized) equivalence group coincides with the entire equivalence groupoid $\mathcal{G}^\sim$ of $\mathcal{L}|_\mathcal{S}$. The normalization of $\mathcal{L}|_\mathcal{S}$ in the usual sense is equivalent to the following conditions. The transformational part φ of each admissible transformation $(\theta',\theta'',\varphi)\in\mathcal{G}^\sim$ does not depend on the fixed initial value θ' of the arbitrary-element tuple θ and, therefore, is appropriate for any initial value of θ.

The normalization properties of the class $\mathcal{L}|_\mathcal{S}$ are usually established via computing its equivalence groupoid $\mathcal{G}^\sim$, which is realized using the direct method. Here one fixes two arbitrary systems from the class, $\mathcal{L}_\theta\colon L(x,u^{(r)},\theta(x,u^{(r)}))=0$ and $\mathcal{L}_{\tilde\theta}\colon L(\tilde x,\tilde u^{(r)},\tilde\theta(\tilde x,\tilde u^{(r)}))=0$, and aims to find the (nondegenerate) point transformations, φ: $\tilde x_i=X^i(x,u)$, $\tilde u^a=U^a(x,u)$, $i=1,\dots,n$, $a=1,\dots,m$, connecting them. For this, one changes the variables in the system $\mathcal{L}_{\tilde\theta}$ by expressing the derivatives $\tilde u^{(r)}$ in terms of $u^{(r)}$ and derivatives of the functions X^i and U^a as well as by substituting X^i and U^a for $\tilde x_i$ and $\tilde u^a$, respectively. The requirement that the resulting transformed system has to be satisfied identically for solutions of $\mathcal{L}_\theta$ leads to the system of determining equations for the components of the transformation φ.

Imposing additional constraints on arbitrary elements of the class, we may single out its subclass whose equivalence group is not contained in the equivalence group of the entire class. Let $\mathcal{L}|_{\mathcal{S}'}$ be the subclass of the class $\mathcal{L}|_\mathcal{S}$, which is constrained by the additional system of equations $\mathcal{S}'(x,u^{(r)},\theta^{(q')})=0$ and inequalities $\Sigma'(x,u^{(r)},\theta^{(q')})\neq0$ with respect to the arbitrary elements $\theta=\theta(x,u^{(r)})$. Here $\mathcal{S}'\subset\mathcal{S}$ is the set of solutions of the united system $\mathcal{S}=0$, $\Sigma\neq0$, $\mathcal{S}'=0$, $\Sigma'\neq0$. We assume that the united system is compatible for the subclass $\mathcal{L}|_{\mathcal{S}'}$ to be nonempty.

Definition 4. The equivalence group $G^\sim(\mathcal{L}|_{\mathcal{S}'})$ of the subclass $\mathcal{L}|_{\mathcal{S}'}$ is called *a conditional equivalence group* of the entire class $\mathcal{L}|_\mathcal{S}$ under the conditions $\mathcal{S}'=0$, $\Sigma'\neq0$. The conditional equivalence group is called *nontrivial* if it is not a subgroup of $G^\sim(\mathcal{L}|_\mathcal{S})$.

Conditional equivalence groups may be trivial not with respect to the equivalence group of the entire class but with respect to other conditional equivalence groups. Indeed, if $\mathcal{S}' \subset \mathcal{S}''$ and $G^{\sim}(\mathcal{L}|_{\mathcal{S}'}) \subset G^{\sim}(\mathcal{L}|_{\mathcal{S}''})$ then the subclass $\mathcal{L}|_{\mathcal{S}'}$ is not interesting from the conditional symmetry point of view. Therefore, the set of additional conditions on the arbitrary elements can be reduced substantially.

Definition 5. The conditional equivalence group $G^{\sim}_{\mathcal{L}|_{\mathcal{S}'}}$ of the class $\mathcal{L}|_{\mathcal{S}}$ under the additional conditions $\mathcal{S}' = 0$, $\Sigma' \neq 0$ is called maximal if for any subclass $\mathcal{L}|_{\mathcal{S}''}$ of the class $\mathcal{L}|_{\mathcal{S}}$ containing the subclass $\mathcal{L}|_{\mathcal{S}'}$ we have $G^{\sim}_{\mathcal{L}|_{\mathcal{S}'}} \not\subset G^{\sim}_{\mathcal{L}|_{\mathcal{S}''}}$.

3. Preliminary analysis of equivalence groupoid. We start studying admissible transformations of the class $\mathcal{F}$ by presenting the equivalence groupoid of its superclass (1) and then descend therefrom to the class under study.

Proposition 6. *The class* (1) *is normalized in the usual sense. Its usual equivalence group* $G^{\sim}_{(1)}$ *consists of the transformations in the joint space of* (t,x,u,θ) *whose* (t,x,u)*-components are of the form*

$$\tilde t = T(t), \quad \tilde x = X(t,x), \quad \tilde u = U^1(t)u + U^0(t,x),$$

where $T = T(t)$, $X = X(t,x)$, $U^1 = U^1(t)$ *and* $U^0 = U^0(t,x)$ *are arbitrary smooth functions of their arguments such that* $T_tX_xU^1 \neq 0$.

Following [6] we can gauge the arbitrary elements $C = 1$ and $A^1 = 0$ by a family of equivalence transformations of the class (1) and obtain the class of reduced general Burgers–KdV equations

$$u_t + uu_x = \sum_{j=2}^{r} A^j(t,x)u_j + A^0(t,x)u + B(t,x). \tag{3}$$

As before, the arbitrary elements run through the set of smooth functions of (t,x) with $A^rC \neq 0$.

Theorem 7. *The class of reduced* (1+1)*-dimensional general* r*th order Burgers–KdV equations* (3) *is normalized in the usual sense. Its usual equivalence group* $G^{\sim}$ *consists of the transformations of the form*

$$\tilde t = T(t), \quad \tilde x = X^1(t)x + X^0(t), \quad \tilde u = \frac{X^1}{T_t}u + \frac{X^1_t}{T_t}x + \frac{X^0_t}{T_t}, \tag{4}$$

$$\tilde{A}^j = \frac{(X^1)^j}{T_t} A^j, \quad \tilde{A}^0 = \frac{1}{T_t}\left(A^0 + 2\frac{X^1_t}{X^1} - \frac{T_{tt}}{T_t}\right), \tag{5}$$

$$\tilde{B} = \frac{X^1}{(T_t)^2} B + \frac{1}{T_t}\left(\frac{X^1_t}{T_t}\right)_t x + \frac{1}{T_t}\left(\frac{X^0_t}{T_t}\right)_t - \left(\frac{X^1_t}{T_t} x + \frac{X^0_t}{T_t}\right)\tilde{A}^0, \tag{6}$$

where $j = 2, \dots, r$, and $T = T(t)$, $X^1 = X^1(t)$ and $X^0 = X^0(t)$ are arbitrary smooth functions of their arguments with $T_t X^1 \neq 0$.

The subclass $\bar{\mathcal{F}}$ of general Burgers–KdV equations with space-dependent coefficients is singled out from the class (1) by the constraints $A^k_t = 0$, $k = 0, \dots, r$, $B_t = 0$ and $C_t = 0$. Therefore, its usual equivalence group $G^\sim_{\bar{\mathcal{F}}}$ is a subgroup of $G^\sim_{(1)}$ that consists of transformations preserving the above constraints.

Proposition 8. *The usual equivalence group $G^\sim_{\bar{\mathcal{F}}}$ of the class $\bar{\mathcal{F}}$ of general Burgers–Korteweg–de Vries equations with space-dependent coefficients consists of the transformations in the joint space of (t, x, u, θ) whose (t, x, u)-components are of the form*

$$\tilde{t} = c_1 t + c_2, \quad \tilde{x} = X(x), \quad \tilde{u} = c_3' u + U^0(x),$$

where c_1, c_2 and c_3' are arbitrary constants and $X = X(x)$ and $U^0 = U^0(x)$ are arbitrary smooth functions of x such that $c_1 X_x c_3' \neq 0$.

The existence of classifying conditions [6]

$$\frac{T_t}{(X_x)^r} X_t \tilde{A}^r_{\tilde{x}} + \left(\frac{T_t}{(X_x)^r}\right)_t \tilde{A}^r = 0, \quad \frac{T_t U^1}{X_x} X_t \tilde{C}_{\tilde{x}} + \left(\frac{T_t U^1}{X_x}\right)_t \tilde{C} = 0,$$

for admissible transformations of the class $\bar{\mathcal{F}}$ implies that it is definitely not normalized in any sense. At the same time, we can gauge the arbitrary elements C and A^1 again by means of equivalence transformations of the class $\bar{\mathcal{F}}$ and produce the class $\mathcal{F}$ of reduced general Burgers–KdV equations with space-dependent coefficients,

$$u_t + u u_x = \sum_{j=2}^{r} A^j(x) u_j + A^0(x) u + B(x).$$

Proposition 9. *The usual equivalence group $G^\sim_{\mathcal{F}}$ of the class $\mathcal{F}$ is four-dimensional and consists of transformations of the form*

$$\tilde{t} = c_1 t + c_2, \quad \tilde{x} = c_3 x + c_4, \quad \tilde{u} = \frac{c_3}{c_1} u,$$

$$\tilde A^j = \frac{(c_3)^j}{c_1}A^j, \quad \tilde A^0 = \frac{1}{c_1}A^0, \quad \tilde B = \frac{c_3}{(c_1)^2}B,$$

where $j=2,\dots,r$, and c's are arbitrary constants with $c_1c_3\neq 0$.

Nor the class $\mathcal F$ neither its superclass $\bar{\mathcal F}$ are normalized in any sense. Thus, the problem of describing the equivalence groupoid $\mathcal G^\sim_{\mathcal F}$ of the class $\mathcal F$ should be considered as the classification of admissible transformations up to $G^\sim_{\mathcal F}$-equivalence, see [9, Sections 2.6 and 3.4]. The class $\mathcal F$ is a subclass of the class (3), whence $\mathcal G^\sim_{\mathcal F}$ is a subgroupoid of the equivalence groupoid of the class (3), and the results of Theorem 7 are valid here, although they should be further specified. This is achieved by differentiating the relations (5)–(6), solved with respect to the source arbitrary elements, with respect to t. This gives the classifying conditions for admissible transformations,

$$\big(X^1_tx+X^0_t\big)\tilde A^j_{\tilde x} + \left(\frac{T_{tt}}{T_t} - j\frac{X^1_t}{X^1}\right)\tilde A^j = 0, \tag{7}$$

$$\big(X^1_tx+X^0_t\big)\tilde A^0_{\tilde x} + \frac{T_{tt}}{T_t}\tilde A^0 = \frac{1}{T_t}\left(2\frac{X^1_t}{X^1} - \frac{T_{tt}}{T_t}\right)_t, \tag{8}$$

$$\begin{aligned}\big(X^1_tx+X^0_t\big)\tilde B_{\tilde x} + \left(2\frac{T_{tt}}{T_t} - \frac{X^1_t}{X^1}\right)\tilde B = -\frac{T_t}{X^1}\big(X^1_tx+X^0_t\big)^2\tilde A^0_{\tilde x}\\ -\frac{X^1}{T_t^2}\left(T_t\frac{X^1_tx+X^0_t}{X^1}\right)_t\tilde A^0 + \frac{X^1}{T_t^2}\left(\frac{T_t}{X^1}\left(\frac{X^1_tx+X^0_t}{T_t}\right)_t\right)_t,\end{aligned} \tag{9}$$

where the initial space variable x should be substituted, after expanding all derivatives, by its expression via $\tilde x$, $x=(\tilde x - X^0)/X^1$. Note that admissible transformations with $T_{tt}=X^0_t=X^1_t=0$ are generated by the usual equivalence group $G^\sim_{\mathcal F}$.

4. Nontrivial conditional equivalence subgroups. In [6] with a help of the method of furcate splitting, cf. [5, 7] the classifying conditions (7)–(9) for admissible transformations of the class $\mathcal F$ were solved, but the obtained admissible transformations were presented superficially. More precisely, they were parameterized by solutions of some ODEs. Here we study the question in more depth and present explicit forms of group parameters of the nontrivial conditional equivalence groups. Besides, following [6] for simplicity we consider only subclasses of the classes $\mathcal F_{\rm I}$ and $\mathcal F_{\rm II}$, defined below, admitting proper subgroups of maximal conditional equivalence groups. In fact, these subgroups are the quotients thereof by the space-translations. Note that given in Theorem 1 are the subclasses admitting maximal nontrivial conditional equivalence subgroups.

I. The class $\mathcal{F}_{\rm I}$ of equations

$$u_t+uu_x=\sum_{j=2}^{r}a_jx^j|x|^\alpha u_j+\big(a_{00}+a_{01}|x|^\alpha\big)u\\ +x\big(b_0+b_1|x|^\alpha+b_2|x|^{2\alpha}\big)$$

with $\alpha a_r\neq0$ naturally partitions into two $\mathcal{G}^{\sim}_{\mathcal{F}_{\rm I}}$-invariant subclasses $\mathcal{F}_{\rm I,0}$ and $\mathcal{F}_{\rm I,1}$ singled out by the conditions $a_{01}=0$ and $a_{01}\neq0$, respectively, since the arbitrary element a_{01} is easily shown to be transformed by the rule $\tilde a_{01}=c_4a_{01}$ under admissible transformations of the class, $c_4\neq0$. The class $\mathcal{F}_{\rm I,1}$ admits additional admissible transformations if and only if $a_{00}=(\alpha{+}2)b_1/a_{01}$ and $b_0=-b_1^2(1{+}\alpha)/a_{01}^2$, so we reduce the arbitrary-elements tuple of the class by a_{00} and b_0 and denote the subclass obtained again by $\mathcal{F}_{\rm I,1}$.

Proposition 10. *The class $\mathcal{F}_{\rm I,1}$ is normalized in the generalized sense. Its generalized equivalence group consists of the point transformations in the relevant space, which are of the form*

$$\tilde t=\bar T,\quad \tilde x=\bar X^1x,\quad \tilde u=\frac{\bar X^1}{\bar T_t}u-\frac{\bar X^1_t}{\bar T_t}x,\\ \tilde\alpha=\alpha,\quad \tilde a_j=\bar c_4a_j,\quad \tilde a_{01}=\bar c_4a_{01},\quad \tilde b_2=\bar c_4^2b_2,\quad \tilde b_1=\bar c_5,$$

where $\bar T$ is a smooth function of t and the arbitrary elements θ,

$$\bar T(t,\theta)=\frac{1}{\bar c_5}\ln\left|\bar c_5\left(c_1\frac{e^{-b_1\alpha t/a_{01}}-1}{-b_1\alpha/a_{01}}+c_2\right)+1\right|,\\ \theta=(\alpha,a_j,a_{01},b_2,b_1),$$

taking the form at the singular points

$$\begin{aligned}&\bar T(t,\theta)=\bar c_1\frac{e^{-b_1\alpha t/a_{01}}-1}{-b_1\alpha/a_{01}}+\bar c_2 &&\textit{if}\quad \bar c_5=0\ \textit{and}\ b_1\neq0,\\ &\bar T(t,\theta)=\frac{1}{\bar c_5}\ln\big|\bar c_5(\bar c_1t+\bar c_2)\big| &&\textit{if}\quad \bar c_5\neq0\ \textit{and}\ b_1=0,\\ &\bar T(t,\theta)=\bar c_1t+\bar c_2 &&\textit{if}\quad (\bar c_5,b_1)=(0,0),\end{aligned}$$

$\bar{c}$'s are arbitrary functions of θ with $\bar{c}_1\bar{c}_4\frac{\partial(\tilde{a}_2,\dots,\tilde{a}_r,\tilde{a}_{01},\tilde{b}_1,\tilde{b}_2)}{\partial(a_2,\dots,a_r,a_{01},b_1,b_2)}\neq 0$ *as well as* $\bar{X}^1(t,\theta)=(\bar{c}_4\bar{T}_t)^{-1/\alpha}$ *if α is odd or rational in the reduced form with an odd numerator and* $\bar{X}^1(t)=\varepsilon|\bar{c}_4\bar{T}_t|^{-1/\alpha}$ *with* $\varepsilon=\pm1$ *and* $\bar{c}_4\bar{T}_t>0$ *otherwise.*

Remark 11. The function T is a solution of an ODE smoothly depending on parameters, so it is a smooth function of these parameters and initial conditions [1, Corollary 6, p. 97] (α, b_1 and a_{01} are the parameters of the equation in this case, c's are the initial conditions). This argumentation is valid for the group parameters in the equivalence groups below, where appropriate, as well. In fact, in these cases it follows from the transformation for $\tilde{A}^0$ (the equation (5)) that the function T satisfies the equation

$$\gamma=\delta\frac{1}{T_t}+\left(\frac{1}{T_t}\right)_t=0$$

for some constants γ and δ, having the general solution

$$T(t)=\frac{1}{\gamma}\ln\left|\gamma\left(c_1\frac{e^{\delta t}-1}{\delta}+c_2\right)+1\right|.$$

The continuity of this function is evident and at the singular points the function takes the form

$$\begin{aligned}
&T(t)=c_1\frac{e^{\delta t}-1}{\delta}+c_2 &&\text{if}\quad \gamma=0 \text{ and } \delta\neq0,\\
&T(t)=\frac{1}{\gamma}\ln|\gamma(c_1t+c_2)+1| &&\text{if}\quad \gamma\neq0 \text{ and } \delta=0,\\
&T(t)=c_1t+c_2 &&\text{if}\quad (\gamma,\delta)=(0,0).
\end{aligned}$$

The transformations in Proposition 10 indeed form a group, which is straightforward to show. Therefore the equivalence group of the class $\mathcal{F}_{\mathrm{I},1}$ is a local Lie group of transformations (all equivalence group here and below in the paper are finite-dimensional so we do not need to talk about Lie pseudogroups). If the function T is of the form $\frac{1}{\gamma}\ln|\gamma(c_1t+c_2)+1|$ and $\gamma c_2=-1$, then $T(t)$ degenerates into an affine function. To avoid this, in all such situations thereafter we implicitly assume otherwise. The notation $\frac{\partial(\cdot,\dots,\cdot)}{\partial(\cdot,\dots,\cdot)}$ stands for the determinant of the corresponding Jacobian matrix. Thereafter, we will not call attention to these facts.

Since the arbitrary element α is invariant under admissible transformations, it is convenient to consider the two subclasses $\mathcal{F}_{\mathrm{I},00}$ and $\mathcal{F}_{\mathrm{I},01}$ of $\mathcal{F}_{\mathrm{I},0}$ singled out by conditions $\alpha=-2$ and $\alpha\neq-2$, respectively. To achieve an extension of a number of admissible transformations in the latter class we need to consider its subclass (denoted again $\mathcal{F}_{\mathrm{I},01}$) singled out by the conditions $b_1=0$ and $b_0=-(\alpha+1)a_{00}^2/(\alpha+2)^2$.

Proposition 12. *The class $\mathcal{F}_{\mathrm{I},01}$ is normalized in the generalized sense. Its generalized equivalence group consists of the point transformations of the form*

$$\tilde t=\bar T(t),\quad \tilde x=\bar X^1(t)x,\quad \tilde u=\frac{\bar X^1}{\bar T_t}u-\frac{\bar X^1_t}{\bar T_t}x,$$
$$\tilde\alpha=\alpha,\quad \tilde a_j=\bar c_4a_j,\quad \tilde a_{00}=\bar c_5,\quad \tilde b_2=\bar c_4^2b_2,$$

where $\bar T$ is a smooth function of t and the arbitrary elements θ,

$$\bar T(t,\theta)=\frac{1}{\bar c_5}\ln\left|\bar c_5\left(\bar c_1\frac{e^{a_{00}\alpha t/(\alpha+2)}-1}{a_{00}\alpha/(\alpha+2)}+\bar c_2\right)+1\right|.$$

The function $\bar T$ takes at the singular points the following forms

$$\bar T(t,\theta)=\bar c_1\frac{e^{a_{00}\alpha t/(\alpha+2)}-1}{a_{00}\alpha/(\alpha+2)}+\bar c_2\quad \text{if}\quad \bar c_5=0\ \text{and}\ a_{00}\neq0,$$
$$\bar T(t,\theta)=\frac{1}{\bar c_5}\ln|\bar c_5(\bar c_1t+\bar c_2)+1|\quad \text{if}\quad \bar c_5\neq0\ \text{and}\ a_{00}=0,$$
$$\bar T(t,\theta)=\bar c_1t+\bar c_2\quad \text{if}\quad (\bar c_5,a_{00})=(0,0).$$

Here $\bar c$'s are arbitrary functions of θ with $\bar c_1\bar c_4\frac{\partial(\tilde a_2,\dots,\tilde a_r,\tilde a_{00},\tilde b_2)}{\partial(a_2,\dots a_r,a_{00},b_2)}\neq0$ as well as $\bar X^1(t,\theta)=(\bar c_4\bar T_t)^{-1/\alpha}$ if α is odd or rational in the reduced form with an odd numerator and $\bar X^1(t,\theta)=\varepsilon|\bar c_4\bar T_t|^{-1/\alpha}$ with $\varepsilon=\pm1$ and $\bar c_4\bar T_t>0$ otherwise.

A description of the equivalence group of the class $\mathcal{F}_{\mathrm{I},00}$ is more complicated and we present its equivalence groupoid first. In accordance with our standard approach we consider its subclass singled out by the conditions $a_{00}=b_1=0$.

Proposition 13. *A point transformation connects the two equations in the class $\mathcal{F}_{\mathrm{I},00}$ if and only if its components are of the form*

$$\tilde t=T(t),\quad \tilde x=X^1(t)x,\quad \tilde u=\frac{X^1}{T_t}u-\frac{X^1_t}{T_t}x,$$

where $(X^1(t))^2 = c_4T_t$ *and the smooth function* T *of* t *satisfies the equation*

$$\left(\frac{T_{tt}}{T_t}\right)_t - \frac{1}{2}\left(\frac{T_{tt}}{T_t}\right)^2 = 2\tilde{b}_0T_t^2 - 2b_0.$$

Here c_4 *is an arbitrary constant and* $c_4T_t > 0$.

The last equation is an autonomous ordinary differential equation on T which can be integrated in quadratures with standard techniques, but proceeding this way one can write an explicit form of the general solution only for specific values of parameters. On the other hand, for any equation in $\mathcal{F}_{\mathrm{I},00}$ there is an equivalent one to it in the subclass $\mathcal{F}_{\mathrm{I},00}^{b_0=0}$ singled out by the condition $b_0 = 0$. The corresponding point transformation is $\tilde{t} = T(t)$, $\tilde{x} = \sqrt{T_t}x$, $\tilde{u} = u/\sqrt{T_t} - T_{tt}x/(2\sqrt{(T_t)^3})$, where a smooth function T of t is a solution of the equation $(T_{tt}/T_t)_t - \frac{1}{2}(T_{tt}/T_t)^2 + 2b_0 = 0$, for which the general solution can be found explicitly, although a particular solution will suffice for our purposes. Thus, if $b_0 = b^2 > 0$, then $T(t) = e^{2bt}$ is a particular solution; if $b_0 = -b^2 < 0$, then $T(t) = \tan(bt)$ is a particular solution, $b > 0$ in both cases.

Proposition 14. *The class* $\mathcal{F}_{\mathrm{I},00}^{b_0=0}$ *is normalized in the usual sense. Its usual equivalence group is constituted by the point transformations of the form*

$$\tilde{t} = T(t), \quad \tilde{x} = X^1(t)x, \quad \tilde{u} = \frac{X^1}{T_t}u - \frac{X^1_t}{T_t}x,$$
$$\tilde{a}_2 = c_4a_2, \quad \tilde{b}_2 = c_4^2b_2,$$

where $X^1(t) = \varepsilon\sqrt{c_4T_t}$ *with* $\varepsilon = \pm 1$, $T = (c_1t + c_2)/(c_3t + c_0)$ *and* c*'s are arbitrary constants, with* $\delta = c_1c_0 - c_2c_3 \neq 0$ *and* c_0, c_1, c_2 *and* c_3 *being defined up to a nonzero constant, and* $c_4\delta > 0$.

On the other hand, any admissible transformation of the class $\mathcal{F}_{\mathrm{I},00}$ can be represented as a composition of an admissible transformation with a source equation in $\mathcal{F}_{\mathrm{I},00}$ and a target equation in $\mathcal{F}_{\mathrm{I},00}^{b_0=0}$, an admissible transformation generated by an equivalence transformation in $\mathcal{F}_{\mathrm{I},00}^{b_0=0}$ and an admissible transformation back. In this way we avoid implicit quadrature expressions arising in a previous approach. Note that the parameter-function T is defined as a solution of a third-order ODE parameterized by b_0 and $\tilde{b}_0$ and thus should be parameterized by three constants to agree with the Picard–Lindelöf theorem. This is indeed the case.

Proposition 15. *The class $\mathcal{F}_{\mathrm{I},00}$ is normalized in the generalized sense. Its effective generalized equivalence group is constituted by the point transformations of the form*

$$
\begin{aligned}
&\tilde t = P^2(T(P^1(t))), \quad \tilde x = \sqrt{P^2_{\bar t}P^1_t}X^1(\hat t)x,\\
&\tilde u = \frac{1}{P^2_{\bar t}}\left(\frac{X^1}{T_{\hat t}P^1_t}u - \left(\frac{X^1P^1_{tt}}{2T_{\hat t}(P^1_t)^{3/2}} + \frac{X^1_{\hat t}\sqrt{P^1_t}}{T_{\hat t}} + \frac{P^2_{\bar t\bar t}X^1\sqrt{P^1_t}}{2P^2_{\bar t}}\right)x\right),\\
&\tilde a_j = c_4a_j, \quad \tilde b_2 = c_4^2b_2,\\
&\tilde b_0 = \frac{1}{(P^2_{\bar t})^2}\left(\frac{1}{(P^1_t)^2}\left(b_0 - \left(\frac{P^1_{tt}}{2P^1_t}\right)^2 + \frac12\left(\frac{P^1_{tt}}{P^1_t}\right)_t\right)\right.\\
&\qquad\left. - \left(\frac{P^2_{\bar t\bar t}}{2P^2_{\bar t}}\right)^2 + \frac12\left(\frac{P^2_{\bar t\bar t}}{P^2_{\bar t}}\right)_{\bar t}\right),
\end{aligned}
$$

where $\hat t = P^1(t)$, $\bar t = T(\hat t)$, $\tilde t = P^2(\bar t)$, $X^1(\hat t) = \varepsilon(c_4T_{\hat t})^{1/2}$, $T = (c_1\hat t + c_2)/(c_3\hat t + c_0)$, with $\delta = c_1c_0 - c_2c_3 \neq 0$, c's are arbitrary constants,

$$
P^1(t) = \begin{cases} t & \text{if} \quad b_0 = 0,\\ \tan(\sqrt{-b_0}t) & \text{if} \quad b_0 < 0,\\ e^{2\sqrt{b_0}t} & \text{if} \quad b_0 > 0; \end{cases}
$$

$P^2(\bar t)$ runs through the set of smooth functions $\left\{\bar t, \frac{1}{c_5}\ln|\bar t|, \frac{1}{2c_5}\arctan\bar t\right\}$, with $c_4\delta > 0$, c_i, $i = 0,1,2,3$, are defined up to a nonzero constant, and $P^2_{\bar t} > 0$ and $\varepsilon = \pm1$.

The arbitrary element $\tilde b_0$ of the target equation takes the value of c_5^2 if $P^2(y) = \frac{1}{c_5}\ln|y|$, of $-c_5^2$ if $P^2(y) = \frac{1}{2c_5}\arctan y$ and of 0 otherwise. The functions $P^2(T(P^1(t)))$ give a three-parameter family of solutions to the nonlinear third-order equation on T above parameterized by b_0 and $\tilde b_0$.

Remark 16. The point transformations in Proposition 15 form a group by construction, and thus constitute an effective generalized equivalence group of the class $\mathcal{F}_{\mathrm{I},00}$. To obtain the entire generalized equivalence group thereof one allows c's to vary through the set of arbitrary smooth functions of the arbitrary elements of the class.

II. A class $\mathcal{F}_{\rm II}$ of differential equations of the form

$$u_t + uu_x = \sum_{j=2}^{r} a_j x^j u_j + (a_{01}\ln|x| + a_{00})u \\ + x\left(-\frac{a_{01}^2}{4}\ln^2|x| + \left(\frac{a_{01}^2}{4} - \frac{a_{00}a_{01}}{2}\right)\ln|x| + b_0\right)$$

is partitioned into two subclasses $\mathcal{F}_{\rm II,0}$ and $\mathcal{F}_{\rm II,1}$ that are singled out by conditions $a_{01} = 0$ and $a_{01} \neq 0$, respectively, and invariant under the admissible transformations of the class $\mathcal{F}_{\rm II}$.

Proposition 17. *The class $\mathcal{F}_{\rm II,0}$ is normalized in the usual sense. Its equivalence group is constituted by the point transformations of the form*

$$\tilde t = c_1 t + c_2, \quad \tilde x = c_4 e^{c_3 t} x, \quad \tilde u = \frac{c_4 e^{c_3 t}}{c_1}(u + c_3 x),$$
$$\tilde a_j = \frac{a_j}{c_1}, \quad \tilde a_{00} = \frac{a_{00} + 2c_3}{c_1}, \quad \tilde b_0 = \frac{b_0 - c_3^2}{(c_1)^2},$$

where c's are arbitrary constants with $c_1c_4 \neq 0$.

The class $\mathcal{F}_{\rm II,0}$ is the only owner of a conditional group normalized in the usual sense.

Proposition 18. *The class $\mathcal{F}_{\rm II,1}$ is normalized in the generalized sense. Its generalized equivalence group $\bar G^{\sim}_{\rm II,1}$ is constituted by the point transformations of the form*

$$\tilde t = \bar c_1 t + \bar c_2, \quad \tilde x = \bar X^1 x, \quad \tilde u = \frac{\bar X^1}{\bar c_1}\left(u + \frac{\bar c_4 a_{01}}{2} e^{a_{01}t/2} x\right),$$
$$\tilde a_j = \frac{a_j}{\bar c_1}, \quad \tilde a_{01} = \frac{a_{01}}{\bar c_1}, \quad \tilde a_{00} = \frac{1}{\bar c_1}(a_{00} - a_{01}\bar c_3),$$
$$\tilde b_0 = \frac{1}{4\bar c_1^2}\left(4b_0 - a_{01}^2(\bar c_3^2 + \bar c_3) + 2a_{00}a_{01}\bar c_3\right),$$

where $\bar X^1 := \exp\left(\bar c_3 + \bar c_4 \exp(\frac{a_{01}t}{2})\right)$, and $\bar c$'s are smooth functions of the arbitrary elements a_{00}, a_{01}, a_j and b_0 with $\bar c_1 \frac{\partial(\tilde a_2,\dots,\tilde a_r,\tilde a_{01},\tilde a_{00},\tilde b_0)}{\partial(a_2,\dots,a_r,a_{01},a_{00},b_0)} \neq 0$.

To extract an effective generalized equivalence group from the generalized equivalence group, we set $\bar c_2 := c_2/a_{01}$, $\bar c_3 := -c_3/a_{01}$ and get rid of the dependence of other $\bar c$'s on the arbitrary elements.

Proposition 19. *An effective generalized equivalence group $\hat{G}^{\sim}_{\mathrm{II},1}$ of the class $\mathcal{F}_{\mathrm{II},1}$ is constituted by the point transformations of the form*

$$\tilde{t} = c_1 t + \frac{c_2}{a_{01}}, \quad \tilde{x} = X^1(t)x, \quad \tilde{u} = \frac{X^1(t)}{c_1}\left(u + \frac{c_4 a_{01}}{2} e^{a_{01}t/2}x\right),$$
$$\tilde{a}_j = \frac{a_j}{c_1}, \quad \tilde{a}_{01} = \frac{a_{01}}{c_1}, \quad \tilde{a}_{00} = \frac{1}{c_1}(a_{00} + c_3),$$
$$\tilde{b}_0 = \frac{1}{4c_1^2}\left(4b_0 + (a_{01} - 2a_{00})c_3 - c_3^2\right),$$

where $X^1(t) := \exp\left(-\frac{c_3}{a_{01}} + c_4 \exp(\frac{a_{01}t}{2})\right)$ *and c's are arbitrary constants with* $c_1 \neq 0$.

The effective generalized equivalence group $\hat{G}^{\sim}_{\mathrm{II},1}$ is not a normal subgroup of $\bar{G}^{\sim}_{\mathrm{II},1}$, which is readily seen after writing the time-transformation out. Therefore, it is not unique as an effective generalized equivalence group as conjugate subgroups in $\bar{G}^{\sim}_{\mathrm{II},1}$ are also effective generalized equivalence groups. Thus, the existence of a class of differential equations with unique nontrivial (proper) effective generalized equivalence group is still a question.

III. A class of differential equations of the form

$$u_t + uu_x = \sum_{j=2}^{r} a_j e^{\alpha x} u_j + (a_{01}e^{\alpha x} + a_{00})u + b_2 e^{2\alpha x} + b_1 e^{\alpha x} + b_0 \quad \text{with } \alpha a_r \neq 0$$

admits additional admissible transformations if and only if

$$b_0 = -\frac{a_{00}^2 + a_{00}}{2\alpha} \quad \text{and} \quad b_1 = -\frac{a_{00}a_{01}}{\alpha}.$$

Proposition 20. *The class $\mathcal{F}_{\mathrm{III}}$ is normalized in the generalized sense. Its generalized equivalence group is constituted by the point transformations of the form*

$$\tilde{t} = \bar{T}, \quad \tilde{x} = \bar{c}_5 x - \frac{\bar{c}_5}{\alpha}\ln|\bar{c}_4 \bar{T}_t|, \quad \tilde{u} = \frac{\bar{c}_5}{\bar{T}_t}\left(u - \frac{\bar{T}_{tt}}{\alpha \bar{T}_t}\right),$$
$$\tilde{\alpha} = \frac{\alpha}{\bar{c}_5}, \quad \tilde{a}_j = \bar{c}_4 \bar{c}_5^j a_j, \quad \tilde{a}_{01} = \bar{c}_4 a_{01}, \quad \tilde{a}_{00} = \bar{c}_3, \quad \tilde{b}_2 = \bar{c}_4^2 \bar{c}_5 b_2,$$

where the function T of t and the arbitrary elements θ is defined by

$$\bar{T}(t,\theta)=\frac{1}{\bar{c}_3}\ln\left|\bar{c}_3\left(\bar{c}_1\frac{e^{a_{00}t}-1}{a_{00}}+\bar{c}_2\right)+1\right|,$$

and takes the following values at the singular points

$$\begin{aligned}
&\bar{T}(t,\theta)=\bar{c}_1\frac{e^{a_{00}t}-1}{a_{00}}+\bar{c}_2 && \textit{if}\quad \bar{c}_3=0 \textit{ and } a_{00}\neq 0,\\
&\bar{T}(t,\theta)=\frac{1}{\bar{c}_3}\ln|\bar{c}_3(\bar{c}_1t+\bar{c}_2)| && \textit{if}\quad a_{00}=0 \textit{ and } \bar{c}_3\neq 0,\\
&\bar{T}(t,\theta)=\bar{c}_1t+\bar{c}_2 && \textit{if}\quad (a_{00},\bar{c}_3)=(0,0),
\end{aligned}$$

$\bar{c}$'s are smooth functions of θ with $\bar{c}_1\bar{c}_4\bar{c}_5\frac{\partial(\bar{\alpha},\tilde{a}_2,\dots,\tilde{a}_r,\tilde{a}_{00},\tilde{a}_{01},\tilde{b}_2)}{\partial(\alpha,a_2,\dots,a_r,a_{00},a_{01},b_2)}\neq 0.$

To find an effective generalized equivalence group of the class $\mathcal{F}_{\rm III}$ we resort to the following heuristic speculation. The arbitrary element $\tilde{a}_{00}$ may take any real value. Thus, it sufficient to parameterize $\tilde{a}_{00}$ to be $a_{00}+c_3$, $c_3\in\mathbb{R}$. We preserve the number of initial conditions parameterizing T and guaranteeing the necessary domain for values of $\tilde{a}_{00}$. To satisfy another condition of an effective generalized equivalence group we drop any dependence of remaining $\bar{c}$'s on the arbitrary elements. In fact, we chose a correct parameterization for them already in the theorem.

Proposition 21. *An effective generalized equivalence group $\hat{G}^{\sim}_{\rm III}$ of the class $\mathcal{F}_{\rm III}$ is constituted by the point transformations of the form*

$$\tilde{t}=T,\quad \tilde{x}=c_5x-\frac{c_5}{\alpha}\ln|c_4T_t|,\quad \tilde{u}=\frac{c_5}{T_t}\left(u-\frac{T_{tt}}{\alpha T_t}\right),\quad \tilde{\alpha}=\frac{\alpha}{c_5},$$
$$\tilde{a}_j=c_4c_5^ja_j,\quad \tilde{a}_{01}=c_4a_{01},\quad \tilde{a}_{00}=a_{00}+c_3,\quad \tilde{b}_2=c_4^2c_5b_2,$$

where the function T is equal to

$$T(t)=\frac{1}{a_{00}+c_3}\ln\left|(a_{00}+c_3)\left(c_1\frac{e^{a_{00}t}-1}{a_{00}}+c_2\right)+1\right|,$$

and takes the following values at the singular points

$$\begin{aligned}
&T(t)=c_1\frac{e^{a_{00}t}-1}{a_{00}}+c_2 && \textit{if}\quad c_3=-a_{00}\neq 0,\\
&T(t)=\frac{1}{c_3}\ln|c_3(c_1t+c_2)| && \textit{if}\quad a_{00}=0 \textit{ and } c_3\neq 0,
\end{aligned}$$

$$T(t) = c_1 t + c_2 \qquad\qquad \textit{if} \quad (a_{00}, c_3) = (0, 0),$$

and c's are arbitrary constants with $c_1 c_4 c_5 \neq 0$.

Guided by the same logic as for the class $\mathcal{F}_{\mathrm{II},1}$, we can show nonuniqueness of effective generalized equivalence groups for $\mathcal{F}_{\mathrm{III}}$ as well.

IV. Finally we discuss the last subclass $\mathcal{F}_{\mathrm{IV}}$ of $\mathcal{F}$ admitting additional admissible transformations. It consists of equations

$$u_t + uu_x = \sum_{j=2}^{r} a_j u_j + a_0 u + b_1 x + b_0.$$

Since the arbitrary elements a_j are scaled under the action of the equivalence group of the class, it is reasonable to single out two subclasses of the class under question: $\mathcal{F}_{\mathrm{IV},0}$ with $a_j = 0$ for all $j = 2, \dots, r-1$, and complementary to it the subclass $\mathcal{F}_{\mathrm{IV},1}$ with at least one a_j nonzero.

Proposition 22. *The class $\mathcal{F}_{\mathrm{IV},1}$ is normalized in the generalized sense. Its generalized equivalence group is constituted by the point transformations of the form*

$$\tilde t = \bar T^1 t + \bar T^0, \quad \tilde x = \bar X^1 x + \bar X^0, \quad \tilde u = \frac{\bar X^1}{\bar T^1} u + \frac{\bar X^0_t}{\bar T^1},$$
$$\tilde a_j = \frac{(\bar X^1)^r}{\bar T^1} a_j, \quad a_0 = \frac{a_0}{\bar T^1}, \quad \tilde b_1 = \frac{b_1}{(\bar T^1)^2},$$
$$\tilde b_0 = \frac{1}{(\bar T^1)^2} \left(\bar X^1 b_0 + \bar c_3 \right),$$

where

$$\bar X^0(t,\theta) = \begin{cases} \bar c_1 e^{\lambda_1 t} + \bar c_2 e^{\lambda_2 t} + \bar c_3 & \textit{if} \quad \lambda_1 \neq 0, \quad D > 0, \\ \bar c_1 t + \bar c_2 e^{\lambda_2 t} + \bar c_3 & \textit{if} \quad \lambda_1 = 0, \quad D > 0, \\ \bar c_1 e^{b_1 t/2} + \bar c_2 t e^{b_1 t/2} + \bar c_3 & \textit{if} \quad b_1 \neq 0, \quad D = 0, \\ \bar c_1 t^2 + \bar c_2 t + \bar c_3 & \textit{if} \quad b_1 = 0, \quad D = 0, \\ e^{b_1 t/2}\big(\bar c_1 \sin(\sqrt{-D}t) & \\ \quad + \bar c_2 \cos(\sqrt{-D}t)\big) + \bar c_3 & \textit{if} \quad D < 0, \end{cases}$$

where $D = b_1^2 + 4a_0$ *and* $\lambda_{1,2} = (b_1 \pm \sqrt{D})/2$ *with* $|\lambda_1| < |\lambda_2|$, $\bar X^1$, $\bar T^0$, $\bar T^1$ *and $\bar c$'s run through the set smooth functions of the arbitrary elements* $\theta = (a_j, a_0, b_1, b_0)$ *with* $\bar X^1 \bar T^1 \frac{\partial(\tilde a_2, \dots, \tilde a_r, \tilde a_0, \tilde b_1, \tilde b_0)}{\partial(a_2, \dots, a_r, a_0, b_1, b_0)} \neq 0$.

Here the function $X^0(t)$ is a solution of the ordinary differential equation $X^0_{ttt} - b_1 X^0_{tt} - a_0 X^0_t = 0$ and thus it smoothly depends on the parameters b_1, a_0 and all the initial conditions.

The equivalence groupoid of the class $\mathcal{F}_{\mathrm{IV},0}$ depends essentially on the order r of equations therein. So we consider both the cases separately. First assume that $r > 2$ and denote the class of such equations $\mathcal{F}^{r>2}_{\mathrm{IV},0}$. This class admits additional admissible transformations if and only if $b_1 = a_0^2(r-1)/(r-2)^2$, so we reduce a tuple of the arbitrary elements thereof by the element b_1.

Proposition 23. *The class $\mathcal{F}^{r>2}_{\mathrm{IV},0}$ is normalized in the generalized sense. Its generalized equivalence group is constituted by the point transformations of the form*

$$\tilde t = \bar T, \quad \tilde x = \bar X^1 x + \bar X^0, \quad \tilde u = \frac{\bar X^1}{\bar T_t} u + \frac{\bar X^1_t}{\bar T_t} x + \frac{\bar X^0_t}{\bar T_t},$$
$$\tilde a_r = \frac{(\bar X^1)^r}{\bar T_t} a_r, \quad \tilde a_0 = \bar c_3, \quad \tilde b_0 = \bar c_5,$$

where the pair of smooth functions $\big(\bar T, \bar X^0\big)$ of t and the arbitrary elements θ equal to

$$(\bar c_1 t + \bar c_2,\ \bar c_7 t^2 + \bar c_6 t + \bar c_5) \quad \text{if} \quad a_0 = 0 \ \text{and}\ \bar c_3 = 0,$$

$$\Bigg(\frac{1}{\bar c_3}\ln|\bar c_3(\bar c_1 t + \bar c_2)|,\ \frac{\bar c_5 r^2}{\bar c_3^2(r-1)} + \frac{\bar c_6 t + \bar c_7}{|t + \bar c_2/(\bar c_1\bar c_3)|^{1/r}} - \frac{\bar c_3^2 b_0 \bar X^1}{2}\left(t + \frac{\bar c_2}{\bar c_1 \bar c_3}\right)^2\Bigg) \ \text{if}\ a_0 = 0 \ \text{and}\ \bar c_3 \neq 0,$$

$$\Bigg(\frac{r-2}{\bar c_3 r}\ln\left|\frac{1}{\bar c_1}e^{\frac{a_0 rt}{r-2}} + \frac{\bar c_2}{\bar c_1}\right|,\ \frac{\bar c_5(r-2)^2}{\bar c_3^2(r-1)} + \frac{\Big(\bar c_6 e^{\frac{a_0 rt}{r-2}} + \bar c_7\Big)}{\Big|\bar c_2/\bar c_1 + e^{\frac{a_0 rt}{r-2}}\Big|^{1/r}} + \frac{(r-2)^2 b_0}{(r-1)a_0^2}\bar X^1\Bigg) \quad \text{if} \quad a_0 \bar c_3 \neq 0,$$

$$\Bigg(\bar c_1 e^{\frac{a_0 rt}{r-2}} + \bar c_2,\ \frac{\bar c_5 \bar c_1^2}{2} e^{\frac{2a_0 rt}{r-2}} + \bar c_6 e^{\frac{a_0 rt}{r-2}} + \bar c_7 - \frac{(r-2)^2 b_0}{(r-1)a_0^2}\bar X^1\Bigg) \quad \text{if} \quad a_0 \neq 0 \ \text{and}\ \bar c_3 = 0,$$

where $\bar c$'s are arbitrary smooth functions of θ with $\bar c_4\bar T_t\frac{\partial(\tilde a_r,\tilde a_0,\tilde b_0)}{\partial(a_r,a_0,b_0)}\neq 0$ as well as $\bar X^1(t,\theta)=\varepsilon(\bar c_4\bar T_t)^{1/r}$ with $\varepsilon=\pm1$ and $\bar c_4\bar T_t>0$ if r is even and $\varepsilon=1$ otherwise.

The function $\bar X^0$ in the second pair in the second set gives a general solution of the linear inhomogeneous equation on $X^0(t)$,

$$\bar c_5=b_0\frac{X^1}{{T_t}^2}+\frac{1}{T_t}\left(\frac{X^0_t}{T_t}\right)_t-\bar c_3\frac{X^0_t}{T_t}-\frac{\bar c_3^2(r-1)}{(r-2)^2}X^0,$$

parameterized by the function T in the set and the corresponding $X^1(t)$. Any particular solution of this equation seems impossible to be found with standard techniques. Here instead, we used a method used for the class $\mathcal F_{\rm I,00}$ with gauging the arbitrary elements a_0 and b_0 to 0 first and composing equivalence transformations thereafter.

Due to the above condition on the arbitrary elements b_1 and a_0, the class $\mathcal F^{r=2}_{\rm IV,0}$ admits additional admissible transformations if and only if $a_0=0$. Abusing notations we denote the subclass singled out by this condition again by $\mathcal F^{r=2}_{\rm IV,0}$.

Proposition 24. *The point transformation of the form*

$$\tilde t=T(t),\quad \tilde x=X^1(t)x+X^0(t),\quad \tilde u=\frac{X^1}{T_t}u+\frac{X^1_t}{T_t}x+\frac{X^0_t}{T_t}$$

connects the source and target equations in the class $\mathcal F^{r=2}_{\rm IV,0}$ if and only if $(X^1)^2/T_t=\mathrm{const}\neq0$, the parameter function T runs through the solution set of the system

$$\left(\frac{T_{tt}}{T_t}\right)_t-\frac12\left(\frac{T_{tt}}{T_t}\right)^2=2\tilde b_1T_t^2-2b_1,$$

and the parameter function X^0 of t satisfies the equation

$$\frac{1}{T_t}\left(\frac{X^0_t}{T_t}\right)_t-\tilde b_1X^0=\tilde b_0-b_0\frac{X^1}{{T_t}^2}.$$

The last equation is linear inhomogeneous with respect to $X^0(T)$ for a given $T(t)$, while the differential equation on T is integrated in quadratures as an autonomous equation on $\ln|T_t|$ with standard techniques. Nonetheless, using the similar trick as was used for the class $\mathcal F_{\rm I,00}$, one can do better. More precisely, we gauge the arbitrary elements b_0 and b_1 to zeros by the point transformation of the form

$$\tilde t = T(t), \quad \tilde x = \sqrt{T_t}x + X^0(t), \quad \tilde u = \frac{u}{\sqrt{T_t}} + \frac{T_{tt}x}{2(T_t)^{3/2}} + \frac{X^0_t}{T_t},$$

where

$$\big(T, X^0\big) = \big(e^{2\sqrt{b_1}t},\ 4b_0(2\sqrt{b_1})^{3/2}e^{\sqrt{b_1}t}\big) \qquad \text{if} \quad b_1 > 0;$$
$$\big(T, X^0\big) = \left(\tan(\sqrt{-b_1}t),\ \frac{-b_0(-b_1)^{3/4}}{\cos\sqrt{-b_1}t}\right) \quad \text{if} \quad b_1 < 0,$$

obtaining the subclass $\mathcal{F}^{r=2}_{\mathrm{IV},00}$ of $\mathcal{F}^{r=2}_{\mathrm{IV},0}$. Thereafter we present the equivalence groupoid of $\mathcal{F}^{r=2}_{\mathrm{IV},0}$ by composing an equivalence transformation within the subclass $\mathcal{F}^{r=2}_{\mathrm{IV},00}$ with point transformations mapping equations in the superclass to equations in the subclass and vice versa.

Proposition 25. *The class $\mathcal{F}^{r=2}_{\mathrm{IV},00}$ is normalized in the usual sense. Its usual equivalence group is constituted by point transformations of the form*

$$\tilde t = T(t), \quad \tilde x = X^1(t)x + X^0, \quad \tilde u = \frac{X^1}{T_t}u + \frac{X^1_t}{T_t}x, \quad \tilde a_2 = c_4a_2,$$

where $X^1(t) = \varepsilon(c_4\delta)^{1/2}/(c_3t + c_0)$, $T = (c_1t + c_2)/(c_3t + c_0)$, X^0 and c's are arbitrary constants with $\delta = c_1c_0 - c_2c_3 \neq 0$, c_0, c_1, c_2 and c_3 being defined up to a nonzero constant, $c_4\delta > 0$ and $\varepsilon = \pm1$.

The point transformation $\mathcal{T}_{\tilde T,\tilde X^0}$ which maps an equation in $\mathcal{F}^{r=2}_{\mathrm{IV},00}$ to an equation in $\mathcal{F}^{r=2}_{\mathrm{IV},0}$ is of the same form as above,

$$t = \tilde T(\tilde t), \quad x = \sqrt{|\tilde T_{\tilde t}|}\tilde x + \tilde X^0(\tilde t), \quad u = \frac{\tilde u}{\sqrt{|\tilde T_{\tilde t}|}} + \frac{\tilde T_{\tilde t\tilde t}\tilde x}{2|\tilde T_{\tilde t}|^{3/2}} + \frac{\tilde X^0_{\tilde t}}{\tilde T_{\tilde t}},$$

where $\tilde T(T(t)) = t$ and $\tilde X^0(\tilde t) = -(X^0/T_t)(\tilde T(\tilde t))$, that is,

$$(\tilde T, \tilde X^0) = \left(\frac{\ln|\tilde t|}{2\sqrt{\tilde b_1}},\ \frac{\tilde b_0}{\tilde b_1}\right) \quad \text{if} \quad \tilde b_1 > 0;$$
$$(\tilde T, \tilde X^0) = \left(\frac{\arctan(\tilde t)}{\sqrt{-\tilde b_1}},\ \frac{\tilde b_0}{\tilde b_1}\right) \quad \text{if} \quad \tilde b_1 < 0.$$

Proposition 26. *The class $\mathcal{F}^{r=2}_{\mathrm{IV},0}$ is normalized in the generalized sense. Its effective generalized equivalence group is constituted by point transformations of the form*

$$
\begin{aligned}
&\tilde t = P^2(T(P^1(t))),\\
&\tilde x = \sqrt{P^2_{\bar t}P^1_t}\,X^1(\hat t)x + \sqrt{P^2_{\bar t}}\big(X^1R^1 + X^0\big) + R^2,\\
&\tilde u = \frac{X^1 u}{\sqrt{P^2_{\bar t}P^1_t T_{\hat t}}} + \left(\frac{X^1P^1_{tt}}{2T_{\hat t}\sqrt{P^2_{\bar t}(P^1_t)^3}} + \frac{X^1_{\hat t}\sqrt{P^1_t}}{\sqrt{P^2_{\bar t}}T_{\hat t}} + \frac{P^2_{\bar t\bar t}X^1}{2(P^2_{\bar t})^2\sqrt{P^1_t}}\right)x\\
&\quad + \frac{X^1R^1_t}{T_{\hat t}\sqrt{P^2_{\bar t}P^1_t}} + \frac{X^1_{\hat t}R^1}{T_{\hat t}} + \frac{P^2_{\bar t\bar t}}{2(P^2_{\bar t})^{3/2}}\big(X^1R^1 + X^0\big) + \frac{R^2_{\bar t}}{P^2_{\bar t}},\\
&\tilde a_2 = c_4a_2, \quad \tilde b_1 = \frac{1}{(P^2_{\bar t})^2}\left(\frac{1}{(P^1_t)^2}\left(b_1 + \frac12\left(\frac{P^1_{tt}}{P^1_t}\right)_t - \frac14\left(\frac{P^1_{tt}}{P^1_t}\right)^2\right)\right.\\
&\qquad\qquad \left. + \frac12\left(\frac{P^2_{\bar t\bar t}}{P^2_{\bar t}}\right)_{\bar t} - \frac14\left(\frac{P^2_{\bar t\bar t}}{P^2_{\bar t}}\right)^2\right),\\
&\tilde b_0 = \frac{1}{(P^2_{\bar t})^{3/2}}\left(\frac{b_0}{(P^1_t)^{3/2}} + \frac{1}{P^1_t}\left(\frac{R^1_t}{P^1_t}\right)_t\right) + \frac{1}{P^2_{\bar t}}\left(\frac{R^2_{\bar t}}{P^2_{\bar t}}\right)_{\bar t} - \tilde b_1R^2,
\end{aligned}
$$

where $\hat t = P^1(t)$, $\bar t = T(\hat t)$, $\tilde t = P^2(\bar t)$, $X^1(\hat t) = \varepsilon(c_4T_{\hat t})^{1/2}$, $T = (c_1\hat t + c_2)/(c_3\hat t + c_0)$ with $\delta = c_1c_0 - c_2c_3 \neq 0$;

$$
(P^1(t), R^1(t)) = \begin{cases}
(t,\ -b_0t^2/2) & \text{if } b_1 = 0,\\
(\tan(\sqrt{-b_1}t),\ -b_0(-b_1)^{3/4}/\cos(\sqrt{-b_1}t)) & \text{if } b_1 < 0,\\
(e^{2\sqrt{b_1}t},\ 4b_0(2\sqrt{b_1})^{-3/2}e^{\sqrt{b_1}t}) & \text{if } b_1 > 0;
\end{cases}
$$

X^0 and c's are arbitrary constants and the pair of smooth functions $(P^2(\bar t), R^2(\bar t))$ runs through the set

$$
\left\{\left(\bar t,\ \frac{c_6\bar t^2}{2}\right),\ \left(\frac{\ln|\bar t|}{2c_5},\ \frac{c_6}{c_5^2}\right),\ \left(\frac{\arctan\bar t}{c_5},\ -\frac{c_6}{c_5^2}\right)\right\},
$$

with c_i, $i = 0,\dots,3$, being defined up to a nonzero constant, $c_4\delta > 0$, $P^2_{\bar t} > 0$ and $\varepsilon = \pm1$.

In the notation of Proposition 26, the point transformation $\mathcal{T}_{P^2,R^2}$ maps an equation in $\mathcal{F}^{r=2}_{\mathrm{IV},00}$ to an equation in $\mathcal{F}^{r=2}_{\mathrm{IV},0}$ with arbitrary-element tuples (b_0, b_1) equal to $(c_6, 0)$, (c_6, c_5^2) and $(c_6, -c_5^2)$, respectively.

Acknowledgments. The author is grateful to Roman O. Popovych for useful discussions and interesting comments, and acknowledges the support of the Natural Sciences and Engineering Research Council of Canada.

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Збірник праць Інституту математики НАН України 2019, т. 16, № 1, 155–163

УДК 517.912:512.816

Поліноміальні розв'язки моделі нелінійного середовища з коливними включеннями

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У статті досліджуються поліноміальні розв'язки нелінійної системи ДРЧП, яка описує динаміку складного середовища з коливними включеннями. Зокрема показано, що коефіцієнти розв'язку при старших мономах задовольняють суттєво нелінійну динамічну систему гамільтонового типу. У цій системі при зміні керуючого параметру моделі існують періодичні, квазіперіодичні та хаотичні режими, вивчення яких здійснювалось на основі аналізу перерізів Пуанкаре та спектру ляпуновських показників.

The paper considers polynomial solutions to a nonlinear system of PDE describing dynamics of complex medium with oscillating inclusions. In particular, it is shown that the coefficients of leading monomials satisfy a strongly nonlinear dynamical system of Hamiltonian type. This system may have periodic, quasiperiodic, and chaotic regimes when the model's control parameter is varied. The observed regimes were studied by means of analysis of Poincaré sections and spectra of Lyapunov exponents.

1. Вступ. З аналізу експериментальних даних стосовно протікання фізичних процесів у гетерогенних середовищах випливає необхідність удосконалення класичних моделей механіки суцільного середовища шляхом врахування внутрішніх часово-просторових масштабів та додаткових ступенів свободи. Така ситуація, зокрема, склалася з описом поширення коротких акустичних хвиль у твердих тілах та твердих полімерах, з описом поведінки полікристалічних тіл та гранульованих середовищ в умовах складного або тривалого навантаження, високих градієнтів тощо [3, 9].

Намагання описати поведінку гранульованих середовищ у континуальному наближенні чи процеси локалізації деформації у гетерогенних матеріалах показали необхідність узагальнення класичних моделей з метою врахування мікроструктури, динаміки структурних елементів та взаємодії між ними. У цій роботі розглядаються моделі, які враховують коливальну динаміку структурних елементів.

2. Врахування коливних ступенів свободи в математичних моделях структурованих середовищ. Як зазначено у роботах [9, 10], структурні елементи природних геосередовищ перебувають у постійному коливальному русі. Для врахування коливної динаміки структурних елементів геосередовища в рамках теорії суцільного середовища можна використати узагальнені рівняння стану [14, 15, 16] або безпосередньо описати динаміку коливань у додатковому рівнянні руху. До останнього випадку належать моделі у вигляді взаємно проникаючих континуумів [1, 2, 7, 11, 12, 15, 16, 17, 22]

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x} - m\frac{\partial^2 w}{\partial t^2}, \quad \frac{\partial^2 w}{\partial t^2} + \omega^2 (w-u) = 0,$$
$$\sigma = \frac{E_1}{\rho}\frac{\partial u}{\partial x} + \frac{E_2}{\rho}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{E_3}{\rho}\left(\frac{\partial u}{\partial x}\right)^3, \tag{1}$$

де u — зміщення основного середовища густини ρ, w — зміщення осцилюючого включення густини $m\rho$ з власною частотою ω.

Останнє рівняння є рівнянням стану несучого середовища. Виконуючи знерозмірення моделі (1) згідно з формулами $t = \tau\overline{t}$, $x = c_0\tau\overline{x}$, $u = c_0\overline{u}$, $w = c_0\overline{w}$, $\overline{\omega} = \omega\tau$, запишемо її у такому вигляді

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x} - m\frac{\partial^2 w}{\partial t^2}, \quad \frac{\partial^2 w}{\partial t^2} + \omega^2 (w-u) = 0,$$
$$\sigma = e_1\frac{\partial u}{\partial x} + e_2\left(\frac{\partial u}{\partial x}\right)^2 + e_3\left(\frac{\partial u}{\partial x}\right)^3, \tag{2}$$

де $e_1 = \left(\frac{c}{c_0}\right)^2$, $e_2 = \frac{E_2}{c_0^2\tau\rho}$, $e_3 = \frac{E_3}{c_0^2\tau^2\rho}$, $c^2 = \frac{E_1}{\rho}$.

Зазначимо, що кубічне рівняння стану є доволі поширеним серед моделей природних матеріалів. Зокрема, коефіцієнти e_i пов'язуються з особливостями графіка рівняння стану наступним чином [20]:

$$\sigma = A_1\frac{\partial u}{\partial x} + A_2\left(\frac{\partial u}{\partial x}\right)^2 + A_3\left(\frac{\partial u}{\partial x}\right)^3,$$

де $A_1 = \frac{\sigma_1}{\varepsilon_1}\frac{5-n}{4}$, $A_2 = \frac{\sigma_1}{\varepsilon_1^2}\frac{1+n}{2}$, $A_3 = \frac{\sigma_1}{\varepsilon_1^3}\frac{3+n}{4}$, $n = \frac{\sigma_2}{\sigma_1} < 0$, параметри σ_1 та ε_1 вибираються із умови, що $\frac{\partial\sigma}{\partial\varepsilon} = 0$ у точці з координатами $(\varepsilon_1, \sigma_1)$. Для багатьох матеріалів $|n| \geq 1$.

Варто також зазначити, що подібні моделі виникають як континуальні наближення дискретних ланцюгів частинок, зв'язаних певними силами взаємодії [8, 13]. Також такі моделі у довгохвильовому наближенні описують розповсюдження збурень у середовищах з пухирцями газу, що не розчиняється [6].

3. Поліноміальні розв'язки моделі взаємно проникаючих континуумів. Зазначимо, що у роботах [2, 11, 15, 16, 17, 22] детально вивчено хвильові розв'язки моделі (1), які описуються нелінійними автономними звичайними диференціальними рівняннями, що дозволяє використати сучасні досягнення в галузі числового та якісного аналізу динамічних систем. Натомість, інші розв'язки вказаної моделі ретельно не досліджувались.

Наразі розглянемо поліноміальні розв'язки моделі (2). У цьому випадку розв'язок системи (2) будемо шукати у наступному вигляді [18]

$$u = a_1 + a_2 x + a_3 x^2, \quad w = b_1 + b_2 x + b_3 x^2, \tag{3}$$

де коефіцієнти $a_i = a_i(t)$ та $b_i = b_i(t)$ є функціями тільки часу.

Підставимо (3) в (2) та випишемо коефіцієнти при мономах x^k. При x^0:

$$-2a_3e_1 - 4a_2a_3e_2 - 6a_2^2a_3e_3 + \frac{\mathrm{d}^2a_1}{\mathrm{d}t^2} + m\frac{\mathrm{d}^2b_1}{\mathrm{d}t^2} = 0,$$
$$\frac{\mathrm{d}^2b_1}{\mathrm{d}t^2} + \omega^2\left(b_1 - a_1\right) = 0.$$

При x^1:

$$-8a_3^2e_2 - 24a_2a_3^2e_3 + \frac{\mathrm{d}^2a_2}{\mathrm{d}t^2} + m\frac{\mathrm{d}^2b_2}{\mathrm{d}t^2} = 0,$$
$$\frac{\mathrm{d}^2b_2}{\mathrm{d}t^2} + \omega^2(b_2 - a_2) = 0.$$

При x^2:

$$-24a_3^3e_3 + \frac{\mathrm{d}^2a_3}{\mathrm{d}t^2} + m\frac{\mathrm{d}^2b_3}{\mathrm{d}t^2} = 0, \quad \frac{\mathrm{d}^2b_3}{\mathrm{d}t^2} + \omega^2(b_3 - a_3) = 0. \tag{4}$$

Для спрощення системи (4) приймемо, що $24e_3 = -\mu < 0$, $a_3 = x$, $b_3 = y$, та виконаємо наступні масштабні перетворення

$$t = \tau \bar{t}, \quad x = \frac{\bar{x}}{q}, \quad y = \frac{\bar{y}}{q}.$$

Якщо параметри $\tau = \frac{1}{\omega}$, $q = \frac{\sqrt{\mu}}{\omega}$, то прийдемо до такої системи рівнянь (нехтуючи рисками над змінними):

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + x^3 = m\left(y - x\right), \quad \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + (y - x) = 0,$$

яка залежить лише від одного параметру зв'язку m.

Легко показати, що отримана система є гамільтоновою. Дійсно, за додаткової заміни змінної $y = \frac{\bar{y}}{\sqrt{m}}$ можна отримати систему

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + x^3 = m\left(\frac{y}{\sqrt{m}} - x\right), \quad \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \sqrt{m}\left(\frac{y}{\sqrt{m}} - x\right) = 0,$$

яка має представлення виду

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\frac{\partial V}{\partial x}, \quad \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -\frac{\partial V}{\partial y},$$

де потенціал $V = \frac{x^4}{4} + \frac{m}{2}\left(\frac{y}{\sqrt{m}} - x\right)^2$, що дозволяє записати функцію Гамільтона у вигляді

$$H = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}p_y^2 + V(x, y), \quad p_x = \frac{\mathrm{d}x}{\mathrm{d}t}, \quad p_y = \frac{\mathrm{d}y}{\mathrm{d}t}, \tag{5}$$

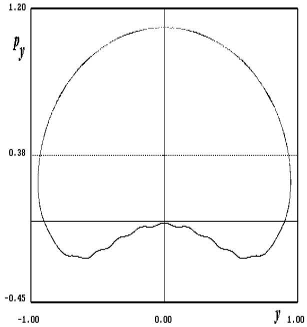
яка на траєкторіях системи набуває сталих значень, тобто $\frac{\mathrm{d}H}{\mathrm{d}t} = 0$, а отже, $H = h = \mathrm{const}$.

Розглянемо структуру фазового простору системи та її зміну при варіювання параметрів системи. Для цього використаємо техніку перерізів Пуанкаре, а у якості керуючого параметру виберемо параметр h — енергія системи. Початкові умови для інтегрування системи виберемо таким чином щоб вони задовольняли гамільтоніан H.

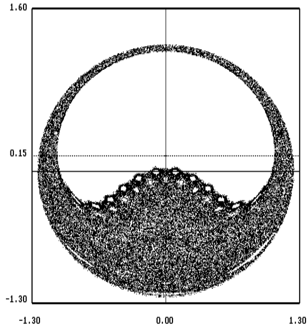
Рівняння $H = h = \mathrm{const}$ являє собою гіперповерхню в 4-вимірному фазовому просторі динамічної системи, на якій інтегральна траєкторія лишається увесь наступний час. Ця траєкторія формує тривимірний фазовий портрет, який можна вивчати за допомогою перерізів Пуанкаре, наприклад, площиною $x = 0$. Тоді у цій площині точки перетину траєкторії з січною площиною формують переріз Пуанкаре.

Зафіксуємо значення параметру $m = 0{,}01$ та значення змінної x у початковий момент часу $x(0) = x_0 = 0{,}45$. Інші початкові умови виберемо у формі $y(0) = \frac{dy}{dt}(0) = 0$ та $\frac{dx}{dt}(0) = \sqrt{2h - mx_0^2 - x_0^4/2}$.

У якості керуючого параметра виберемо h. При малих h коливання близькі до гармонічних, але при зростанні h їх форма починає усе більше відрізнятись від гармонічних під впливом нелінійності. Типовий переріз Пуанкаре, який відповідає квазіперіодичній траєкторії, зображено на рис. 1а. Подальше зростання h спричиняє ускладнення геометричної структури перерізів, що супроводжується розділенням замкнутої кривої на фрагменти, утворення на місці цих фрагментів нових замкнутих кривих і, нарешті, появою областей з хаотично заповненими точками. Зокрема, на рис. 1б представлено такий переріз Пуанкаре з хаотичною областю, побудований при $h = 0{,}79$.



(а)



(б)

Рис. 1. Переріз Пуанкаре при $h = 0{,}705$ (а) та $h = 0{,}79$ (б).

4. Алгоритм обчислення СЛП. Для встановлення характеру виявлених режимів використаємо ляпуновські показники [5], які є узагальненням власних значень стаціонарних точок динамічної системи та мультиплікаторів граничних циклів на випадок більш загальних траєкторій. Вони характеризують стійкість цієї траєкторії, тобто структуру фазового простору поблизу виділеної траєкторії. Тому аналіз цієї частини фазового простору можна робити у лінійному наближенні, яке задовольняє варіаційне рівняння $\frac{d\Phi}{dt} = DF\Phi$.

Зокрема, для стаціонарної точки, коли траєкторією є сталий розв'язок, рівняння у варіаціях є системою із сталими коефіцієнтами з розв'язком, що є суперпозицією частинних розв'язків $m_i = \exp(\lambda_i t)$, де λ_i — власні значення матриці лінеаризації.

Тоді ляпуновський показник $\lambda_i = \lim\limits_{t\to\infty} \frac{1}{t}\ln(|m_i|)$. В інших випадках $m_i(t)$ не є строго експоненційною функцією, але завдяки осередненню за великий проміжок часу ця величина збігається до експоненційної майже завжди (для майже всіх початкових умов).

Зазначимо, що визначення показників Ляпунова зіштовхується із проблемами числового характеру [21], оскільки не зберігається лінійна незалежність при числовому інтегруванні. Для вирішення цієї проблеми використовують процедуру ортогоналізації Грама–Шмідта [21]. У якості перевірки правильності обчислення показників використаємо той факт, що сума показників дорівнює середньому за часом значенню сліду матриці лінеаризації.

Зокрема, виберемо значення параметра $h = 0{,}705$ та початкову точку $\{0{,}45; 0; 1{,}1779; 0\}$ на профілі досліджуваної траєкторії, фазовий портрет якої зображено на рис. 1(а). У результаті роботи програми отримаємо спектр $\lambda = \left\{\pm 3\cdot 10^{-4}; \pm 5\cdot 10^{-4}\right\}$.

Оскільки система гамільтонова, то сума показників повинна бути нульовою, що для цієї системи забезпечується з точністю $\sim 10^{-6}$.

Для траєкторії рис. 1(б), побудованої при значенні параметра $h = 0{,}79$ з початкової точки $\{0{,}45; 0; 1{,}2479; 0\}$, спектр показників Ляпунова $\lambda = \left\{\pm 0{,}0176; \pm 7\cdot 10^{-4}\right\}$. Наявність додатного показника вказує на нестійкість траєкторії, що супроводжується появою хаотичного перерізу Пуанкаре.

5. Випадок слабко зв'язаної системи. Розглянемо детально випадок системи, коли параметр $m = \varepsilon m \ll 1$ малий і систему

$$\begin{aligned} &\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + x^3 = m\varepsilon(y - x), \\ &\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + (y - x) = 0 \end{aligned} \tag{6}$$

можна вважати слабко зв'язаною.

Для дослідження розв'язків такої системи використаємо результати робіт [19, 23]. Виконаємо у системі (6) наступну заміну змінних

$$\begin{aligned} &\psi_1 = \dot{x} + \mathrm{i}x, \quad \psi_2 = \dot{y} + \mathrm{i}y, \\ &\psi_1^\star = \dot{x} - \mathrm{i}x, \quad \psi_2^\star = \dot{y} - \mathrm{i}y, \quad \mathrm{i} = \sqrt{-1}. \end{aligned}$$

У нових змінних система набуде вигляду

$$\dot{\psi}_1 - \tfrac{\mathrm{i}}{2}\left(\psi_1 + \psi_1^\star\right) + \tfrac{\mathrm{i}}{8}\left(\psi_1 - \psi_1^\star\right)^3 + m\varepsilon\,\tfrac{\mathrm{i}}{2}\left(\psi_2 - \psi_2^\star - \psi_1 + \psi_1^\star\right) = 0,$$
$$\dot{\psi}_2 - \mathrm{i}\psi_2 + \tfrac{\mathrm{i}}{2}\left(\psi_1 - \psi_1^\star\right) = 0.$$

Наближений розв'язок отриманої системи шукатимемо у вигляді

$$\psi_1 = \varphi_1 \mathrm{e}^{\mathrm{i}t}, \quad \psi_2 = \varphi_2 \mathrm{e}^{\mathrm{i}t},$$

який описує резонансний випадок коливань поблизу відповідної частоти. У результаті осереднення за швидкою змінною отримаємо амплітудне рівняння

$$\begin{aligned}
&\dot{\varphi}_1 + \tfrac{\mathrm{i}}{2}\left(1 - m\varepsilon\right)\varphi_1 - \tfrac{3\mathrm{i}}{8}|\varphi_1|^2\varphi_1 + \tfrac{\mathrm{i}}{2}m\varepsilon\varphi_2 = 0,\\
&\dot{\varphi}_1^\star - \tfrac{\mathrm{i}}{2}\left(1 - m\varepsilon\right)\varphi_1^\star + \tfrac{3\mathrm{i}}{8}|\varphi_1|^2\varphi_1^\star - \tfrac{\mathrm{i}}{2}m\varepsilon\varphi_2^\star = 0,\\
&\dot{\varphi}_2 + \tfrac{\mathrm{i}}{2}\varphi_1 = 0, \quad \dot{\varphi}_2^\star - \tfrac{\mathrm{i}}{2}\varphi_1^\star = 0.
\end{aligned} \tag{7}$$

Легко переконатись, що отримана система має перший інтеграл

$$|\varphi_1|^2 + m\varepsilon|\varphi_2|^2 = N = \mathrm{const}.$$

Це дозволяє ввести нові змінні згідно із співвідношеннями

$$\varphi_1 = N\sin\theta \mathrm{e}^{\mathrm{i}\delta_1}, \quad \varphi_2 = \frac{N}{\sqrt{m\varepsilon}}\sin\theta \mathrm{e}^{\mathrm{i}\delta_2},$$

підстановка яких у (7) приводить до системи відносно θ та δ:

$$\begin{aligned}
&\dot{\theta} + \tfrac{1}{2}(1 - m\varepsilon) - \tfrac{3}{8}N^2\sin^2\theta + \sqrt{m\varepsilon}\,\mathrm{ctg}\,2\theta\cos\delta = 0,\\
&\dot{\delta} + \tfrac{1}{2}\sqrt{m\varepsilon}\sin\delta = 0.
\end{aligned} \tag{8}$$

Отримана система має стаціонарні точки з координатами $\delta = 0$ та θ, що задовольняє рівняння

$$G(\theta) \equiv \tfrac{1}{2}\left(1 - m\varepsilon\right) - \tfrac{3}{8}N^2\sin^2\theta + \sqrt{m\varepsilon}\,\mathrm{ctg}\,2\theta = 0.$$

Аналізуючи графіки функції $G(\theta)$ (рис. 2) протягом її періоду π можна переконатись, що кількість стаціонарних точок і їх тип залежать від величини N. Дійсно, при $N = 1{,}1$ функція $G(\theta)$ має два корені, які є центрами, оточеними замкнутими траєкторіями

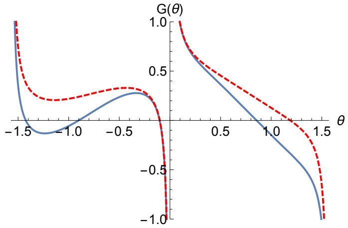


Рис. 2. Графіки функції $G(\theta)$ при $N = 1{,}1$ (пунктирна лінія) та $N = 1{,}5$ (суцільна крива).

системи (8). При зростанні N спостерігається деформування графіка, що спричинює появу додаткових двох коренів. Корінь, в якому похідна $G'_\theta(\theta) > 0$, відповідає сідловій стаціонарній точці, через яку проходять її сепаратриси. Наявність у фазовому просторі системи (8) сідлової точки служить ознакою того, що в системі (6) можуть реалізуватись хаотичні режими, що виникають унаслідок утворення стохастичного шару [4].

6. Висновки. Таким чином, у роботі показано, що модель складного середовища має поліноміальні розв'язки, поведінка яких у часі може бути як періодичною, квізіперіодичною так і хаотичною. Хаотичні режими пов'язані з утворенням стохастичного шару в околі сепаратрисних контурів сідлових точок системи.

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Збірник праць Інституту математики НАН України 2019, т. 16, № 1, 164–173

УДК 517.912:512.816

Симетрійні властивості та точні розв'язки (2+1)-вимірного лінійного рівняння ціноутворення азійських опціонів

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Використовуючи класичний метод Лі–Овсяннікова, знайдено максимальну алгебру інваріантності рівняння, яке випливає з рівняння ціноутворення азійських опціонів. За допомогою операторів цієї алгебри проведено симетрійну редукцію й побудовано інваріантні точні розв'язки як цього рівняння, так і, відповідно, рівняння ціноутворення азійських опціонів.

Using the classical Lie–Ovsyannikov method, a maximal invariance algebra was found for a equation that follows from the pricing equation of Asian options. Using the operators of that algebra symmetric reduction is carried out and invariant exact solutions are constructed for this equation, as well as for the pricing equation of Asian options, respectively.

1. Вступ. Традиційною моделлю в теорії фінансових ринків є модель Блека–Шоулза, яка описується лінійним диференціальним рівнянням в частинних похідних другого порядку з двома незалежними змінними [8]. Однак практичні дослідження вказують на те, що ця модель відповідно до зроблених припущень далека від адекватності реальним процесам, які відбуваються на фінансових ринках. Тому в останні десятиліття дослідники перейшли до більш складних моделей динаміки фінансових ринків, які описуються рівняннями з більшою кількістю незалежних змінних або нелінійними рівняннями. Методи дослідження таких моделей досить різні, зокрема часто використовують чисельні методи [10]. Як завжди, коли мова йде про процеси, що моделюються диференціальними рівняннями, важливо мати точні розв'язки таких рівнянь. Одним з найбільш ефективних методів, що дозволяють здійснити пошук розв'язків, є методи групового аналізу [4, 5]. Перші дослідження групових властивостей лінійного рівняння Блека–Шоулза було проведено в роботі [12]. В останні роки методами симетрійного аналізу досліджуються різні лінійні та нелінійні модифікації рівняння Блека–Шоулза [1, 9, 11, 13, 14, 15].

Ця робота присвячена симетрійному аналізу та побудові точних розв'язків лінійного рівняння ціноутворення азійських опціонів в неперервному часі $\tau \in [0; T]$ [7]:

$$\frac{\partial V}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + S\frac{\partial V}{\partial A} - rV = 0, \tag{1}$$

де T — термін дії контракту; $V = V(\tau, S, A)$ — функція вартості опціону; S — вартість базового активу; A — усереднене значення всіх наявних цін базових активів S до моменту часу τ; r і σ — сталі, що описують безризикову відсоткову ставку і волатильність акції відповідно. Рівняння (1) за допомогою заміни

$$V(\tau, S, A) = f(\tau, S, A)u(t(\tau, S, A), x(\tau, S, A), y(\tau, S, A)), \tag{2}$$

де функція $f(\tau, S, A)$ і нові незалежні змінні t, x, y визначаються, відповідно, формулами

$$\begin{gathered} f = s^{-m}\mathrm{e}^{-\frac{q\sigma^2}{2}(T-\tau)}, \quad t = \frac{\sigma^2}{2}(T-\tau), \quad x = S, \\ y = \frac{\sigma^2}{2}A, \quad m = \frac{r}{\sigma^2}, \quad q = m^2 + m, \end{gathered} \tag{3}$$

зводиться до рівняння

$$\frac{\partial u}{\partial t} = x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial y}, \tag{4}$$

де $u = u(t, x, y)$.

2. Симетрійні властивості. Для дослідження симетрійних властивостей рівняння (4) використаємо класичний метод Лі–Овсяннікова [4]. У результаті отримаємо:

Теорема 1. *Максимальна алгебра Лі інваріантності рівняння* (4) *генерується такими диференціальними операторами:*

$$\begin{aligned}&\langle X_1 = \partial_t, \quad X_2 = \partial_y, \quad X_3 = u\partial_u, \quad X_4 = x\partial_x + y\partial_y,\\ &X_5 = xy\partial_x + \frac{1}{2}y^2\partial_y + \frac{1}{2}xu\partial_u, \quad X_\infty = \beta(t,x,y)\partial_u\rangle,\end{aligned} \tag{5}$$

де функція $\beta(t,x,y)$ *є довільним розв'язком рівняння* (4).

Далі не враховуватимемо оператор симетрії $X_\infty = \beta(t,x,y)\partial_u$, який притаманний лінійним рівнянням і обумовлює принцип суперпозиції. Задача опису таких операторів еквівалентна пошуку загального розв'язку таких рівнянь. Зазначимо, що диференціальні оператори симетрії (5) X_i, $i = 1,\ldots,5$ утворюють базис 5-вимірної алгебри Лі L_5, яка є прямою сумою алгебр Лі $\langle X_1\rangle$, $\langle X_3\rangle$ і $\langle X_2, X_4, X_5\rangle$, тобто

$$L_5 = X_1 \oplus X_3 \oplus \langle X_2, X_4, X_5\rangle.$$

Алгебра L_5 ізоморфна алгебрі

$$A_5 = \langle e_1, e_2, e_3, e_4, e_5\rangle = \langle e_1, e_2, e_3\rangle \oplus \langle e_4, e_5\rangle = \mathfrak{sl}(2,\mathbb{R}) \oplus 2A_1,$$

яка є прямою сумою підалгебр $\langle e_1, e_2, e_3\rangle = \mathfrak{sl}(2,\mathbb{R})$ та $\langle e_4, e_5\rangle = 2A_1$.

Ізоморфізм між алгебрами L_5 і A_5 встановлюється лінійними перетвореннями:

$$\begin{aligned}&e_1 = 2X_4 = 2x\partial_x + 2y\partial_y,\\ &e_2 = -2X_5 = -2xy\partial_x - y^2\partial_y - xu\partial_u,\\ &e_3 = X_2 = \partial_y, \quad e_4 = X_1 = \partial_t, \quad e_5 = X_3 = u\partial_u.\end{aligned}$$

3. Симетрійна редукція. Одним із застосувань симетрійних властивостей диференціальних рівнянь з частинними похідними є симетрійна редукція рівнянь з нетривіальною симетрією до рівнянь з меншою кількістю незалежних змінних. Метод редукції за оптимальною системою підалгебр максимальної алгебри Лі інваріантності добре відомий і достатньо алгоритмічний (див., напр., [3, 4, 5]).

Для того, щоб використати можливі редукції рівняння (4), необхідно знайти нееквівалентні підалгебри алгебри A_5. Зокрема, одновимірним підалгебрам буде відповідати редукція рівняння (4) до рівняння з частинними похідними від двох незалежних змінних. Симетрійна редукція рівняння (4) до звичайних диференціальних рівнянь передбачає наявність списку двовимірних підалгебр алгебри A_5.

У роботі [5] наведено метод класифікації підалгебр дійсних алгебр Лі з точністю до перетворень, які визначають групи внутрішніх автоморфізмів цих алгебр Лі. Згідно з цим методом одновимірні підалгебри алгебри A_5 вичерпуються алгебрами

$$\begin{aligned}&\langle e_4+\alpha e_5\rangle, \quad \langle e_5\rangle, \quad \langle e_1+\alpha e_4+\beta e_5\rangle,\\ &\langle e_2+\alpha e_4+\beta e_5\rangle, \quad \langle e_2-e_3+\alpha e_4+\beta e_5\rangle,\end{aligned} \tag{6}$$

а двовимірні підалгебри – такими алгебрами:

$$\begin{aligned}&\langle e_4, e_5\rangle, \quad \langle e_1+\alpha e_5, e_4+\beta e_5\rangle, \quad \langle e_2+\alpha e_5, e_4+\beta e_5\rangle,\\ &\langle e_1+\alpha e_4+\beta e_5, e_2\rangle, \quad \langle e_2-e_3+\alpha e_4, e_5\rangle, \quad \langle e_2+\alpha e_4, e_5\rangle,\\ &\langle e_2-e_3+\alpha e_5, e_4+\beta e_5\rangle, \quad \langle e_1+\alpha e_4, e_5\rangle, \quad \alpha,\beta\in\mathbb{R}.\end{aligned} \tag{7}$$

Використаємо оператори (6) і (7) для побудови точних розв'язків рівняння (4). Одновимірним підалгебрам (6) буде відповідати редукція рівняння (4) до рівняння з частинними похідними від двох незалежних змінних. Використовуючи перелік одновимірних підалгебр (6), насамперед відбираємо ті підалгебри, які задовольняють необхідну умову існування редукції [3]. Отже, такими підалгебрами будуть

$$\begin{aligned}&\langle e_4+\alpha e_5\rangle, \quad \langle e_1+\alpha e_4+\beta e_5\rangle,\\ &\langle e_2+\alpha e_4+\beta e_5\rangle, \quad \langle e_2-e_3+\alpha e_4+\beta e_5\rangle.\end{aligned} \tag{8}$$

Для кожної з одновимірних підалгебр (8) подано, відповідно, анзац та редуковане рівняння:

1) $\langle e_4+\alpha e_5\rangle\colon\ u=\mathrm{e}^{\alpha t}f(\omega_1,\omega_2), \quad \omega_1=x, \quad \omega_2=y,$
$\omega_1^2 f_{\omega_1\omega_1}+\omega_1 f_{\omega_2}-\alpha f=0;$

2) $\langle e_1+\alpha e_4+\beta e_5\rangle\colon$ якщо $\alpha\neq 0,$ то $u=\exp\left(\dfrac{\beta t}{\alpha}\right)f(\omega_1,\omega_2),$

$$\omega_1=\frac{x}{y}, \quad \omega_2=y\exp\left(\frac{-2t}{\alpha}\right),$$

$$\omega_1^2 f_{\omega_1\omega_1}+\omega_2\left(\frac{2}{\alpha}+\omega_1\right)f_{\omega_2}-\omega_1^2 f_{\omega_1}-\frac{\beta}{\alpha}f=0;$$

якщо $\alpha=0,$ то $u=y^{\beta/2}f(\omega_1,\omega_2), \quad \omega_1=t, \quad \omega_2=\dfrac{x}{y},$

$$\omega_2^2 f_{\omega_2\omega_2} - \omega_2^2 f_{\omega_2} - f_{\omega_1} + \frac{\beta}{2}\omega_2 f = 0;$$

3) $\langle e_2 + \alpha e_4 + \beta e_5 \rangle$:

$$u = \exp\left(\frac{\beta + x}{y}\right) f(\omega_1, \omega_2), \quad \omega_1 = t - \frac{\alpha}{y}, \quad \omega_2 = \frac{x}{y^2},$$

$$\omega_2^2 f_{\omega_2\omega_2} + (\alpha\omega_2 - 1) f_{\omega_1} - \beta\omega_2 f = 0;$$

4) $\langle e_2 - e_3 + \alpha e_4 + \beta e_5 \rangle$:

$$u = \exp\left(-\beta \operatorname{arctg} y + \frac{xy}{y^2 + 1}\right) f(\omega_1, \omega_2),$$

$$\omega_1 = t + \alpha \operatorname{arctg} y, \quad \omega_2 = \frac{x}{y^2 + 1},$$

$$\omega_2^2 f_{\omega_2\omega_2} + (\alpha\omega_2 - 1) f_{\omega_1} + (\omega_2 - \beta)\omega_2 f = 0.$$

Використовуючи інваріанти операторів симетрій рівняння (4), які відповідають знайденим двовимірним підалгебрам, можна провести редукцію цього рівняння до звичайних диференціальних рівнянь.

Для кожної із двовимірних підалгебр (7), які задовольняють необхідній умові існування редукції, подано, відповідно, анзац та редуковане рівняння:

1) $\langle e_1 + \alpha e_5, e_4 + \beta e_5 \rangle$: $u = y^{\alpha/2} \mathrm{e}^{\beta t} f(\omega), \quad \omega = \dfrac{x}{y},$

$$\omega^2 \ddot{f} - \omega^2 \dot{f} + \left(\frac{\alpha}{2}\omega - \beta\right) f = 0;$$

2) $\langle e_2 + \alpha e_5, e_4 + \beta e_5 \rangle$: $u = \exp\left(\beta t + \dfrac{\alpha + x}{y}\right) f(\omega), \quad \omega = \dfrac{\sqrt{x}}{y},$

$$\omega^2 \ddot{f} - \omega \dot{f} - 4(\alpha\omega^2 + \beta) f = 0;$$

3) $\langle e_1 + \alpha e_4 + \beta e_5, e_2 \rangle$:

якщо $\alpha \neq 0$, то $u = \exp\left(\dfrac{\beta t}{\alpha} + \dfrac{x}{y}\right) f(\omega),$

$$\omega = \frac{x}{y^2} \exp\left(\frac{2t}{\alpha}\right), \quad \alpha\omega^2 \ddot{f} - 2\omega \dot{f} - \beta f = 0;$$

якщо $\alpha = 0$, то $u = \left(\dfrac{y}{\sqrt{x}}\right)^{\beta} \exp\left(\dfrac{x}{y}\right) f(\omega), \quad \omega = t,$

$$\dot{f} - \left(\frac{\beta^2 + 2\beta}{2}\right) f = 0;$$

4) $\langle e_2 - e_3 + \alpha e_5, e_4 + \beta e_5 \rangle$:

$$u = \exp\left(\beta t - \alpha \operatorname{arctg} y + \frac{xy}{y^2+1}\right) f(\omega), \quad \omega = \frac{x}{y^2+1},$$

$$\omega^2 \ddot{f} + (\omega^2 - \alpha\omega - \beta) f = 0,$$

де крапка визначає диференціювання функції f за змінною ω.

4. Точні розв'язки лінійного рівняння ціноутворення азійських опціонів. Побудуємо інваріантні розв'язки рівняння (4). Розглянемо одновимірну підалгебру $\langle e_2 + \alpha e_4 + \beta e_5 \rangle$, якій відповідає редуковане рівняння

$$\omega_2^2 f_{\omega_2\omega_2} + (\alpha\omega_2 - 1) f_{\omega_1} - \beta\omega_2 f = 0.$$

Якщо $\alpha = \beta = 0$, рівняння має вигляд

$$\omega_2^2 f_{\omega_2\omega_2} - f_{\omega_1} = 0,$$

частинними розв'язками якого будуть функції [6]

$$f_(\omega_1, \omega_2) = (C_1 \ln \omega_2 + C_2)\sqrt{\omega_2} \exp\left(-\frac{\omega_1}{4}\right),$$

$$f_(\omega_1, \omega_2) = \left(2\omega_1 + \ln^2 \omega_2\right)\sqrt{\omega_2} \exp\left(-\frac{\omega_1}{4}\right),$$

$$f_(\omega_1, \omega_2) = \omega_2^{\mu} \exp\left(\left(\mu^2 - \mu\right)\omega_1\right),$$

де $C_1, C_2, \mu \in \mathbb{R}$.

Підставляючи ці функції у відповідний анзац для залежної змінної u, отримаємо розв'язки рівняння (4):

$$u = \left(C_1 \ln \frac{x}{y^2} + C_2\right) \frac{\sqrt{x}}{y} \exp\left(\frac{x}{y} - \frac{t}{4}\right),$$

$$u = \left(2t + \ln^2 \frac{x}{y^2}\right) \frac{\sqrt{x}}{y} \exp\left(\frac{x}{y} - \frac{t}{4}\right), \tag{9}$$

$$u = \left(\frac{x}{y^2}\right)^{\mu} \exp\left(\frac{x}{y} + (\mu^2 - \mu)t\right).$$

Використовуючи заміну змінних (2), (3) для функцій (9) отримаємо точні розв'язки рівняння (1):

$$V(\tau, S, A) = \left(C_1 \ln \frac{4S}{\sigma^4 A^2} + C_2\right) \frac{2S^{-r\sigma^{-2}}\sqrt{S}}{\sigma^2 A}$$

$$\times \exp\left(\frac{2S}{\sigma^2 A} - \frac{\sigma^2(T-\tau)}{8} + \left(r^2\sigma^{-2}+r\right)\frac{(\tau-T)}{2}\right),$$

$$V(\tau,S,A) = \left(\sigma^2(T-\tau) + \ln^2\frac{4S}{\sigma^4A^2}\right)\frac{2S^{-r\sigma^{-2}}\sqrt{S}}{\sigma^2 A}$$
$$\times \exp\left(\frac{2S}{\sigma^2 A} - \frac{\sigma^2(T-\tau)}{8} + \left(r^2\sigma^{-2}+r\right)\frac{(\tau-T)}{2}\right),$$

$$V(\tau,S,A) = S^{-r\sigma^{-2}}\left(\frac{4S}{\sigma^4A^2}\right)^{\mu}\exp\left(\frac{2S}{\sigma^2 A} + \left(\mu^2-\mu\right)\frac{\sigma^2(T-\tau)}{2}\right.$$
$$\left. + (r^2\sigma^{-2}+r)\frac{(\tau-T)}{2}\right).$$

Далі, розглянемо двовимірну підалгебру $\langle e_1 + \alpha e_4 + \beta e_5, e_2\rangle$.

У випадку $\alpha = 0$ редуковане рівняння має вигляд $\dot{f} - \left(\frac{\beta^2+2\beta}{2}\right)f = 0$. Підставляючи розв'язок цього рівняння у анзац для залежної змінної, отримаємо розв'язок рівняння (4):

$$u = C\left(\frac{y}{\sqrt{x}}\right)^{\beta}\exp\left(\frac{x}{y} + \frac{\beta^2+2\beta}{2}t\right), \tag{10}$$

де C — довільна стала. Після підстановки (10) в (2) отримаємо точні розв'язки рівняння (1):

$$V(\tau,S,A) = CS^{-r\sigma^{-2}}\left(\frac{\sigma^2 A}{2\sqrt{S}}\right)^{\beta}$$
$$\times \exp\left(\frac{2S}{\sigma^2 A} + \left(\frac{(\beta^2+\beta)\sigma^2}{4} - \frac{r^2}{\sigma^2} - r\right)(T-\tau)\right).$$

Якщо $\alpha \neq 0$, тоді двовимірній підалгебрі відповідає редуковане рівняння

$$\omega^2\ddot{f} - \frac{2\omega}{\alpha}\dot{f} - \frac{\beta}{\alpha}f = 0. \tag{11}$$

Рівняння (11) є рівнянням Ейлера і воно має такі розв'язки [2]:

а) якщо $\left(1+\dfrac{2}{\alpha}\right)^2 > -\dfrac{4\beta}{\alpha}$, то

$f(\omega) = \omega^{(\alpha+2)/2\alpha}\left(C_1\omega^{\mu} + C_2\omega^{-\mu}\right);$

b) якщо $\left(1+\dfrac{2}{\alpha}\right)^2 = -\dfrac{4\beta}{\alpha}$, то

$f(\omega) = \omega^{(\alpha+2)/2\alpha}(C_1 + C_2 \ln\omega)$;

c) якщо $\left(1+\dfrac{2}{\alpha}\right)^2 < -\dfrac{4\beta}{\alpha}$, то

$f(\omega) = \omega^{(\alpha+2)/2\alpha}(C_1 \sin(\mu\ln\omega) + C_2\cos(\mu\ln\omega))$,

де $\mu = \frac{1}{2}\sqrt{\left|\left(1+\frac{2}{\alpha}\right)^2 + \frac{4\beta}{\alpha}\right|}$; C_1 і C_2 — довільні сталі.

Наведемо частині точні розв'язкі, які відповідають випадкам a), b), c).

Якщо $\alpha = \beta = -2$ (випадок a)), маємо розв'язок

$$f(\omega) = C_1\omega + C_2\omega^{-1}.$$

Підставляючи цю функцію у відповідний анзац для залежної змінної, отримаємо розв'язок рівняння (4):

$$u = \left(C_1\frac{x}{y^2\mathrm{e}^t} + C_2\frac{y^2\mathrm{e}^t}{x}\right)\exp\left(\frac{x}{y}+t\right). \tag{12}$$

Після підстановки (12) в (2) отримаємо точний розв'язок рівняння (1):

$$\begin{aligned} V(\tau,S,A) = \exp&\left(\frac{2S}{\sigma^2 A} + \frac{\sigma^2(T-\tau)}{2} + \left(r^2\sigma^{-2}+r\right)\frac{\tau-T}{2}\right)\\ &\times\left(4C_1 S^{-r\sigma^{-2}+1}\sigma^{-4}A^{-2}\exp\left(\frac{\sigma^2(\tau-T)}{2}\right)\right.\\ &\left.+\frac{C_2}{4}S^{-r\sigma^{-2}-1}\sigma^4A^2\exp\left(\frac{\sigma^2(T-\tau)}{2}\right)\right), \end{aligned}$$

де C_1 і C_2 — довільні сталі.

Якщо $\alpha = 2$, $\beta = -2$ (випадок b)), маємо розв'язок

$$f(\omega) = \omega(C_1 + C_2\ln\omega).$$

Підставляючи цю функцію у відповідний анзац для залежної змінної, отримаємо розв'язок рівняння (4):

$$u = \left(C_1 + C_2\left(\ln\frac{x}{y^2}+t\right)\right)\frac{x}{y^2}\exp\left(\frac{x}{y}\right). \tag{13}$$

Після підстановки (13) в (2) отримаємо точний розв'язок рівняння (1):

$$V(\tau,S,A)=4\sigma^{-4}A^{-2}S^{1-r\sigma^{-2}}\exp\left(\frac{2S}{\sigma^2 A}+\left(r^2\sigma^{-2}+r\right)\frac{\tau-T}{2}\right)$$
$$\times\left(C_1+C_2\ln\left(4S\sigma^{-4}A^{-2}\exp\left(\frac{\sigma^2(T-\tau)}{2}\right)\right)\right),$$

де C_1 і C_2 — довільні сталі.

Якщо $\alpha=-2$, $\beta=2$ (випадок c)), маємо розв'язок

$$f(\omega)=C_1\sin(\ln\omega)+C_2\cos(\ln\omega).$$

Підставляючи цю функцію у відповідний анзац для залежної змінної, отримаємо розв'язок рівняння (4):

$$u=\left(C_1\sin\left(\ln\frac{x}{y^2}-t\right)+C_2\cos\left(\ln\frac{x}{y^2}-t\right)\right)\exp\left(\frac{x}{y}-t\right). \quad (14)$$

Після підстановки (14) в (2) отримаємо точний розв'язок рівняння (1):

$$V(\tau,S,A)=S^{-r\sigma^{-2}}\left(C_1\sin\left(\ln\left(4S\sigma^{-4}A^{-2}\right)+\sigma^2(\tau-T)/2\right)\right.$$
$$\left.+C_2\cos\left(\ln\left(4S\sigma^{-4}A^{-2}\right)+\sigma^2(\tau-T)/2\right)\right)$$
$$\times\exp\left(\frac{2S}{\sigma^2 A}-\frac{\sigma^2(T-\tau)}{2}+(r^2\sigma^{-2}+r)\frac{\tau-T}{2}\right),$$

де C_1 і C_2 — довільні сталі.

Висновки. У роботі лінійне рівняння ціноутворення азійських опціонів за допомогою заміни змінних було зведено до рівняння, яке є простішим у використанні. Досліджено симетрійні властивості отриманого рівняння. Симетрійні властивості використані для побудови інваріантних анзаців, які редукують рівняння до диференціальних рівнянь відносно меншої кількості незалежних змінних. У результаті розв'язування деяких редукованих рівнянь побудовано точні розв'язки рівняння. Виконуючи зворотну заміну змінних, отримані точні розв'язки лінійного рівняння ціноутворення азійських опціонів. У майбутньому планується більш детально дослідити властивості отриманих розв'язків щодо їх можливого застосування.

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Збірник праць Інституту математики НАН України 2019, т. 16, № 1, 174–180

УДК 517.912:512.816

Lie–Bäcklund symmetry reduction of nonlinear and non-evolution equations

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Досліджено застосування операторів симетрії Лі–Беклунда, які допускаються звичайним диференціальним рівнянням, для редукції диференціальних рівнянь з частинними похідними. Анзаци для залежної змінної побудовано інтегруванням звичайних диференціальних рівнянь. Показано, що метод можна застосовувати для рівнянь еволюційного і нееволюційного типу. У рамках цього підходу знайдено ров'язок, що залежить від довільної функції одного аргументу.

The application of Lie–Bäcklund symmetry operators admitted by ordinary differential equations for reducing partial differential equations are studied. The ansatze for dependent variable are constructed by integrating ordinary differential equations. We show that the method is applicable for nonlinear evolution and non-evolution types equations. In the framework of the approach we construct the solution depending on arbitrary smooth function on one variable.

1. Introduction. It is a known fact that the symmetry groups of nonlinear PDEs are being used for finding special solutions invariant with respect to a certain subgroup of the complete symmetry group of the equation. Invariant solutions are constructed by solving a reduced equation with smaller number of independent variables than the initial equation, an ODE in particular. Conditional symmetry is a generalization of a classical Lie symmetry of differential equations and substantially extends the possibilities of construction of solutions of nonlinear PDEs. It must be noted, that the conditional symmetry method can be effectively used both for integrable (in some sense) and non-integrable equations. In [1, 5] concept of conditional Lie–Bäcklund symmetry of evolution equation, which is a generalization of point conditional symmetry, is proposed. In the framework of this approach we obtain reduced system of ODEs. The relationship of generalized conditional symmetry of evolution equations to compatibility of systems of differential equations is studied in [2]. In [3] Svirshchevskii used Lie–Bäcklund symmetry of linear homogeneous ODEs for reducing evolution equations to a system of ODEs. To apply this method we have to solve the inverse symmetry problem, namely to find linear homogeneous ODEs which admit given Lie–Bäcklund symmetry operator. We study the reduction of nonlinear generalization of the heat equation and modified Korteweg–de Vries equation by using Lie–Bäcklund symmetry property of linear and nonlinear ODEs [4]. It allows us to construct solutions for equations of evolution and non-evolution types.

2. Application of the symmetry reduction method. In this section we discuss the relationship between the Lie–Bäcklund symmetry of ordinary differential and reduction of generalized version of Korteweg–de Vries equation and nonlinear heat equation.

Example 1. We show how to apply the Lie–Bäcklund symmetry reduction using mKdV equation as an example. First step is finding an ODE or ODEs invariant under the operator $K[u] = K\big(x, u, \frac{\partial u}{\partial x}, \dots, \frac{\partial^p u}{\partial x^p}\big)$, in which case $K[u]$ is the right-hand side of the mKdV equation. Let p be a positive integer. Consider the ODE to be of the form $u_{xx}+g(u,u_x)=0$, where g is a differentiable function of u and u_x. Invariance condition for such ODE reads as

$$X^\infty\big(u_{xx} + g(u,u_x)\big)\big|_{u_{xx}+g(u,u_x)=0} = 0, \tag{1}$$

where X^∞ is a prolongation of the vector field $X = (u_{xxx} + u^p u_x)\frac{\partial}{\partial u}$ on the jet space. After necessary substitutions equation (1) becomes

$$\begin{aligned} &pu^{p-1}u_x(g_{u_x}u_x - 3g) - u_x^3 g_{uuu}\\ &\quad + 3u_x(u_x g_u g_{uu_x} + u_x g g_{uuu_x} + g g_{uu})\\ &\quad - 3g(u_x g_{u_x} g_{uu_x} + u_x g_u g_{u_x u_x} + u_x g g_{uu_x u_x} + g g_{uu_x})\\ &\quad + g^2(3g_{u_x} g_{u_x u_x} + g g_{u_x u_x u_x}) + p(p-1)u^{p-2}u_x^3 = 0. \end{aligned} \tag{2}$$

The subscripts u and u_x denote differentiation with respect to u and u_x. We assume that $g(u,u_x) = \sum\limits_{i=j}^{k} \lambda_i(u)u_x^i$ for some integers j and k. In that case the left-hand side of the equation (2) becomes a power series of u_x. For every $k \geq 3$ and $j \leq -1$ the coefficients for the highest and lowest powers of u_x are $2k(k-1)(2k-1)\lambda_k^4$ and $2j(j-1)(2j-1)\lambda_j^4$ respectively, which implies $\lambda_i = 0$ for $i \notin \{0,1,2\}$, meaning $g = \lambda_2(u)u_x^2 + \lambda_1(u)u_x + \lambda_0(u)$. The six remaining coefficients in the now power series (2) become the determining equations. They are

$$12\lambda_2^4 - 30\lambda_2'\lambda_2^2 + 6\lambda_2'^2 + 9\lambda_2''\lambda_2 - \lambda_2''' = 0, \tag{3}$$

$$30\lambda_1\lambda_2^3 - 48\lambda_2'\lambda_1\lambda_2 - 15\lambda_1'\lambda_2^2 + 9\lambda_2'\lambda_1' + 9\lambda_2''\lambda_1 + 6\lambda_1''\lambda_2 - \lambda_1''' = 0, \tag{4}$$

$$24\lambda_0\lambda_2^3 + 23\lambda_1^2\lambda_2^2 - 42\lambda_2'\lambda_0\lambda_2 - 18\lambda_2'\lambda_1^2 - 21\lambda_1'\lambda_1\lambda_2 - 6\lambda_0'\lambda_2^2 + 6\lambda_2'\lambda_0' + 3\lambda_1'^2 - 9\lambda_2''\lambda_0 + 6\lambda_1''\lambda_1 + 3\lambda_0''\lambda_2 - \lambda_0''' - pu^{p-1}\lambda_2 + p(p-1)u^{p-2} = 0, \tag{5}$$

$$36\lambda_0\lambda_1\lambda_2^2 + 6\lambda_1^3\lambda_2 - 30\lambda_2'\lambda_0\lambda_1 - 18\lambda_1'\lambda_0\lambda_2 - 6\lambda_1'\lambda_1^2 - 6\lambda_0'\lambda_1\lambda_2 + 3\lambda_1'\lambda_0' + 6\lambda_1''\lambda_0 + 3\lambda_0''\lambda_1 - 2pu^{p-1}\lambda_1 = 0, \tag{6}$$

$$12\lambda_0^2\lambda_2^2 + 12\lambda_0\lambda_1^2\lambda_2 - 12\lambda_2'\lambda_0^2 - 9\lambda_1\lambda_1'\lambda_0 - 6\lambda_0'\lambda_0\lambda_2 + 3\lambda_0''\lambda_0 - 3pu^{p-1}\lambda_0 = 0, \tag{7}$$

$$6\lambda_0^2\lambda_1\lambda_2 - 3\lambda_0^2\lambda_1' = 0. \tag{8}$$

Based on equations (3)–(8) we will consider four cases:

Case (i): $\lambda_0 = 0$, $\lambda_2 = \frac{\omega}{u}$, $\omega \in \left\{-1, -\frac{1}{2}, 0\right\}$. Because we restricted p to be a nonzero natural number, any assumptions other than $\lambda_1 = \lambda_2 = 0$, $p = 1$ lead to contradictions, therefore a solution exists only for $p = 1$ and it is $\lambda_i = 0$, which means that the invariant equations is

$$u_{xx} = 0. \tag{9}$$

Case (ii): $\lambda_2 = 0$, $\lambda_1 = \kappa = \text{const}$. Here $\kappa = 0$, $\lambda_0 = \frac{1}{p+1}u^{p+1} + \alpha_1 u + \alpha_2$, $\alpha_i \in \mathbb{R}$, therefore the invariant equation is

$$u_{xx} + \tfrac{1}{p+1}u^{p+1} + \alpha_1 u + \alpha_2 = 0. \tag{10}$$

Case (iii): $\lambda_2 = -\frac{1}{2u}$, $\lambda_1 = \frac{\kappa}{u}$, $\kappa = \text{const}$. Here $\kappa = 0$, $\lambda_0 = \frac{1}{p+2}u^{p+1} + \beta_1 u + \frac{\beta_2}{u}$, $\beta_i \in \mathbb{R}$, therefore the invariant equation is

$$u_{xx} - \tfrac{u_x^2}{2u} + \tfrac{1}{p+2}u^{p+1} + \beta_1 u + \tfrac{\beta_2}{u} = 0. \tag{11}$$

Case (iv): $\lambda_2 = -\frac{1}{u}$, $\lambda_1 = \frac{\kappa}{u^2}$, $\kappa = \text{const}$. Here $\kappa = 0$, $\lambda_0 = \frac{p}{(p+1)(p+2)}u^{p+1} + \gamma_1 + \frac{\gamma_2}{u}$, $\gamma_i \in \mathbb{R}$, therefore the invariant equation is

$$u_{xx} - \tfrac{u_x^2}{u} + \tfrac{p}{(p+1)(p+2)}u^{p+1} + \gamma_1 + \tfrac{\gamma_2}{u} = 0. \tag{12}$$

Second step is the variation of the parameters for the solution of the ODE for time dependence. Outcome of this step is an ansatz for the PDE, and in this case, the mKdV equation. Equation (9) is the only linear one and its solution is a trivial ansatz $u(x,t) = c_1(t)x + c_2(t)$. Equations (10)–(12) however, are nonlinear and generate implicit ansatzes ($\varepsilon = \pm 1$)

$$\varepsilon \int_0^{u(x,t)} \frac{\mathrm{d}a}{\sqrt{c_1(t) - \alpha_1 a^2 - 2\alpha_2 a - \frac{2}{(p+1)(p+2)} a^{p+2}}} = x + c_2(t),$$

$$\varepsilon \int_0^{u(x,t)} \frac{\mathrm{d}a}{\sqrt{c_1(t)a - 2\beta_1 a^2 + 2\beta_2 - \frac{2}{(p+1)(p+2)} a^{p+2}}} = x + c_2(t),$$

$$\varepsilon \int_0^{u(x,t)} \frac{\mathrm{d}a}{\sqrt{c_1(t)a^2 + 2\gamma_1 a + \gamma_2 - \frac{2}{(p+1)(p+2)} a^{p+2}}} = x + c_2(t),$$

respectively. For certain parameters the ansatzes can be written in an explicit form. For example when $\alpha_1 = 0$ and $p = 1$ the equation

$$u_{xx} + \tfrac{1}{2}u^2 + \alpha_2 = 0 \tag{13}$$

produces an explicit ansatz

$$u(x,t) = -12\wp\big(x + c_1(t), -\tfrac{1}{3}\alpha_2, c_2(t)\big),$$

where $\wp$ denotes the Weierstrass function $\wp(z, g_2, g_3)$.

Third step of the method would be substitution of the ansatz to the equation we wish to reduce. Let us consider equation (13) with $\alpha_2 = 0$, meaning

$$u_{xx} + \tfrac{1}{2}u^2 = 0. \tag{14}$$

It is not linearizable and it admits trivial symmetries $u_t\partial_u$, $(u_{xxx} + uu_x)\partial_u$. On the grounds of the aforementioned findings, the solution to (14) provides an ansatz and a reduction for the KdV equation $u_t = u_{xxx} + uu_x$. Before we proceed with the reduction, we can indulge in a side challenge of finding some other PDEs sharing the same ansatz. For this purpose we introduce independent variable z. It can identified with the time variable t or viewed as a second space variable. One can show the equation (14) is invariant under LBS operators

$$\big(u^2 u_x u_z + 2u_{xz}u_x^2\big)\partial_u, \quad \big(u^2 u_z^2 + 2u_{xz}u_x u_z\big)\partial_u.$$

Since a linear combination of symmetry operators is itself a symmetry operator, we can use the solution of equation (14) to reduce (1+2)-dimensional equations

$$u_t = u_{xxx} + uu_x + u^2 u_x u_z + 2u_{xz}u_x^2,$$
$$u_t = u_{xxx} + uu_x + u^2 u_z^2 + 2u_{xz}u_x u_z,$$

or a non-evolution equation in one of the forms

$$\varepsilon u_t + u^2 u_t u_x + 2u_{tx}u_x^2 + u_{xxx} + uu_x = 0, \tag{15}$$
$$\varepsilon u_t + u^2 u_t^2 + 2u_{tx}u_t u_x + u_{xxx} + uu_x = 0, \tag{16}$$

where ε is an arbitrary constant.

The solution of the ODE (14) is the Weierstrass elliptic function $u(x) = -12\wp(x + c_1, 0, c_2)$, meaning the ansatz we substitute into the presented PDEs for reduction is

$$u(x,t) = -12\wp(x + c_1(t), 0, c_2(t)).$$

After such substitution, equation (15) for example, reduces to a system

$$\varepsilon c_2' = 0, \quad \varepsilon c_1' - 144c_2' = 0.$$

This means that for $\varepsilon = 0$ equation (15) has a class of solutions

$$u(x,t) = -12\wp(x + c_1(t), 0, c_2), \quad c_2 = \text{const}$$

and for $\varepsilon \neq 0$ there is only a stationary solution

$$u(x,t) = -12\wp(x + c_1, 0, c_2), \quad c_1, c_2 = \text{const}.$$

For equation (16) the reduced equations are

$$c_1'(144c_2' - \varepsilon) = 0, \quad c_2'(144c_2' - \varepsilon) = 0.$$

The solutions to this system are

$$c_1 = \text{const}, \quad c_2 = \text{const}$$

or

$$c_1(t) \text{ is arbitrary function}, \quad c_2 = \tfrac{\varepsilon}{144}t + c_0, \quad c_0 = \text{const}.$$

This means that the equation (16) has a class of solutions

$$u(x,t) = -12\wp\big(x + c_1(t), 0, \tfrac{\varepsilon}{144}t + c_0\big), \quad c_0 = \text{const}$$

as well as a stationary solution

$$u(x,t) = -12\wp(x + c_1, 0, c_2), \quad c_1, c_2 = \text{const}.$$

Example 2. Equations

$$u_x = \tfrac{1-u^2}{\sqrt{2}} \tag{17}$$

and

$$u_{xx} + u - u^3 = 0 \tag{18}$$

share the same kink solution

$$u = \tanh\big(\tfrac{x+c}{\sqrt{2}}\big), \quad c = \text{const}. \tag{19}$$

It can be easily shown that both $\big(u_{xx} + u - u^3\big)\partial_u$ and $u_t\partial_u$ are Lie–Bäcklund symmetry operators of equation (17). It follows that the substitution

$$u = \tanh\big(\tfrac{x+c(t)}{\sqrt{2}}\big) \tag{20}$$

reduces equation

$$u_t = u_{xx} + u\big(1 - u^2\big) \tag{21}$$

to a first-order differential equation $c'(t) = 0$, which means that equation (21) admits a stationary solution

$$u(x,t) = u(x) = \tanh\big(\tfrac{x+c}{\sqrt{2}}\big).$$

The second-order ODE (18) is a differential consequence of equation (17), therefore the right-hand side of the reduced equation vanishes on the ansatz solution. To obtain non-stationary solutions using this particular ansatz and this particular first-order ODE, we can add to the evolutionary equation first-order terms corresponding to the contact symmetries of (17). Contact symmetry of (17) in general form can be written as

$$f\big(\tfrac{u_x}{1-u^2}, \tfrac{1-u}{1+u}\mathrm{e}^{\sqrt{2}x}\big)u_x\partial_u,$$

where f is an arbitrary smooth function of two arguments. Substitution of (20) into

$$u_t = u_{xx} + u\big(1 - u^2\big) + f\big(\tfrac{u_x}{1-u^2}, \tfrac{1-u}{1+u}\mathrm{e}^{\sqrt{2}x}\big)u_x \tag{22}$$

reduces this equation to a simple first-order ODE

$$c'(t) = f\big(\tfrac{1}{\sqrt{2}}, \mathrm{e}^{-\sqrt{2}c(t)}\big).$$

Equation (22) will have a kink solution if $\frac{\partial f}{\partial c(t)} = 0$ and $f \neq 0$.

3. Conclusion. We show the application of the Lie–Bäcklund symmetry method for reducing the generalized version of Korteweg–de Vries equation of and nonlinear heat equation. We construct the class of ordinary differential equations which admit given Lie–Bäcklund symmetry operator and the corresponding ansatze reducing the equation under study to the system of two ordinary differential equations. The method enables us to find solutions which contain arbitrary functions on one variable for the equations (15), (16) and the solution generalizing the kink solution for nonlinear heat equation.

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Збірник праць Інституту математики НАН України 2019, т. 16, № 1, 181–192

УДК 517.912:512.816

Adjoint solutions and superposition principle for linearizable Krichever–Novikov equation

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Наявність операторної рівності для рівнянь, пов'язаних нелокальними перетвореннями, дозволила запропонувати метод знаходження іншого розв'язку вихідного рівняння, який приєднаний до відомого його розв'язку. Цей підхід застосовано для побудови точних розв'язків лінеаризованого рівняння Кричевера–Новікова та відповідного лінійного рівняння. Виведено формулу нелінійної нелокальної суперпозиції розв'язків, яку використано для розмноження точних розв'язків цього нелінійного рівняння.

Existence of an operator equality for equations connected by nonlocal transformations allowed us to propose a method of finding of a new solution of the initial equation adjoint to its known solution. This approach is used for construction of exact solutions for the linearizable Krichever–Novikov equation and for the corresponding linear equation. The formula of nonlinear nonlocal superposition of solutions for this nonlinear equation is derived and applied to generation of its solutions.

1. Introduction. A wide range of efficient methods for study of nonlinear partial differential equations are being developed at the moment. A considerable part of them are based on a fundamental idea of symmetry and, in particular, on the group-theoretical method suggested by Lie [6, 14, 16]. The most important generalizations of the basic symmetry group approach are realized in the concepts of conditional (nonclassical) symmetries, weak symmetries [5, 9, 15] and nonlocal symmetries of differential equations [1, 2, 3, 4, 7, 10, 12, 13, 17, 18, 21, 23, 24]. Therefore, development of other approaches to seek for new symmetries and methods for investigation of these equations is of importance and stays relevant.

Finite nonlocal transformations are efficiently used to study and solve nonlinear partial differential equations for a long time [8, 10, 21, 23, 24]. In particular, a number of interesting results for nonlinear equations connected among themselves by means of the *nonlocal transformations of variables* were obtained and formulae generating solutions or nonlocal nonlinear superposition were derived [11, 19, 22, 23].

Let us remind here the main concepts and terminology of the nonlocal transformations method. Assume that a given nonlocal transformation

$$\mathcal{T}\colon\ x^i = h^i(y, v_{(k)}), \quad u^K = H^K(y, v_{(k)}),$$
$$i = 1, \dots, n, \qquad K = 1, \dots, m, \tag{1}$$

maps an initial (*source*) equation

$$F_0(x, u_{(n)}) = 0 \tag{2}$$

into an equation $\Phi(y, v_{(q)}) = 0$ of order $q = n + k$ that admits factorization to another equation which we call a *target* equation

$$F_1(y, v_{(s)}) = 0, \tag{3}$$

i.e.,

$$\Phi(y, v_{(q)}) = \lambda F_1(y, v_{(s)}). \tag{4}$$

Here λ is a differential operator of order $n + k - s$. This results in algorithms for finding solutions of (2) via known solutions of (3). Existence of factorization equation (4) gives rise to a technique of finding of a special solution to the initial equation (2) from a known solution of the equation $\Phi(y, v_{(q)}) = 0$. The symbol $u_{(r)}$ denotes the tuple of partial derivatives of the function u from order zero up to order r. In the case of two independent variables, we use the special notation of the variables: $x_1 = x$, $x_2 = t$ and thus $u_t = \partial u/\partial t = \partial_t u$, $u_x = \partial u/\partial x = \partial_x u$.

The paper is organized as follows. In the next section we begin with some preliminary remarks on the concept of adjoint solution of the initial equation. Then we apply it to the linearisable Krichever–Novikov equation derived from the linear one via the known nonlocal transformation. In Section 3 this concept is applied to the case of the nonlocal invariance of the linearizable Krichever–Novikov equation. Examples of adjoint solutions are constructed.

2. Adjoint solution of the initial equation. This section is devoted to construction of solutions of the initial equation generated from known solutions of the appropriate inhomogeneous target equation. Existence of a factorization equation (4) gives rise to a technique [20] of construction of the special solution to the initial equation (2). Further we call it an *adjoint* solution.

We assume that a given function $v = f(y)$ *is not a solution* of equation (3), that is, substituting this function into (3), we get another equation with *discrepancy* $w(y)$

$$F_1(y, v_{(s)}) = w(y). \tag{5}$$

Suppose, nevertheless, that equation (4) holds and the equation

$$\lambda(y, v_{(s)})F_1(y, v_{(s)}) = \lambda(y, v_{(s)})w(y) = 0 \tag{6}$$

appears to be true. Here $w(y)$ runs through the set of solutions of a *linear* equation $\lambda(y, v_{(s)})w(y) = 0$ with variable coefficients of spatial form. Solving (6) with respect to the unknown function $w(y, v_{(k)}(y))$, one can find its solution as a function depending on y, $v_{(k)}(y)$

$$w = W(y, v_{(k)}). \tag{7}$$

After substitution of (7) into the equation (5) we obtain an *inhomogeneous* equation for the dependent variable v:

$$F_1(y, v_{(s)}) = W(y, v_{(k)}). \tag{8}$$

Hence the result of transformation (1) takes a form

$$\Phi(y, v_{(q)}) = \lambda F_2(y, v_{(s)}) = \lambda(F_1(y, v_{(s)}) - W(y, v_{(k)}). \tag{9}$$

The function $v(y)$ determined by (9) satisfies the equation $\mathcal{T}F_0(x, u_{(n)}) = \Phi(y, v_{(q)})$. Therefore, substituting $v(y)$ obtained in this way into the formulae of nonlocal transformation $\mathcal{T}$, one can find an appropriate solution of the given equation (2). Moreover, having the information on symmetries of the inhomogeneous equation (8), one can construct a r-parametrical family of solutions for it and, consequently, find the corresponding parametrical sets of solutions to the initial equation (2). If we let in (8) $W(y, v_{(k)}) = 0$ the equality (9) returns us to the connection of solutions of the initial equation and (3). That is why further we call such a special solution an *adjoint*. In what follows, we illustrate this approach by some examples.

3. Adjoint solutions constructed via the linearization. In this section we use the nonlocal transformation that connects the linear equation and the Krichever–Novikov equation of a special form and illustrate the proposed approach by some examples.

It is well-known [8] that the Krichever–Novikov equation

$$u_t + \tfrac{3}{4}u_x^{-1}u_{xx}^2 - u_{xxx} = 0 \tag{10}$$

can be obtained by applying the nonlocal transformation

$$w = \sqrt{u_x} \tag{11}$$

to the homogeneous linear third-order partial differential equation

$$w_t - w_{xxx} = 0. \tag{12}$$

The operator equation (4) connecting these two equations has the form

$$-4u_x^2\partial_x\big(u_t + \tfrac{3}{4}u_x^{-1}u_{xx}^2 - u_{xxx}\big) = 0. \tag{13}$$

Suppose a function $S(x,t)$ (*discrepancy*) is defined such that the inhomogeneous equation

$$u_t + \tfrac{3}{4}u_x^{-1}u_{xx}^2 - u_{xxx} = S(x,t)$$

is satisfied. Then the condition $\partial_x S(x,t) = 0$ follows from (13). In particular, if we let $S(x,t)$ be a linear function of time, i.e., $S(x,t) = ht$, the corresponding equation takes the form

$$u_t + \tfrac{3}{4}u_x^{-1}u_{xx}^2 - u_{xxx} = ht. \tag{14}$$

This equation admits the Lie algebra spanned by the following operators

$$\begin{gathered} X_1 = \partial_x, \quad X_2 = \partial_t + ht\partial_u, \quad X_3 = \partial_u, \\ X_4 = (u - ht)\partial_u, \quad X_5 = \tfrac{1}{3}x\partial_x + t\partial_t + ht^2\partial_u. \end{gathered} \tag{15}$$

1) We obtain a simple group-invariant solution of equation (14) solving the characteristic equation generated by the sum of the first two

operators of the algebra (15), $u_x + u_t - ht = 0$. The corresponding Lie ansatz is

$$u(x,t) = -\tfrac{1}{2}hx^2 + hxt + f(t-x).$$

Substituting this expression into (14), we find the reduced ordinary differential equation

$$4f'''(h\omega - f') + 3f''^2 - 6hf'' - 4f'^2 + 8h\omega f' - 4h^2\omega^2 + 3h^2 = 0,$$

where $\omega = t - x$. The general solution of this equation allows us to write down the required solution of (14)

$$\begin{aligned} u(x,t) = \tfrac{1}{2}ht^2 &+ \tfrac{1}{16}c_2 \sin 2(t-x+c_1) \\ &+ \tfrac{1}{2}c_2 \sin(t-x+c_1) + \tfrac{3}{8}c_2(t-x+c_1) + c_3, \end{aligned}$$

where c_1, c_2, c_3 are arbitrary constants.

Applying the nonlocal transformation (11) to the obtained solution, we get the corresponding solution of the linear equation:

$$w(x,t) = \tfrac{1}{4}\sqrt{-2c_2\cos 2(t-x+c_1) - 8c_2\cos(t-x+c_1) - 6c_2}.$$

One can compare the above solution with another solution of (12) being obtained in the form $w(x,t) = f(ct-x)$ determining a wave of unchanging profile moving at the constant velocity c:

$$w(x,t) = \bar{c}_1 + \bar{c}_2 \sin(ct-x) + \bar{c}_3 \cos(ct-x).$$

2) Another group-invariant solution of the equation (14) corresponding to the operator X_5 of the Lie algebra (15) has an implicit form

$$\begin{aligned} u(x,t) &= \tfrac{1}{2}ht^2 + \int^{t/x^3} \exp Q(k)\,\mathrm{d}k + c_3, \\ Q(k) &= \tfrac{2}{9}\sqrt{3}c_1B_1 - \tfrac{16}{3}\sqrt{3}B_2 + \tfrac{4}{3}B_3 + \tfrac{2}{9}\sqrt{3}B_4 \\ &\quad - \tfrac{16}{3}\sqrt{3}c_1B_5 + \tfrac{4}{3}c_1B_6 + c_2, \end{aligned} \tag{16}$$

where

$$B_1 = \int^k \frac{Y_{\frac{2}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)}{b^{3/2}\left(c_1Y_{-\frac{1}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right) + J_{-\frac{1}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)\right)}\,\mathrm{d}b,$$

$$B_2=\int^k \frac{J_{\frac{2}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)}{b^{1/2}\left(c_1Y_{-\frac{1}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)+J_{-\frac{1}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)\right)}\,\mathrm{d}b,$$

$$B_3=\int^k \frac{4\sqrt{3b}J_{\frac{2}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)-J_{-\frac{1}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)}{b\left(c_1Y_{-\frac{1}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)+J_{-\frac{1}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)\right)}\,\mathrm{d}b,$$

$$B_4=\int^k \frac{J_{\frac{2}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)}{b^{3/2}\left(c_1Y_{-\frac{1}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)+J_{-\frac{1}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)\right)}\,\mathrm{d}b,$$

$$B_5=\int^k \frac{Y_{\frac{2}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)}{b^{1/2}\left(c_1Y_{-\frac{1}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)+J_{-\frac{1}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)\right)}\,\mathrm{d}b,$$

$$B_6=\int^k \frac{4\sqrt{3b}Y_{\frac{2}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)-Y_{-\frac{1}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)}{b\left(c_1Y_{-\frac{1}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)+J_{-\frac{1}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)\right)}\,\mathrm{d}b,$$

where c_1, c_2, c_3 are arbitrary constants, $J_\alpha(x)$ and $Y_\alpha(x)$ are Bessel functions of the first and the second kinds respectivelly. Applying the formula (11) to this solution, we get such nonstationary solution of the linear equation (12):

$$w(x,t)=\pm\frac{1}{x^2}\sqrt{3t}\mathrm{e}^{\frac{1}{9}\sqrt{3}c_1\bar{B}_1-\frac{8}{3}\sqrt{3}\bar{B}_2+\frac{2}{3}\bar{B}_3+\frac{1}{9}\sqrt{3}\bar{B}_4-\frac{8}{3}c_1\bar{B}_5+\frac{2}{3}c_1\bar{B}_6+\frac{c_2}{2}}.$$

$\bar{B}_i$, $i=1,\ldots,6$, are the same as introduced above with $k=\frac{t}{x^3}$.

4. Adjoint solutions found via the nonlocal invariance. Beside a nonlocal linearization, the Krichever–Novikov equation (10) admits the auto-Bäcklund transformation [8]:

$$\begin{aligned}&u_x=v_x^{-1}v_{xx}^2,\\ &u_t=2v_x^{-1}v_{xx}v_{xxxx}-2v_x^{-2}v_{xx}^2v_{xxx}+\tfrac{5}{4}v_x^{-3}v_{xx}^4-v_x^{-1}v_{xxx}^2,\end{aligned}\qquad(17)$$

where $v(x,t)$ is another solution of the same equation

$$v_t+\tfrac{3}{4}v_x^{-1}v_{xx}^2-v_{xxx}=0.$$

In other words, the equation (10) stays invariant under the nonlocal transformation (17). This connection is realized by means of the operator equality

$$\left(-4v_{xx}v_x^2\partial_x+8v_x^3\partial_{xx}\right)\cdot\left(v_t+\tfrac{3}{4}v_x^{-1}v_{xx}^2-v_{xxx}\right)=0.\qquad(18)$$

We assume existence of the function $v(x,t)$ that is a solution of the inhomogeneous equation

$$v_t + \tfrac{3}{4} v_x^{-1} v_{xx}^2 - v_{xxx} = W(x,t). \tag{19}$$

Solving the partial differential equation generated by (18)

$$-4v_{xx}v_x^2 W_x(x,t) + 8v_x^3 W_{xx}(x,t) = 0$$

with respect to $W(x,t,v,v_x,v_{xx})$, we obtain the general solution

$$W(x,t) = f_1(t) + f_2(t)\int \sqrt{v_x}\,\mathrm{d}x. \tag{20}$$

To exclude an integral term in (20), we differentiate (19) with respect to x and set for simplicity $f_1(t) = 0$, $f_2(t) = K$ in (20), where K is a constant. So, instead of (19) we consider the equation

$$\partial_x\left(v_t + \tfrac{3}{4} v_x^{-1} v_{xx}^2 - v_{xxx}\right) - K\sqrt{v_x} = 0. \tag{21}$$

This inhomogeneous equation admits an infinite-dimensional Lie algebra spanned by the following operators

$$X_1 = \partial_t + F_1(t)\partial_v, \quad X_2 = \partial_x + F_2(t)\partial_v,$$
$$X_3 = \tfrac{1}{3}\partial_x + t\partial_t + \left(\tfrac{7}{3}v + F_3(t)\right)\partial_v. \tag{22}$$

Here $F_i(t)$, $i = 1,2,3$, are arbitrary functions of the time variable. This algebra allows us to get a wide range of group-invariant solutions of the equation (21). We choose $v(x,t)$ in the traveling wave solution form $v(x,t) = G(\omega)$, $\omega = x - ht$, where h is a fixed constant. Substituting this expression into (21), we get the reduced equation

$$4G'(\omega)^2 G''''(\omega) - 6G'(\omega)G''(\omega)G'''(\omega)$$
$$+ 3G''(\omega)^3 + 4hG'(\omega)^2 G''(\omega) + 4KG'(\omega)^{5/2} = 0, \tag{23}$$

which admits a solution

$$v(x,t) = G(\omega) = \int Y(\omega)\,\mathrm{d}\omega + c_4,$$

where $Y(\omega) = Z(\omega) + \omega + c_3$ and $Z(\omega)$ is the function determined by the equation

$$\int^{Z(\omega)} H^{-1}\,\mathrm{d}f = 0. \tag{24}$$

$H(Z(\omega), f)$ is an implicit solution of any of two equations

$$\mp\big(2\sqrt{f} - c_2 \mp Ph\big)h^{3/2} + Kh\arctan\left(\frac{Hh + K\sqrt{f}}{\sqrt{hf}P}\right) = 0, \qquad (25)$$

$$P = \sqrt{-\frac{hH^2 + 2\sqrt{f}KH - c_1 f}{f}}.$$

To use the formula (17) and to verify a new solution, we need an expression for u_x to be a function of $\omega = x - ht$. First we find a solution of the equation (10) differentiated with respect to x. We set $u(x,t) = L(\omega)$ and substitute that into the equation

$$\partial_x\big(u_t + \tfrac{3}{4}u_x^{-1}u_{xx}^2 - u_{xxx}\big) = 0. \qquad (26)$$

Implementation of the reduction procedure leads to the ordinary differential equation

$$4L'(\omega)^2L''''(\omega) - 6L'(\omega)L''(\omega)L'''(\omega) \\ + 3L''(\omega)^3 + 4hL'(\omega)^2L''(\omega) = 0. \qquad (27)$$

An implicit solution of this equation is determined by the integral equation

$$\int^{L(\omega)} Q(a)^{-1}\,\mathrm{d}a - \omega - \bar{c}_4 = 0,$$

where function $Q(a)$ is defined by the equations

$$\mp\int^{Q(a)} \frac{hf}{\sqrt{hf(\bar{c}_1 - 4fh^2 + 4\sqrt{f}h^2\bar{c}_2 - h^2\bar{c}_2^2)}}\,\mathrm{d}f + a + \bar{c}_3 = 0,$$

where $\bar{c}_1$, $\bar{c}_2$, $\bar{c}_3$ are arbitrary constants.

Now we apply the formula (17) to the obtained solution of the equation (23) and find the corresponding expression for $u_x(x,t)$:

$$u_x(x,t) = \hat{L}'(\omega) = G'(\omega)^{-1}G''(\omega)^2.$$

After simplification we get

$$\hat{L}'(\omega) = \frac{B^2}{Y}, \quad Y(\omega) = Z(\omega) + \omega + c_3.$$

Here $B(\omega, Y)$ are the implicit functions determined by the equations

$$\mp(2\sqrt{Y} - c_2)h^{5/2} + Th^{3/2} + Kh\arctan\left(\frac{Bh + K\sqrt{Y}}{\sqrt{hY}T}\right) = 0,$$

$$T = \sqrt{-\frac{hB^2 + 2\sqrt{Y}KB - c_1Y}{Y}}.$$

The function $Z(\omega)$ was implicitly determined above by equations (24), (25). Substitution of this solution into the equation (27) takes it to zero.

Knowing the Lie algebra (22) of the inhomogeneous equation (21) we can construct a wide family of group-invariant solutions, and, therefore, obtain various solutions of the equation (10). The new solution of the equation (26) constructed above obviously can be generated via the invariance algebra admitted by this equation or by means of its any other symmetry. The symmetry solutions of the special inhomogeneous target equation allow us to generate different solutions for the initial equation. What type of the symmetry of initial equations do we have in this case? As a target equation is broken by a discrepancy appearance, it seems naturally to call it a forced symmetry.

5. The superposition formula and generation of solutions. We return to the homogeneous linear third-order differential equation (12) and the nonlocal transformation (11), connecting this equation with (10). We choose a linear superposition principle for (12) setting

$$w^{\mathrm{III}}(x,t) \equiv w(x,t) = w^{\mathrm{I}}(x,t) + w^{\mathrm{II}}(x,t).$$

Here $w^{\mathrm{I}}(x,t)$, $w^{\mathrm{II}}(x,t)$ are some known solutions of the linear equation. As equations are connected by the nonlocal transformation (11), the corresponding principle of nonlinear nonlocal superposition of solutions for equation (10) can be constructed.

Theorem 1. *The nonlinear nonlocal superposition formula of solutions for equation* (10) *has the form*

$$u(x,t) = u^{\mathrm{I}}(x,t) + u^{\mathrm{II}}(x,t) + 2\int \sqrt{u^{\mathrm{I}}(x,t)}\sqrt{u^{\mathrm{II}}(x,t)}\,\mathrm{d}x + s(t), \tag{28}$$

where the arbitrary function $s(t)$ is defined by the equation

$$u_t = 2\left(\sqrt{u^{\mathrm{I}}(x,t)} + \sqrt{u^{\mathrm{II}}(x,t)}\right)\partial_x^2\left(\sqrt{u^{\mathrm{I}}(x,t)} + \sqrt{u^{\mathrm{II}}(x,t)}\right)$$

$$- \left(\partial_x \left(\sqrt{u^{\mathrm{I}}(x,t)} + \sqrt{u^{\mathrm{II}}(x,t)}\right)\right)^2. \tag{29}$$

Given solutions u^{I} and u^{II}, the new solution of (10) is found integrating the third term of (28). We get the specialization of the function $s(t)$ substituting the expression (28) into (29) and solving the equations obtained with respect to s.

We illustrate utilization of the proposed superposition formula for generation of solutions of the equation (10).

1) It can be easily verified that

$$u^{\mathrm{I}} = \tfrac{1}{1280c_1}\left(x^5 + 5c_2x^4 + 10c_2^2x^3 + 10c_2^3x^2 + 5c_2^4x + c_2^5\right) + c_3,$$
$$u^{\mathrm{II}} = k_1x^5$$

are time-independent solutions of the equation (10). Applying the formula (28) adduced in Theorem 1 we find a time-dependent solution

$$u^{\mathrm{III}} = \tfrac{1}{3840{c_1}^{3/2}}\left(a_1x^5 + a_2x^4 + a_3x^3 + 30\sqrt{c_1}c_2^3x^2 + 15\sqrt{c_1}c_2^4x + 960c_1c_2^2\sqrt{5k_1}t + \kappa\right),$$
$$a_1 = 3\sqrt{c_1} + 96\sqrt{5k_1}c_1 + 3840k_1{c_1}^{3/2},$$
$$a_2 = 15c_2\sqrt{c_1} + 240\sqrt{5k_1}c_1c_2,$$
$$a_3 = 160\sqrt{5k_1}c_1c_2^2 + 30\sqrt{c_1}c_2^2,$$
$$\kappa = 3\sqrt{c_1}{c_2}^5 + 380c_1^{3/2}(c_3 + c_4).$$

2) Choosing $c_1 = 0$ in (16), we obtain a simpler solution

$$u^{\mathrm{I}} = \int^{\frac{y}{x^3}} \exp\left(-\tfrac{16}{3}\sqrt{3}\tilde{B}_1 + \tfrac{4}{3}\tilde{B}_2 + \tfrac{2}{9}\sqrt{3}\tilde{B}_3 + c_2\right) \mathrm{d}k + c_3,$$
$$\tilde{B}_1 = \int^k J_{\frac{2}{3}}\left(\tfrac{1}{9}\tfrac{\sqrt{3}}{\sqrt{b}}\right) b^{-1/2} J_{-\frac{1}{3}}\left(\tfrac{1}{9}\tfrac{\sqrt{3}}{\sqrt{b}}\right)^{-1} \mathrm{d}b,$$
$$\tilde{B}_2 = \int^k \frac{4\sqrt{3b}J_{\frac{2}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right) - J_{-\frac{1}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)}{b\left(J_{-\frac{1}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)\right)} \mathrm{d}b,$$
$$\tilde{B}_3 = \int^k J_{\frac{2}{3}}\left(\tfrac{1}{9}\tfrac{\sqrt{3}}{\sqrt{b}}\right) b^{-3/2} J_{-\frac{1}{3}}\left(\tfrac{1}{9}\tfrac{\sqrt{3}}{\sqrt{b}}\right)^{-1} \mathrm{d}b. \tag{30}$$

Let the second solution be

$$u^{\mathrm{II}} = k_1 x^5.$$

Then

$$u_x^{\mathrm{III}} = -3tx^{-4}\exp\left(-\tfrac{16}{3}\sqrt{3}\hat{B}_1 + \tfrac{4}{3}\hat{B}_2 + \tfrac{2}{9}\sqrt{3}\hat{B}_3 + c_2\right) + 5k_1x^4 + 2\sqrt{5}\sqrt{k_1x^4}\sqrt{-3tx^{-4}\exp\left(-\tfrac{16}{3}\sqrt{3}\hat{B}_1 + \tfrac{4}{3}\hat{B}_2 + \tfrac{2}{9}\sqrt{3}\hat{B}_3 + c_2\right)},$$

where $\hat{B}_i$, $i = 1, 2, 3$, are the same as those introduced in (30) but $k = \frac{t}{x^3}$. One can easily verify that obtained expression satisfies (26). More solutions may be constructed by utilization of the previous theorem and application of the Lie symmetry transformations or any other formula generating solutions.

6. Conclusion. The concept of an adjoint solution of the initial equation was developed in this paper, and used for construction of new solutions of linearizable Krichever–Novikov equation and for the connected linear one. Some of them were obtained in an explicit form, while others have a parametrical representations with functional parameters given in implicit form. The Lie symmetry solutions of the special inhomogeneous target equation allowed us to generate appropriate solutions for the given initial equation. The superposition formula was derived in the present paper and applied for the generation of solutions to the equation (10). All the found solutions can be naturally extended by means of the Lie symmetry transformations or any other formula generating new solutions. The results obtained for the equation (10) can be extended to similar classes of equations.

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Збірник праць Інституту математики НАН України 2019, т. 16, № 1, 193–202

УДК 512.813:517.957.6

On symmetry reduction of the Euler–Lagrange–Born–Infeld equation to linear ODEs

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Вивчається зв'язок між структурними властивостями тривимірних підалгебр алгебри Пуанкаре $\mathfrak{p}(1,4)$ і симетрійною редукцією рівняння Ейлера–Лагранжа–Борна–Інфельда. Основну увагу зосереджено на редукцях за тривимірними підалгебрами, що зводять рівняння Ейлера–Лагранжа—Борна–Інфельда до лінійних диференціальних рівнянь.

Connections between structure properties of three-dimensional subalgebras of the Poincaré algebra $\mathfrak{p}(1,4)$ and Lie reductions of the Euler–Lagrange–Born–Infeld equation are studied. We concentrate our attention on Lie reductions with respect to three-dimensional subalgebras that reduce the Euler–Lagrange–Born–Infeld equation to linear ordinary differential equations.

1. Introduction. Symmetry reduction is the most universal tool for finding exact solutions of partial differential equations (PDEs). We focus our attention on some applications of the classical Lie method to investigation of PDEs with non-trivial symmetry groups. In 1895, Lie [19] considered solutions of PDEs that are invariant with respect to symmetry groups admitted by these PDEs. It turned out that the problem of symmetry reduction and construction of independent invariant solutions for a PDE with a non-trivial symmetry group is reduced to the algebraic problem of classification of inequivalent subalgebras of the Lie invariance algebra of this equation [23, 24].

In 1975, Patera, Winternitz, and Zassenhaus [25] proposed a general method for describing inequivalent subalgebras of Lie algebras with nontrivial ideals. It turned out that reduced equations obtained from inequivalent subalgebras of the same dimension were of different types. Grundland, Harnad and Winternitz [17] were the first who pointed out and studied this phenomenon. Further details can be found in [6, 8, 11, 15, 16, 21, 22]. The results obtained cannot be explained using only the dimension of subalgebras of Lie invariance algebras.

To explain a difference in properties of reduced equations for PDEs with nontrivial symmetry groups, we investigate the relation between structure properties of inequivalent subalgebras of the same dimension of the Lie invariance algebras of those PDEs and properties of the respective reduced equations. By now, we have studied this relation for the case of low-dimensional ($\dim L \leq 3$) inequivalent subalgebras of the same dimension of the algebra $\mathfrak{p}(1,4)$, which is the Lie algebra of the Poincaré group $P(1,4)$, and the eikonal equation [8].

This paper is devoted to the study of the relation between structural properties of low-dimensional ($\dim L \leq 3$) inequivalent subalgebras of the same rank of the algebra $\mathfrak{p}(1,4)$ and properties of reduced equations for the Euler–Lagrange–Born–Infeld (ELBI) equation. By now, this relation has been investigated for three-dimensional subalgebras. We obtained the following types of reduced equations: identities, linear ordinary differential equations, nonlinear ordinary differential equations, partial differential equations. For some subalgebras, it is impossible to construct ansatzes that reduce the ELBI equation.

We focus our attention on reduction of the ELBI equation to linear ODEs. More precisely, we only present the results of symmetry reduction for those types of subalgebras that provide us reductions to linear ODEs.

2. Lie algebra of the Poincaré group $P(1,4)$ and its nonequivalent subalgebras. The group $P(1,4)$ is the group of rotations and translations of the five-dimensional Minkowski space $M(1,4)$. It is the minimal group that contains, as subgroups, the extended Galilei group $\widetilde{G}(1,3)$ [12] and the Poincaré group $P(1,3)$, which are underlying groups of classical and relativistic physics, respectively.

The Lie algebra $\mathfrak{p}(1,4)$ of the group $P(1,4)$ is spanned by 15 basis elements $M_{\mu\nu} = -M_{\nu\mu}$, $\mu,\nu = 0,1,2,3,4$, and P_μ, $\mu = 0,1,2,3,4$, which satisfy the commutation relations

$$[P_\mu, P_\nu] = 0, \quad [M_{\mu\nu}, P_\sigma] = g_{\nu\sigma}P_\mu - g_{\mu\sigma}P_\nu,$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = g_{\mu\sigma}M_{\nu\rho} + g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\rho},$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \neq \nu$.

We consider the canonical realization [13, 14] of $\mathfrak{p}(1,4)$,

$$P_0 = \frac{\partial}{\partial x_0}, \quad P_1 = -\frac{\partial}{\partial x_1}, \quad P_2 = -\frac{\partial}{\partial x_2}, \quad P_3 = -\frac{\partial}{\partial x_3},$$
$$P_4 = -\frac{\partial}{\partial u}, \quad M_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu, \quad x_4 \equiv u.$$

Hereafter we use the following basis elements

$$G = M_{04}, \quad L_1 = M_{23}, \quad L_2 = -M_{13}, \quad L_3 = M_{12},$$
$$P_a = M_{a4} - M_{0a}, \quad C_a = M_{a4} + M_{0a},$$
$$X_0 = \tfrac{1}{2}(P_0 - P_4), \quad X_k = P_k, \quad X_4 = \tfrac{1}{2}(P_0 + P_4), \quad a, k = 1, 2, 3.$$

Subalgebras of the Lie algebra $\mathfrak{p}(1,4)$ were studied up to $P(1,4)$-conjugation in [4, 5, 10], in particular, the classification of subalgebras of $\mathfrak{p}(1,4)$ of dimensions up to three was given in [7]. Note that the Lie algebra of the extended Galilei group $\widetilde{G}(1,3)$ is spanned by L_1, L_2, L_3, P_1, P_2, P_3, X_0, X_1, X_2, X_3 and X_4.

3. Classification of symmetry reductions for the Euler–Lagrange–Born–Infeld equation. Born–Infeld-like equations arise in fluid dynamics, theory of continuous medium, general relativity, field theory, theory of minimal surfaces, nonlinear electrodynamics, etc. [1, 2, 3, 18, 26].

We consider the Euler–Lagrange–Born–Infeld (ELBI) equation

$$\Box u \left(1 - u_\nu u^\nu\right) + u^\mu u^\nu u_{\mu\nu} = 0, \tag{1}$$

where $u = u(x)$, $x = (x_0, x_1, x_2, x_3) \in M(1,3)$, $u_\mu \equiv \frac{\partial u}{\partial x_\mu}$, $u_{\mu\nu} \equiv \frac{\partial^2 u}{\partial x_\mu \partial x_\nu}$, $u^\mu = g^{\mu\nu}u_\nu$, $\mu, \nu = 0, 1, 2, 3$, and $\Box$ is the d'Alembert operator.

In 1984, Fushchych and Serov [13] studied symmetry properties of the multidimensional nonlinear Euler–Lagrange equation. These results imply that the Lie invariance algebra of the equation (1) contains, as a subalgebra, the Poincaré algebra $\mathfrak{p}(1,4)$.

We carry out Lie symmetry reductions of the ELBI equation to linear ODEs using subalgebras of $\mathfrak{p}(1,4)$ of the following types: $3A_1$, $A_2 \oplus A_1$, $A_{3,1}$, $A_{3,2}$, $A_{3,3}$, $A_{3,6}$. The notation of three-dimensional algebras is according to Mubarakzyanov's classification of low-dimensional Lie algebras [20].

Among inequivalent subalgebras of the Poincaré algebra $\mathfrak{p}(1,4)$ listed in [7], we select only such subalgebras that do reduce the ELBI equation to linear ODEs with nonlinear solutions since linear solutions are considered to be trivial. For each of the selected subalgebras, we construct an ansatz for u, the corresponding reduced equation, its general solution and the associated family of invariant solutions of the ELBI equation.

Proposition 1. *The Lie algebra* $\mathfrak{p}(1,4)$ *contains* 31 *three-dimensional inequivalent subalgebras of the type* $3A_1$*.*

1. $\langle P_1\rangle \oplus \langle P_2\rangle \oplus \langle X_3\rangle$:
   the ansatz is $x_0^2 - x_1^2 - x_2^2 - u^2 = \varphi(\omega)$, $\omega = x_0 + u$;
   the reduced equation is $\omega^2\varphi'' - 6\omega\varphi' + 6\varphi = 0$;
   the solution of the reduced equation is $\varphi(\omega) = c_1\omega^6 + c_2\omega$;
   the solution of the ELBI equation is
   $$x_0^2 - x_1^2 - x_2^2 - u^2 = c_1(x_0+u)^6 + c_2(x_0+u).$$

2. $\langle P_3\rangle \oplus \langle X_1\rangle \oplus \langle X_2\rangle$:
   the ansatz is $x_0^2 - x_3^2 - u^2 = \varphi(\omega)$, $\omega = x_0 + u$;
   the reduced equation is $\omega^2\varphi'' - 4\omega\varphi' + 4\varphi = 0$;
   the solution of the reduced equation is $\varphi(\omega) = c_2\omega^4 + c_1\omega$;
   the solution of the ELBI equation is
   $$x_0^2 - x_3^2 - u^2 = c_2(x_0+u)^4 + c_1(x_0+u).$$

3. $\langle P_1\rangle \oplus \langle P_2\rangle \oplus \langle P_3\rangle$:
   the ansatz is $x_0^2 - x_1^2 - x_2^2 - x_3^2 - u^2 = \varphi(\omega)$, $\omega = x_0 + u$;
   the reduced equation is $\omega^2\varphi'' - 8\omega\varphi' + 8\varphi = 0$;
   the solution of the reduced equation is $\varphi(\omega) = c_1\omega^8 + c_2\omega$;
   the solution of the ELBI equation is
   $$x_0^2 - x_1^2 - x_2^2 - x_3^2 - u^2 = c_1(x_0+u)^8 + c_2(x_0+u).$$

4. $\langle P_1\rangle \oplus \langle P_2 - X_2\rangle \oplus \langle X_3\rangle$:
   the ansatz is $\frac{x_0^2 - x_1^2 - u^2}{x_0+u} - \frac{x_2^2}{x_0+u+1} = \varphi(\omega)$, $\omega = x_0 + u$;
   the reduced equation is $(\omega+1)^5\omega^5\left(\omega(\omega+1)\varphi'' - 2(2\omega+1)\varphi'\right) = 0$;
   the solution of the reduced equation is

$\varphi(\omega) = c_2\omega^3\big(6\omega^2 + 15\omega + 10\big) + c_1$;
the solution of the ELBI equation is

$$\frac{x_0^2 - x_1^2 - u^2}{x_0 + u} - \frac{x_2^2}{x_0 + u + 1} = c_2(x_0 + u)^3\big(6(x_0 + u)^2 + 15(x_0 + u) + 10\big) + c_1.$$

5. $\langle P_1\rangle \oplus \langle P_2 - \alpha X_2, \alpha > 0\rangle \oplus \langle P_3 - \gamma X_3, \gamma \neq 0\rangle$:
the ansatz is $2u + \frac{x_1^2}{x_0+u} + \frac{x_2^2}{x_0+u+\alpha} + \frac{x_3^2}{x_0+u+\gamma} = \varphi(\omega),\ \omega = x_0 + u$;
the reduced equation is
$(\omega+\gamma)^5\omega^5(\omega+\alpha)^5\big[\omega\big(\omega^2+(\alpha+\gamma)\omega+\alpha\gamma\big)\varphi'' - 2(3\omega^2+2(\alpha+\gamma)\omega+\alpha\gamma)(\varphi'-1)\big] = 0$;
the solution of the reduced equation is
$\varphi(\omega) = c_1\big[\frac{1}{7}\omega^4 + \frac{1}{3}(\alpha+\gamma)\omega^3 + \frac{1}{5}(\alpha^2 + 4\alpha\gamma + \gamma^2)\omega^2 + \frac{1}{2}\alpha\gamma(\alpha+\gamma)\omega + \frac{1}{3}\alpha^2\gamma^2\big]\omega^3 + \omega + c_2$;
the solution of the ELBI equation is

$$\begin{aligned} &2u + \frac{x_1^2}{x_0 + u} + \frac{x_2^2}{x_0 + u + \alpha} + \frac{x_3^2}{x_0 + u + \gamma} \\ &= c_1\left[\tfrac{1}{7}(x_0+u)^4 + \tfrac{1}{3}(\alpha+\gamma)(x_0+u)^3 + \tfrac{1}{5}(\alpha^2 + 4\alpha\gamma + \gamma^2)\right. \\ &\quad \left.\times (x_0+u)^2 + \tfrac{1}{2}\alpha\gamma(\alpha+\gamma)(x_0+u) + \tfrac{1}{3}\alpha^2\gamma^2\right] \\ &\times (x_0+u)^3 + x_0 + u + c_2. \end{aligned}$$

6. $\langle P_1\rangle \oplus \langle P_2 - \alpha X_2, \alpha > 0\rangle \oplus \langle P_3\rangle$:
the ansatz is $2u + \frac{x_1^2+x_3^2}{x_0+u} + \frac{x_2^2}{x_0+u+\alpha} = \varphi(\omega)$, $\omega = x_0 + u$;
the reduced equation is
$(\omega+\alpha)^5\omega^5(\omega(\omega+\alpha)\varphi'' - 2(3\omega+2\alpha)(\varphi'-1)) = 0$;
the solution of the reduced equation is
$\varphi(\omega) = c_1\big(\frac{1}{7}\omega^2 + \frac{\alpha}{3}\omega + \frac{\alpha^2}{5}\big)\omega^5 + \omega + c_2$;
the solution of the ELBI equation is

$$\begin{aligned} &2u + \frac{x_1^2 + x_3^2}{x_0 + u} + \frac{x_2^2}{x_0 + u + \alpha} \\ &= c_1\left(\tfrac{1}{7}(x_0+u)^2 + \tfrac{\alpha}{3}(x_0+u) + \tfrac{\alpha^2}{5}\right)(x_0+u)^5 \\ &\quad + x_0 + u + c_2. \end{aligned}$$

7. $\langle P_3 - 2X_0\rangle \oplus \langle X_1\rangle \oplus \langle X_2\rangle$:
the ansatz is

$\frac{1}{6}(x_0+u)^3+x_3(x_0+u)+x_0-u=\varphi(\omega)$, $\omega=(x_0+u)^2+4x_3$;
the reduced equation is $2\omega\varphi''-\varphi'=0$;
the solution of the reduced equation is
$\varphi(\omega)=c_2\omega^{3/2}+c_1$;
the solution of the ELBI equation is

$$\frac{1}{6}(x_0+u)^3+x_3(x_0+u)+x_0-u$$
$$=c_2\big((x_0+u)^2+4x_3\big)^{3/2}+c_1.$$

8. $\langle P_3-2X_0\rangle\oplus\langle X_1\rangle\oplus\langle X_4\rangle$:
   the ansatz is $(x_0+u)^2+4x_3=\varphi(\omega)$, $\omega=x_2$;
   the reduced equation is $\varphi''=0$;
   the solution of the reduced equation is $\varphi(\omega)=c_1\omega+c_2$;
   the solution of the ELBI equation is

$$u=\varepsilon(c_1x_2-4x_3+c_2)^{1/2}-x_0,\quad \varepsilon=\pm1.$$

Proposition 2. *The Lie algebra* $\mathfrak{p}(1,4)$ *contains* 10 *three-dimensional inequivalent subalgebras of the type* $A_2\oplus A_1$.

1. $\langle-(G+\alpha X_2),X_4,\alpha>0\rangle\oplus\langle X_1\rangle$:
   the ansatz is $x_2-\alpha\ln(x_0+u)=\varphi(\omega)$, $\omega=x_3$;
   the reduced equation is $\varphi''=0$;
   the solution of the reduced equation is $\varphi(\omega)=c_1\omega+c_2$;
   the solution of the ELBI equation is

$$x_2-\alpha\ln(x_0+u)=c_1x_3+c_2.$$

Proposition 3. *The Lie algebra* $\mathfrak{p}(1,4)$ *contains* 17 *three-dimensional inequivalent subalgebras of the type* $A_{3,1}$.

1. $\langle 2\mu X_4,P_3-2X_0,X_1+\mu X_3,\mu>0\rangle$:
   the ansatz is $(x_0+u)^2+4x_3-4\mu x_1=\varphi(\omega)$, $\omega=x_2$;
   the reduced equation is $\varphi''=0$;
   the solution of the reduced equation is $\varphi(\omega)=c_1\omega+c_2$;
   the solution of the ELBI equation is

$$u=\varepsilon\left(4\mu x_1+c_1x_2-4x_3+c_2\right)^{1/2}-x_0,\quad \varepsilon=\pm1.$$

Proposition 4. *The Lie algebra* $\mathfrak{p}(1,4)$ *contains* 3 *three-dimensional nonconjugate subalgebras of the type* $A_{3,2}$.

1. $\langle 2\beta X_4, P_3, G+\alpha X_1+\beta X_3, \alpha>0, \beta>0\rangle$:
   the ansatz is $x_1-\alpha\ln(x_0+u)=\varphi(\omega)$, $\omega=x_2$;
   the reduced equation is $\varphi''=0$;
   the solution of the reduced equation is $\varphi(\omega)=c_1\omega+c_2$;
   the solution of the ELBI equation is

$$x_1-\alpha\ln(x_0+u)=c_1x_2+c_2.$$

Proposition 5. *The Lie algebra $\mathfrak{p}(1,4)$ contains five three-dimensional inequivalent subalgebras of the type $A_{3,3}$.*

1. $\langle P_3, X_4, G+\alpha X_1, \alpha>0\rangle$:
   the ansatz is $x_1-\alpha\ln(x_0+u)=\varphi(\omega)$, $\omega=x_2$;
   the reduced equation is $\varphi''=0$;
   the solution of the reduced equation is $\varphi(\omega)=c_1\omega+c_2$;
   the solution of the ELBI equation is

$$u=\exp\left(\frac{x_1-c_1x_2-c_2}{\alpha}\right)-x_0.$$

Proposition 6. *The Lie algebra $\mathfrak{p}(1,4)$ contains* 18 *three-dimensional inequivalent subalgebras of the type $A_{3,6}$.*

1. $\langle P_1-X_1, P_2-X_2, -P_3+L_3\rangle$:
   the ansatz is $\frac{x_1^2+x_2^2}{x_0+u+1}+\frac{x_3^2}{x_0+u}+2u=\varphi(\omega)$, $\omega=x_0+u$;
   the reduced equation is
   $\omega^5(\omega+1)^5[\omega(\omega+1)\varphi''-2(3\omega+1)(\varphi'-1)]=0$;
   the solution of the reduced equation is
   $\varphi(\omega)=\frac{c_1}{7}\omega^7+\frac{2}{3}c_1\omega^6+\frac{6}{5}c_1\omega^5+c_1\omega^4+\frac{c_1}{3}\omega^3+\omega+c_2$;
   the solution of the ELBI equation is

$$\frac{x_1^2+x_2^2}{x_0+u+1}+\frac{x_3^2}{x_0+u}+2u=\tfrac{c_1}{7}(x_0+u)^7+\tfrac{2}{3}c_1(x_0+u)^6$$
$$+\tfrac{6}{5}c_1(x_0+u)^5+c_1(x_0+u)^4+\tfrac{c_1}{3}(x_0+u)^3+x_0+u+c_2.$$

2. $\langle P_1, -P_2, -(L_3+\alpha X_3), \alpha>0\rangle$:
   the ansatz is $x_0^2-x_1^2-x_2^2-u^2=\varphi(\omega)$, $\omega=x_0+u$;
   the reduced equation is $\omega^2\varphi''-6\omega\varphi'+6\varphi=0$;
   the solution of the reduced equation is $\varphi(\omega)=c_1\omega^6+c_2\omega$;
   the solution of the ELBI equation is

$$x_0^2-x_1^2-x_2^2-u^2=c_1(x_0+u)^6+c_2(x_0+u).$$

3. $\langle X_1, -X_2, P_3 - L_3 \rangle$:
the ansatz is $x_0^2 - x_3^2 - u^2 = \varphi(\omega)$, $\omega = x_0 + u$;
the reduced equation is $\omega^2\varphi'' - 4\omega\varphi' + 4\varphi = 0$;
the solution of the reduced equation is $\varphi(\omega) = c_1\omega^4 + c_2\omega$;
the solution of the ELBI equation is

$$x_0^2 - x_3^2 - u^2 = c_1(x_0 + u)^4 + c_2(x_0 + u).$$

4. $\langle P_1, P_2, L_3 - P_3 \rangle$:
the ansatz is $x_0^2 - x_1^2 - x_2^2 - x_3^2 - u^2 = \varphi(\omega)$, $\omega = x_0 + u$;
the reduced equation is $\omega^2\varphi'' - 8\omega\varphi' + 8\varphi = 0$;
the solution of the reduced equation is $\varphi(\omega) = c_1\omega^8 + c_2\omega$;
the solution of the ELBI equation is

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 - u^2 = c_1(x_0 + u)^8 + c_2(x_0 + u).$$

5. $\langle X_1, -X_2, P_3 - L_3 - 2\alpha X_0, \alpha > 0 \rangle$:
the ansatz is $(x_0 + u)^3 + 6\alpha x_3(x_0 + u) + 6\alpha^2(x_0 - u) = \varphi(\omega)$, $\omega = (x_0 + u)^2 + 4\alpha x_3$;
the reduced equation is $2\omega\varphi'' - \varphi' = 0$;
the solution of the reduced equation is $\varphi(\omega) = c_2\omega^{3/2} + c_1$;
the solution of the ELBI equation is

$$\begin{aligned}(x_0 + u)^3 &+ 6\alpha x_3(x_0 + u) + 6\alpha^2(x_0 - u) \\ &= c_2\big((x_0 + u)^2 + 4\alpha x_3\big)^{3/2} + c_1.\end{aligned}$$

4. Conclusions. In this paper we focused our attention on Lie reductions of the ELBI equation to linear ODEs. More precisely, we presented results for such three-dimensional subalgebras of $\mathfrak{p}(1,4)$ that give reductions of the ELBI equation to linear ODEs with nonlinear solutions.

It is known [7] that the Lie algebra $\mathfrak{p}(1,4)$ contains three-dimensional inequivalent subalgebras of the following types: $3A_1$, $A_2 \oplus A_1$, $A_{3,1}$, $A_{3,2}$, $A_{3,3}$, $A_{3,4}$, $A_{3,6}$, $A_{3,7}^a$, $A_{3,8}$, $A_{3,9}$. Results of the paper imply that all the above Lie reductions of the ELBI equation to linear ODEs can be obtained using subalgebras of the types $3A_1$, $A_2 \oplus A_1$, $A_{3,1}$, $A_{3,2}$, $A_{3,3}$ and $A_{3,6}$. Moreover, all the subalgebras considered in the paper are also subalgebras of the Lie algebra of the extended Galilei group $\widetilde{G}(1,3)$.

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Зміст

Наукове видання

Збірник праць Інституту математики НАН України

Том 16 № 1

Симетрія та інтегровність рівнянь математичної фізики

Комп'ютерний оригінал-макет *В.М. Бойко, О.В. Локазюк, О.А. Почекета*

Підписано до друку 03.09.2019. Формат 60×84/16. Папір офс. Офс. друк.
Фіз. друк. арк. 12,8. Умов. друк. арк. 11,9. Зам. № 41. Тираж 70 пр.

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