On the theory of inner Riesz balayage and its recent extensions and applications

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We mainly deal with the theory of potentials on $\mathbb{R}^n$, $n \geqslant 2$, with respect to the Riesz kernel $\kappa_\alpha(x,y) := |x-y|^{\alpha-n}$ of order $\alpha \in (0,n)$, $\alpha \leqslant 2$, $|x-y|$ being the Euclidean distance between $x, y \in \mathbb{R}^n$. To be precise, we are mainly concerned with the theory of *inner* Riesz balayage of *any* positive (Radon) measure on $\mathbb{R}^n$ to *any* set $A \subset \mathbb{R}^n$, established in [13–15, 18]. Its recent extensions and applications, given in [16, 17, 19–27], will also be briefly mentioned. It is worth noting that such a theory, being substantially based on the concept of energy and a pre-Hilbert structure on suitable spaces of signed measures, is not covered by those developed in the setting of balayage spaces [1] or H-cones [2], both dealing with the theory of *outer* balayage.

Let $\mathfrak{M} = \mathfrak{M}(\mathbb{R}^n)$ be the linear space of all real-valued measures on $\mathbb{R}^n$, equipped with the *vague* topology of pointwise convergence on the class $C_0(\mathbb{R}^n)$ of all continuous functions $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ of compact support, and $\mathfrak{M}^+ = \mathfrak{M}^+(\mathbb{R}^n)$ the convex cone of all positive $\mu \in \mathfrak{M}$, where $\mu \in \mathfrak{M}$ is *positive* if $\mu(\varphi) \geqslant 0$ for all positive $\varphi \in C_0(\mathbb{R}^n)$.

For any $\mu, \nu \in \mathfrak{M}$, the *mutual energy* $I(\mu,\nu)$ and the *potential* U^μ are given by

$$I(\mu,\nu) := \int \kappa_\alpha(x,y)\,\mathrm{d}(\mu\otimes\nu)(x,y) \quad \text{and} \quad U^\mu(\cdot) := \int \kappa_\alpha(\cdot,y)\,\mathrm{d}\mu(y),$$

respectively, provided the value on right is well defined as a finite number or $\pm\infty$. For $\mu = \nu$, the mutual energy $I(\mu,\nu)$ defines the *energy* $I(\mu) := I(\mu,\mu)$.

When speaking of $\mu \in \mathfrak{M}^+$, we always mean that $U^\mu \not\equiv +\infty$ on $\mathbb{R}^n$, or equivalently

$$\int_{|y|>1} \frac{\mathrm{d}\mu(y)}{|y|^{n-\alpha}} < \infty,$$

see [10, Section I.3.7]. This does hold if μ is *bounded* (i.e., $\mu(\mathbb{R}^n) < \infty$), or of finite energy [8, Corollary to Lemma 3.2.3]. Actually, then (and only then) the potential of any $\mu \in \mathfrak{M}$ is well defined and finite quasi-everywhere (q.e.) on $\mathbb{R}^n$, cf. [10, Section III.1.1], whence nearly everywhere (n.e.) on $\mathbb{R}^n$. As for the terminology used, a proposition $P(x)$ is said to hold *quasi-everywhere* (resp. *nearly everywhere*) on $A \subset \mathbb{R}^n$ if the set E of all $x \in A$ where $P(x)$ fails, is of outer (resp. inner) capacity zero. Regarding the *outer* and *inner* capacities, denoted by $c^*(\cdot)$ and $c_*(\cdot)$, respectively, see [10, Section II.2.6]. If A is *capacitable*, that is, if $c_*(A) = c^*(A)$, then $c(A) := c^*(A)$ is simply termed the *capacity*; this occurs, e.g., if A is Borel [10, Theorem 2.8].

A crucial fact discovered by Riesz [11, Chapter I, equation (13)] is that the kernel κ_α is *strictly positive definite*, which means that $I(\mu) \geqslant 0$ for every $\mu \in \mathfrak{M}$, and moreover $I(\mu) = 0 \iff \mu = 0$. This implies that all (signed) $\mu \in \mathfrak{M}$ with $I(\mu) < \infty$ form a pre-Hilbert space $\mathcal{E}$ with the inner product $\langle \mu,\nu\rangle := I(\mu,\nu)$ and the energy norm $\|\mu\| := \sqrt{I(\mu)}$, cf. [8, Lemma 3.1.2]. The topology on $\mathcal{E}$ defined by this norm is said to be *strong*. Moreover, due to Deny [6] (for $\alpha = 2$, cf. also Cartan [4]), the cone $\mathcal{E}^+ := \mathcal{E} \cap \mathfrak{M}^+$ is *complete* in the induced strong topology, and the strong topology on $\mathcal{E}^+$ is *finer* than the induced vague topology on $\mathcal{E}^+$; that is, the kernel κ_α is *perfect* (Fuglede [8, Section 3.3]). Thus any strong Cauchy sequence (net) $(\mu_j) \subset \mathcal{E}^+$ converges both strongly and vaguely to the same (unique) limit $\mu_0 \in \mathcal{E}^+$, the strong topology on $\mathcal{E}$ as well as the vague topology on $\mathfrak{M}$ being Hausdorff.[1] Along with the perfectness of the Riesz kernel, the following so-called *domination principle* is also crucial to the theory in question: *For any given $\nu \in \mathcal{E}^+$ and $\mu \in \mathfrak{M}^+$, if $U^\nu \leqslant U^\mu$ ν-a.e., then the same inequality holds true on all of $\mathbb{R}^n$.*

For any $A \subset \mathbb{R}^n$, let $\mathfrak{M}^+(A)$ stand for the class of all $\mu \in \mathfrak{M}^+$ *concentrated on* A, which means that $A^c := \mathbb{R}^n \setminus A$ is μ-negligible, or equivalently that the set A is μ-measurable and $\mu = \mu|_A$, $\mu|_A$ being the trace of μ to A, cf. [3, Section V.5.7]. If A is closed, then $\mu \in \mathfrak{M}^+$ is concentrated on A if and only if $S(\mu) \subset A$, where $S(\mu)$ denotes the support of μ; while otherwise, $\mu \in \mathfrak{M}^+(A)$ does not imply that $S(\mu) \subset A$.

Let $\mathcal{E}'(A)$ be the closure of $\mathcal{E}^+(A) := \mathcal{E} \cap \mathfrak{M}^+(A)$ in the strong topology on $\mathcal{E}$. Being a strongly closed subcone of the strongly complete cone $\mathcal{E}^+$, *the convex cone $\mathcal{E}'(A)$ is likewise strongly complete.* We also denote

$$\Gamma_{A,\zeta} := \bigl\{\mu \in \mathfrak{M}^+ \mid \ U^\mu \geqslant U^\zeta \ \text{ n.e. on } \ A\bigr\}. \tag{1}$$

The concept of inner balayage ζ^A of any $\zeta \in \mathfrak{M}^+$ to any $A \subset \mathbb{R}^n$ is introduced by means of the following Theorems 1 and 2.

Theorem 1. *For any $\zeta := \sigma \in \mathcal{E}^+$, there is precisely one $\sigma^A \in \mathcal{E}'(A)$, called the inner balayage of σ to A, that is determined by any one of the following* (i)–(iii)*.*

(i) *There exists the unique $\sigma^A \in \mathcal{E}'(A)$ having the property*[2]

$$\|\sigma - \sigma^A\| = \min_{\mu \in \mathcal{E}'(A)} \|\sigma - \mu\|.$$

(ii) *There exists the unique $\sigma^A \in \mathcal{E}'(A)$ satisfying the equality*

$$U^{\sigma^A} = U^\sigma \quad \textit{n.e. on} \ \ A.$$

(iii) *σ^A is the unique solution to the problem of minimizing the energy over the class $\Gamma_{A,\sigma}$, introduced by* (1) *with $\zeta := \sigma$. That is, $\sigma^A \in \Gamma_{A,\sigma}$ and*

$$I(\sigma^A) = \min_{\mu \in \Gamma_{A,\sigma}} I(\mu).$$

Proof. See [13, Section 3], [14, Section 4], and [18, Section 3]. □

Theorem 2. *For any $\zeta \in \mathfrak{M}^+$, there exists precisely one $\zeta^A \in \mathfrak{M}^+$, called the inner balayage of ζ to A, that is determined by any one of the following* (i_1)–(iii_1)*.*

(i_1) *ζ^A is the unique solution to the problem of minimizing the potential U^μ over the class $\Gamma_{A,\zeta}$, that is, $\zeta^A \in \Gamma_{A,\zeta}$ and*

$$U^{\zeta^A} = \min_{\mu \in \Gamma_{A,\zeta}} U^\mu \quad \textit{on} \ \ \mathbb{R}^n.$$

(ii_1) *There exists the unique $\zeta^A \in \mathfrak{M}^+$ satisfying the symmetry relation*

$$I(\zeta^A, \sigma) = I(\zeta, \sigma^A) \quad \textit{for all} \ \ \sigma \in \mathcal{E}^+,$$

where σ^A is uniquely determined by means of Theorem 1.

[1] The whole pre-Hilbert space $\mathcal{E}$ is, in general, *strongly incomplete* (see a counterexample by Cartan [4] pertaining to the Newtonian kernel κ_2). For conditions ensuring the strong completeness of convex subsets of $\mathcal{E}$, see the author's result [12, Theorem 9.1], even dealing with an arbitrary perfect kernel on a locally compact Hausdorff space and infinite dimensional vector measures.

[2] That is, the inner balayage σ^A of $\sigma \in \mathcal{E}^+$ to A is, actually, *the orthogonal projection* of σ in the pre-Hilbert space $\mathcal{E}$ onto the (convex, strongly complete) cone $\mathcal{E}'(A)$, cf. [7, Theorem 1.12.3].

(iii$_1$) *There exists the unique* $\zeta^A \in \mathfrak{M}^+$ *satisfying either of the two limit relations*

$$\sigma_j^A \to \zeta^A \quad \text{vaguely in } \mathfrak{M}^+ \text{ as } j \to \infty,$$
$$U^{\sigma_j^A} \uparrow U^{\zeta^A} \quad \text{pointwise on } \mathbb{R}^n \text{ as } j \to \infty,$$

where $(\sigma_j) \subset \mathcal{E}^+$ *denotes an arbitrary sequence having the property*[3]

$$U^{\sigma_j} \uparrow U^{\zeta} \quad \text{pointwise on } \mathbb{R}^n \text{ as } j \to \infty,$$

while σ_j^A *is uniquely determined by means of Theorem 1.*

Proof. See [13, Sections 3 and 4]. □

In spite of being in agreement with the theory of inner Newtonian balayage by Cartan [5], the results thus obtained require substantially different approaches, for in the case in question, useful specific features of Newtonian potentials fail to hold.

Remark 3. Although for $\zeta \in \mathfrak{M}^+$, we still have (see [13, Theorem 3.10])

$$U^{\zeta^A} = U^{\zeta} \quad \text{n.e. on } A, \tag{2}$$

this equality no longer determines ζ^A uniquely within $\mathcal{E}'(A)$ (as it does for $\zeta := \sigma \in \mathcal{E}^+$, cf. Theorem 1 (ii)), which can be seen by taking $\zeta := \varepsilon_y$, y being an inner α-*irregular point* of A. See [13, 15] for a definition and a detailed explanation.

Remark 4. The term "*inner* balayage" is justified by the limit relations

$$\zeta^K \to \zeta^A \quad \text{vaguely}, \quad U^{\zeta^K} \uparrow U^{\zeta^A} \quad \text{pointwise on } \mathbb{R}^n,$$

where K ranges over the upward partially ordered family $\mathfrak{C}_A$ of all *compact* sets in A. For $\zeta \in \mathcal{E}^+$, $\zeta^K \to \zeta^A$ also strongly. (See [13, Theorem 4.5]; cf. [13, Theorem 4.8], where A is the intersection of a lower partially ordered family of closed sets.)

Along with (2), the following properties of the inner balayage ζ^A, $\zeta \in \mathfrak{M}^+$ and $A \subset \mathbb{R}^n$ being arbitrary, are often useful.

(a) $U^{\zeta^A} \leqslant U^{\zeta}$ everywhere on $\mathbb{R}^n$ (see [13, Theorem 3.10]).

(b) Principle of positivity of mass: $\zeta^A(\mathbb{R}^n) \leqslant \zeta(\mathbb{R}^n)$ (see [13, Corollary 4.9]).

(c) Balayage "with a rest": $\zeta^A = (\zeta^Q)^A$ for any $Q \supset A$ (see [13, Corollary 4.2]).

Theorem 5. *For any* $\zeta \in \mathfrak{M}^+$ *and* $A \subset \mathbb{R}^n$*, the integral representation holds:*

$$\zeta^A = \int \varepsilon_x^A \, \mathrm{d}\zeta(x),$$

where ε_x *denotes the unit Dirac measure at* $x \in \mathbb{R}^n$*.*

Proof. See [15, Theorem 5.1]. □

Theorem 5 is particularly useful in applications, for ε_x^A, *the inner* α*-harmonic measure* of a set A at a point x, serves as the main tool in solving the generalized Dirichlet problem for α-harmonic functions, associated with the fractional Laplacian.

The concept of inner α-harmonic measure of A is closely related to that of *inner* α*-equilibrium measure* γ_A of A, treated in an extended sense where both $\gamma_A(\mathbb{R}^n)$ and $I(\gamma_A)$ might be $+\infty$,

[3] Such a sequence $(\sigma_j) \subset \mathcal{E}^+$ does exist (see, e.g., [10, p. 272] or [5, p. 257, footnote]).

as well as to that of *inner α-thinness of A at infinity*. To be precise, A is said to be α-thin at infinity if for some (equivalently, every) $y \in \mathbb{R}^n$,

$$\sum_{j\in\mathbb{N}} \frac{c_*(A_j)}{q^{j(n-\alpha)}} < \infty,$$

where $q \in (1, \infty)$ and $A_j := A \cap \{x \in \mathbb{R}^n \mid q^j \leqslant |x-y| < q^{j+1}\}$; whereas the inner α-equilibrium measure γ_A is uniquely determined by the limit relation

$$\gamma_K \to \gamma_A \quad \text{vaguely in } \mathfrak{M}^+ \text{ as } K \text{ ranges through } \mathfrak{C}_A,$$

where γ_K is the unique measure in $\mathcal{E}^+(K)$ such that $U^{\gamma_K} = 1$ n.e. on K.

Theorem 6. *For arbitrary $A \subset \mathbb{R}^n$, the following* (i$_2$)–(v$_2$) *are equivalent:*

- (i$_2$) *There exists the* (*unique*) *inner equilibrium measure γ_A of A, treated in the above-mentioned extended sense.*
- (ii$_2$) *A is inner α-thin at infinity.*
- (iii$_2$) *There exists $\nu \in \mathfrak{M}^+$ having the property*

$$\operatorname{ess\,inf}_{x\in A}\, U^\nu(x) > 0,$$

the infimum being taken over all of A except for a subset of $c_*(\cdot) = 0$.

- (iv$_2$) *For some* (*equivalently, every*) *$y \in \mathbb{R}^n$, $\varepsilon_y^{A_y^*}$ is C-absolutely continuous, where A_y^* is the inverse of A with respect to the sphere $S_{y,1} := \{z \mid |z-y| = 1\}$.*[4]
- (v$_2$) *There exists $\chi \in \mathfrak{M}^+$ such that $\chi^A(\mathbb{R}^n) < \chi(\mathbb{R}^n)$.* (Compare with (b).)

Furthermore, if any one of these (i$_2$)–(v$_2$) *is fulfilled, then for every $y \in \mathbb{R}^n$,*

$$\varepsilon_y^{A_y^*} = (\gamma_A)^*,$$

where $(\gamma_A)^*$ is the Kelvin transform of $\gamma_A \in \mathfrak{M}^+$ with respect to the sphere $S_{y,1}$.[5]

Proof. See [15, Theorem 2.1]. □

Another application of the above results leads to the following surprising *generalization of Deny's principle of positivity of mass* (compare with [9, Theorem 3.11]).

Theorem 7. *Given $\mu, \nu \in \mathfrak{M}^+$, assume there exists $A \subset \mathbb{R}^n$ which is not inner α-thin at infinity, and such that $U^\mu \leqslant U^\nu$ n.e. on A. Then $\mu(\mathbb{R}^n) \leqslant \nu(\mathbb{R}^n)$.*

Proof. See [17, Theorem 1.2]. Compare with [22, Theorem 5.1], generalizing this to the associated α-Green kernels g_D^α, D being an (open, connected) domain in $\mathbb{R}^n$. □

Some of the above-mentioned results have recently been generalized to suitable kernels κ on a locally compact space X, suitable measures μ (not necessarily positive), and suitable sets $A \subset X$. See [14, 18–20, 23, 24, 27]. Furthermore, those results and their generalizations were shown to be a powerful tool in minimum energy problems in the presence of external fields, see [16, 20, 21, 23–27].

[4] $\mu \in \mathfrak{M}^+$ is said to be *C-absolutely continuous* if $\mu(K) = 0$ for every compact set $K \subset \mathbb{R}^n$ with $c(K) = 0$. This certainly occurs if $I(\mu) < \infty$, but not conversely (see Landkof [10, pp. 134–135]).

[5] For the concept of Kelvin transformation, see [11, Section 14] as well as [10, Section IV.5.19].

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