On the theory of inner Riesz balayage and its recent extensions and applications

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Symmetry and Integrability of Equations of Mathematical Physics

Main notions and notations

- $\mathbb{R}^n$, $n \geqslant 2$, with the Euclidean distance $|x-y|$;
- $C_0(\mathbb{R}^n)$ — the set of all $\varphi \in C(\mathbb{R}^n)$ of compact support;
- $\mathfrak{M} = \mathfrak{M}(\mathbb{R}^n)$ — the space of all Radon measures μ on $\mathbb{R}^n$ with the vague topology of pointwise convergence on $C_0(\mathbb{R}^n)$.

That is, a sequence $(\mu_j) \subset \mathfrak{M}$ is said to converge to μ_0 vaguely if

$$\mu_j(\varphi) \to \mu_0(\varphi) \quad \text{for every } \varphi \in C_0(\mathbb{R}^n).$$

Remainder:

- A Radon measure μ on $\mathbb{R}^n$ is thought to be a linear functional on $C_0(\mathbb{R}^n)$ that is continuous in the following sense:

  for every compact set $K \subset \mathbb{R}^n$, if

  $$\operatorname{Supp}\varphi_j,\ \operatorname{Supp}\varphi_0 \subset K \quad \text{and} \quad \varphi_j \to \varphi_0 \quad \text{uniformly on } K,$$

  then

  $$\mu(\varphi_j) \to \mu(\varphi_0) \quad \text{as } j \to \infty.$$

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- $\mathfrak{M}^+ = \mathfrak{M}^+(\mathbb{R}^n)$ — the convex cone of all positive $\mu \in \mathfrak{M}$, i.e.

$$\mu(\varphi) \geqslant 0 \quad \text{whenever } \varphi \geqslant 0.$$

By the Hahn–Jordan decomposition theorem,

$$\mathfrak{M} = \mathfrak{M}^+ - \mathfrak{M}^+.$$

Main notions and notations

- $\kappa_\alpha(x,y) := |x-y|^{\alpha-n}$, $\alpha \in (0,n)$, — the Riesz kernel in $\mathbb{R}^n$.
- For $\alpha = 2$, κ_2 is referred to as the Newtonian kernel, and it is actually the fundamental solution to the Laplace equation in $\mathbb{R}^n$:

$$-\Delta * \kappa_2 = C_{n,\alpha}\varepsilon, \quad \text{where } \varepsilon \text{ is the unit Dirac measure.}$$

- For $\alpha \neq 2$, κ_α is associated with the so-called fractional Laplace equation, a major subject in the probabilistic approach to the modern potential theory, a point of interest for American, Chinese, German, Polish, etc. mathematical schools.

See, e.g., the books:

- Bliedtner, J., Hansen, W., Potential Theory. An Analytic and Probabilistic Approach to Balayage. Springer, Berlin, 1986.
- Bogdan, K., Byczkowski, T., Kulczycki, T., Ryznar, M., Song, R., Vondraček, Z., Potential Analysis of Stable Processes and its Extensions. Springer, Berlin, 2009.

Main notions and notations

- $\kappa_\alpha(x,y) := |x-y|^{\alpha-n}$, $\alpha \in (0,n)$, — the Riesz kernel in $\mathbb{R}^n$.
- The (Riesz) potential of $\mu \in \mathfrak{M}$ at $x \in \mathbb{R}^n$ is given by means of

$$U^\mu(x) := \int \kappa_\alpha(x,y)\, d\mu(y),$$

  provided the value on the right is well defined in $[-\infty, +\infty]$.
- The mutual energy of $\mu, \nu \in \mathfrak{M}$ is defined by

$$I(\mu,\nu) := \int U^\mu \, d\nu = \int \kappa_\alpha(x,y)\, d(\mu \otimes \nu)(x,y).$$

  For $\mu = \nu$, $I(\mu,\mu) =: I(\mu)$ is termed the energy of $\mu \in \mathfrak{M}$.

For integration w. resp. to $\mu \in \mathfrak{M}$, we refer to Bourbaki's books.

Crucial facts

- κ_α is strictly positive definite, which means that $I(\mu) \geqslant 0$ for every (signed) $\mu \in \mathfrak{M}$, and moreover $I(\mu) = 0$ only for $\mu = 0$ (M. Riesz, 1938). Denote $\mathcal{E} := \mathcal{E}(\mathbb{R}^n) := \{\mu \in \mathfrak{M} : I(\mu) < \infty\}$.
- Therefore, $\mathcal{E}$ is a pre-Hilbert space with the inner product $\langle \mu, \nu \rangle := I(\mu, \nu)$ and the energy norm $\|\mu\| := \sqrt{I(\mu)}$. The topology on $\mathcal{E}$ defined by means of $\|\cdot\|$ is said to be strong.

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- By J. Deny (1950), $\mathcal{E}^+ := \mathcal{E}^+(\mathbb{R}^n) := \mathcal{E} \cap \mathfrak{M}^+$ is complete in the induced strong topology, and moreover the strong topology on $\mathcal{E}^+$ is finer than the vague topology. By B. Fuglede (1960), a kernel possessing these two properies, is said to be perfect.

Conclusion. Thus, if a sequence (a net) $(\mu_j) \subset \mathcal{E}^+$ is strongly Cauchy, then (μ_j) converges both strongly and vaguely to one and the same limit measure $\mu_0 \in \mathcal{E}^+$, the strong topology on $\mathcal{E}$ as well as the vague topology on $\mathfrak{M}$ being Hausdorff.

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- Note that the whole pre-Hilbert space $\mathcal{E}$ is strongly incomplete; see a counterexample by H. Cartan (1945), pertaining to the Newtonian kernel. A quite general criterion ensuring the strong completeness of convex subsets of $\mathcal{E}$, was found by N. Zorii (Potential Anal., 2013), and this was done even for infinite dimensional vector Radon measures on a locally compact space.

Inner and outer capacities of sets $A \subset \mathbb{R}^n$

For any A, let $\mathfrak{M}^+(A)$ consist of all $\mu \in \mathfrak{M}^+$ concentrated on A, which means that $A^c := \mathbb{R}^n \setminus A$ is μ-negligible, i.e. $\mu^*(A^c) = 0$; or equivalently that A is μ-measurable and $\mu = \mu|_A$.

For closed $A =: F$, $\mathfrak{M}^+(F)$ is the class of $\mu \in \mathfrak{M}^+$ with $\operatorname{Supp}\mu \subset F$; whereas this is not the case otherwise.

Example. The unit mass, uniformly distributed over the open unit ball $B_{0,1}$, belongs to $\mathfrak{M}^+(B_{0,1})$, whereas its support equals $\overline{B}_{0,1}$.

Henceforth, the following notations will often be in use:

$$\mathfrak{M}^1(A) := \bigl\{\mu \in \mathfrak{M}^+(A) : \ \mu(A) = 1\bigr\},$$
$$\mathcal{E}^+(A) := \mathcal{E} \cap \mathfrak{M}^+(A),$$
$$\mathcal{E}^1(A) := \mathcal{E} \cap \mathfrak{M}^1(A).$$

For any $A \subset \mathbb{R}^n$, the inner (Riesz) capacity $c_*(A)$ is defined by

$$1/c_*(A) := \inf_{\mu \in \mathcal{E}^1(A)} I(\mu), \tag{1}$$

while the outer $c^*(A)$ by

$$c^*(A) := \inf_{G \supset A} c_*(G) \quad \bigl(G \text{ open}\bigr). \tag{2}$$

If $c_*(A) = c^*(A)$, $c(A) := c_*(A)$ is simply termed the capacity of A; this occurs if A is Borel (see B. Fuglede, 1960 or Landkof's book).

It follows easily from (1) that (compare with (2))

$$c_*(A) = \sup_{K \subset A} c(K) \quad \bigl(K \text{ compact}\bigr),$$

or equivalently $c(K) \uparrow c_*(A)$, where K ranges over the upward partially ordered family $\mathfrak{C}_A$ of all compact subsets of A.

Negligible (exceptional) sets

A proposition $P(x)$, $x \in \mathbb{R}^n$, is said to hold quasi-everywhere (q.e.) on $A \subset \mathbb{R}^n$ if $c^*(E) = 0$, where $E := \{x \in A : \ P(x) \ \text{fails}\}$.

Replacing here $c^*(E) = 0$ by $c_*(E) = 0$ we arrive at the concept of nearly everywhere (n.e.) on A.

If $P(x)$ q.e. on A, then it also does n.e. on A, for $c^*(E) \geqslant c_*(E)$.

General requirement on $\mu \in \mathfrak{M}^+$

U^μ, $\mu \in \mathfrak{M}^+$, is always meant to be not identically infinite on $\mathbb{R}^n$

$$\Updownarrow$$

$$\int_{|x|\geqslant 1} \frac{d\mu(x)}{|x|^{n-\alpha}} < \infty$$

$$\Updownarrow$$

$$U^\mu < \infty \quad \text{q.e. on } \mathbb{R}^n$$

$$\Updownarrow$$

$$U^\nu = U^{\nu^+} - U^{\nu^-},\ \nu \in \mathfrak{M}, \quad \text{is well-defined and finite q.e. on } \mathbb{R}^n$$

$$\Uparrow$$

ν is bounded, i.e. $|\nu|(\mathbb{R}^n) = \nu^+(\mathbb{R}^n) + \nu^-(\mathbb{R}^n) < \infty$; or $\nu \in \mathcal{E}$

Fundamental problems

Inner balayage problem. For any $\zeta \in \mathfrak{M}^+$ and any $A \subset \mathbb{R}^n$, does there exist $\zeta^A \in \mathfrak{M}^+(\overline{A})$ such that $U^{\zeta^A} = U^\zeta$ n.e. on A?

Such ζ^A (if it $\exists!$) is termed the inner balayage of ζ to A.

Inner equilibrium problem. For any $A \subset \mathbb{R}^n$, does there exist $\gamma_A \in \mathfrak{M}^+(\overline{A})$ such that $U^{\gamma_A} = 1$ n.e. on A?

Such γ_A (if it $\exists!$) is termed the inner equilibrium measure for A.

These two problems will be shown to be deeply related to one another via the Kelvin transformation. We also intend to apply the concept of inner balayage to the inner Gauss variational problem, the problem of minimizing $I(\mu) - 2\int U^\zeta \, d\mu$ when μ ranges over $\mathcal{E}^1(A)$. A generalization of the results obtained to suitable function kernels on a locally compact space is also planned to be discussed.

Remarks

- For $\alpha = 2$, these problems were solved by Cartan (1946). However, his methods are not applicable for $\alpha \neq 2$, since the specific features of Newtonian potentials, based on harmonicity, then fail to hold.

- The balayage problem is deeply related to the Dirichlet problem. In fact, let $\alpha = 2$, $D \subset \mathbb{R}^n$ be a bounded (open, connected) domain with Lipschitz boundary ∂D, and ε_z be the unit Dirac measure at $z \in D$. Then for any $f \in C(\partial D)$,

$$h_f(z) := \int f(x)\, d\varepsilon_z^{\partial D}(x) = \int f(x)\, d\varepsilon_z^{D^c}(x)$$

is harmonic in D, and moreover

$$\lim_{z \to x \in \partial D} h_f(z) = f(x).$$

- The latter remark remains valid for $\alpha \in (0, 2)$ with "a harmonic function" replaced by the so-called "an α-harmonic function".

A permanent requirement

From now on, let $0 < \alpha \leqslant 2$; for if not, the balayage and equilibrium problems are both unsolvable — even for $\overline{B}_{0,1}$, which is caused by the fact that for $\alpha > 2$, U^μ, where $\mu \in \mathfrak{M}^+$, is superharmonic on $\mathbb{R}^n$.

Then the two maximum principles are fulfilled, often referred to as the first and the second maximum principles, respectively:

Frostman's maximum principle. For any $\mu \in \mathfrak{M}^+$ such that $U^\mu \leqslant 1$ on $\operatorname{Supp}\mu$, the same inequality holds true on all of $\mathbb{R}^n$.

The domination principle. For any $\nu \in \mathcal{E}^+$ and $\mu \in \mathfrak{M}^+$ with $U^\nu \leqslant U^\mu$ on $\operatorname{Supp}\nu$, the same inequality holds true on all of $\mathbb{R}^n$.

Let $\mathcal{E}'(A)$ be the closure of $\mathcal{E}^+(A)$ in the strong topology on $\mathcal{E}$. Also,

$$\Gamma_{A,\zeta} := \{\mu \in \mathfrak{M}^+ : \ U^\mu \geqslant U^\zeta \ \text{ n.e. on } \ A\}. \tag{3}$$

Theorem 1. For any $\zeta := \sigma \in \mathcal{E}^+$, there is precisely one $\sigma^A \in \mathcal{E}'(A)$, called the inner balayage of σ to A, given by any one of (i)–(iii).

(i) There is the unique $\sigma^A \in \mathcal{E}'(A)$ s. th.

$$\|\sigma - \sigma^A\| = \min_{\mu \in \mathcal{E}'(A)} \|\sigma - \mu\|.$$

(ii) There is the unique $\sigma^A \in \mathcal{E}'(A)$ s. th.

$$U^{\sigma^A} = U^\sigma \quad \text{n.e. on } \ A. \tag{4}$$

(iii) σ^A is the unique solution to the problem of minimizing the energy over the class $\Gamma_{A,\sigma}$, cf. (3). That is, $\sigma^A \in \Gamma_{A,\sigma}$ and

$$I(\sigma^A) = \min_{\mu \in \Gamma_{A,\sigma}} I(\mu).$$

Remarks

- Equality (4) can be refined as follows:

$$U^{\sigma^A} = U^\sigma \quad \text{on } A \setminus A^r, \tag{5}$$

where A^r is the set of all inner α-regular points for A. That is,

$$y \in A^r \iff \sum_{j\in\mathbb{N}} \frac{c_*(A_j)}{q^{j(n-\alpha)}} = \infty,$$

where $q \in (0,1)$ and $A_j := A \cap \{x \in \mathbb{R}^n : \ q^{j+1} \leqslant |x-y| < q^j\}$. If $A^i := \overline{A} \setminus A^r$ denotes the set of all inner α-irregular points for A, then, by the Kellog–Evans type theorem,

$$c_*(A \cap A^i) = 0,$$

whence (5) is finer than (4), indeed.

Remarks

- If $A =: F$ is closed, then $\mathfrak{M}^+(F)$ is vaguely closed (Bourbaki), which implies, by use of the perfectness of the Riesz kernel κ_α, the strong closedness of the class $\mathcal{E}^+(F)$. Thus

$$\mathcal{E}'(F) = \mathcal{E}^+(F),$$

whence the above (ii) can be specified as follows:

(ii′) For any $\sigma \in \mathcal{E}^+$ and any closed F, the inner balayage σ^F is uniquely determined within $\mathcal{E}^+(F)$ by the equality

$$U^{\sigma^F} = U^\sigma \quad \text{n.e. on} \quad F.$$

Theorem 2. For any $\zeta \in \mathfrak{M}^+$, there is precisely one $\zeta^A \in \mathfrak{M}^+$, called the inner balayage of ζ to A, determined by any one of (i$_1$)–(iii$_1$):

(i$_1$) ζ^A is the unique solution to the problem of minimizing U^μ over the class $\Gamma_{A,\zeta}$, that is, $\zeta^A \in \Gamma_{A,\zeta}$ and

$$U^{\zeta^A} = \min_{\mu \in \Gamma_{A,\zeta}} U^\mu \quad \text{on} \quad \mathbb{R}^n.$$

(ii$_1$) There is the unique $\zeta^A \in \mathfrak{M}^+$ meeting the symmetry relation

$$I(\zeta^A, \sigma) = I(\zeta, \sigma^A) \quad \text{for all} \quad \sigma \in \mathcal{E}^+,$$

where σ^A is uniquely determined by means of Theorem 1.

(iii$_1$) There is the unique $\zeta^A \in \mathfrak{M}^+$ meeting either of the two relations

$$\sigma_j^A \to \zeta^A \quad \text{vaguely in } \mathfrak{M}^+, \qquad U^{\sigma_j^A} \uparrow U^{\zeta^A} \quad \text{pointwise on } \mathbb{R}^n,$$

where $(\sigma_j) \subset \mathcal{E}^+$ is an arbitrary sequence such that

$$U^{\sigma_j} \uparrow U^\zeta \quad \text{pointwise on } \mathbb{R}^n. \tag{6}$$

Remarks

- A sequence $(\sigma_j) \subset \mathcal{E}^+$ meeting (6) does exist. One can take e.g.

$$U^{\sigma_j}(x) := \min\left\{U^\zeta(x),\, jU^\gamma(x)\right\}, \quad x \in \mathbb{R}^n,$$

where $\gamma \in \mathcal{E}^+$ is a fixed bounded measure.

- Similarly as it was in the case of $\sigma \in \mathcal{E}^+$, for $\zeta \in \mathfrak{M}^+$, we have

$$U^{\zeta^A}(x) = U^\zeta(x) \quad \text{for all } x \in A^r \quad \text{(whence, n.e. on } A). \tag{7}$$

However, this no longer determines ζ^A uniquely — even for closed A, which can be seen by taking $\zeta := \varepsilon_z$ with $z \in A^i$. (See below for a detailed explanation of this phenomenon.)

Some further properties of the inner balayage ζ^A

Along with equality (7), we have:

(a) $U^{\zeta^A} \leqslant U^{\zeta}$ on all of $\mathbb{R}^n$.

(b) $\zeta^A(\mathbb{R}^n) \leqslant \zeta(\mathbb{R}^n)$ (by Deny's principle of positivity of mass).

(c) $\zeta^A = (\zeta^Q)^A$ for any $Q \supset A$ (balayage "with a rest").

(d) $\zeta^A = \int \varepsilon_z^A \, d\zeta$ (the integral representation formula).

ε_z^A is referred to as the inner α-harmonic measure of A at z. Actually,

$$\varepsilon_z^A \begin{cases} = \varepsilon_z \quad \text{if} \quad z \in A^r, \\ \text{is } C\text{-absolutely continuous otherwise,} \end{cases} \tag{8}$$

the latter means that $\varepsilon_z^A(E) = 0$ for any $E \subset \mathbb{R}^n$ with $c_*(E) = 0$.

Why $U^{\zeta^A} = U^\zeta$ n.e. on A does not determine ζ^A uniquely ?
(Unless, of course, $\zeta \in \mathcal{E}^+$ while A is closed.)

Let $A =: F$ be closed, $\zeta := \varepsilon_z$, where $z \in F^i$. Then according to (7),

$$U^{\varepsilon_z^F} = U^{\varepsilon_z} \quad \text{n.e. on } F, \tag{9}$$

while by the latter formula in (8), $\varepsilon_z^F \in \mathfrak{M}^+(F)$ is C-abs. continuous. Noting that (9) also holds for $\varepsilon_z \in \mathfrak{M}^+(F)$ in place of ε_z^F, whereas ε_z is certainly not C-absolutely continuous, we arrive at the claim.

Conclusion. Thus the definition of ζ^F as a measure supported by F and s. th. $U^{\zeta^F} = U^\zeta$ n.e. on F is incorrect. Such a wrong definition is spread broadly in papers where potential theory is used as a tool.

A description of $\operatorname{Supp}\zeta^F$

For any closed $F \subset \mathbb{R}^n$, let $\check{F}$ denote the reduced kernel of F, that is, the set of all $x \in F$ such that $c(B_{x,r} \cap F) > 0$ for all $r > 0$.

Theorem 3. Assume that $\alpha < 2$. Then for any nonzero $\zeta \in \mathfrak{M}^+$,

$$\operatorname{Supp}\zeta^F = \check{F}.$$

Proof. This follows from the integral representation for ζ^F in view of the relationship between the swept and equilibrium measures, to be discussed below.

Convergence theorems for monotone families of sets

Theorem 4. For any A and any $\zeta \in \mathfrak{M}^+$,

$$\zeta^K \to \zeta^A \quad \text{vaguely},$$

$$U^{\zeta^K} \uparrow U^{\zeta^A} \quad \text{pointwise on } \mathbb{R}^n,$$

K ranging over the upward partially ordered family of all compact subsets of A. If moreover $\zeta \in \mathcal{E}^+$, then also $\zeta^K \to \zeta^A$ strongly.

Theorem 5. If (F_s) is a lower partially ordered family of closed sets with the intersection F, then for any $\zeta \in \mathfrak{M}^+$,

$$\zeta^{F_s} \to \zeta^F \quad \text{vaguely}.$$

If moreover $\zeta \in \mathcal{E}^+$, then also

$$\zeta^{F_s} \to \zeta^F \quad \text{strongly},$$

$$U^{\zeta^{F_s}} \downarrow U^{\zeta^F} \quad \text{pointwise q.e. on } \mathbb{R}^n.$$

The theory of inner Riesz balayage as well as its generalization to any perfect kernel on a locally compact space X, satisfying the domination principle, and any $\sigma \in \mathcal{E}^+(X)$ and $A \subset X$ is given in:

- Zorii, N., A theory of inner Riesz balayage and its applications, Bull. Pol. Acad. Sci. Math. 68 (2020), 41–67, arXiv:1910.09946.
- Zorii, N., Harmonic measure, equilibrium measure, and thinness at infinity in the theory of Riesz potentials. Potential Anal. 57 (2022), 447–472, arXiv:2006.12364.
- Zorii, N., Balayage of measures on a locally compact space, Anal. Math. 48 (2022), 249–277, arXiv:2010.07199.
- Zorii, N., On the theory of capacities on locally compact spaces and its interaction with the theory of balayage, Potential Anal. 59 (2023), 1345–1379, arXiv:2202.01996.
- Zorii, N., On the theory of balayage on locally compact spaces, Potential Anal. 59 (2023), 1727–1744, arXiv:2108.13224.

Inner α-thinness of a set at infinity

$A \subset \mathbb{R}^n$ is inner α-thin at infinity if for some (equiv., every) $y \in \mathbb{R}^n$,

$$\sum_{j\in\mathbb{N}} \frac{c_*(A_j)}{q^{j(n-\alpha)}} < \infty,$$

where $q \in (1,\infty)$ and $A_j := A \cap \bigl\{x \in \mathbb{R}^n : \ q^j \leqslant |x-y| < q^{j+1}\bigr\}$;

or alternatively, if for some (equivalently, every) $y \in \mathbb{R}^n$,

$$y \in (A^*_y)^i,$$

A^*_y being the inverse of A with respect to $S_{y,1}$.

Remark. A set A is not inner α-thin at at infinity $\Rightarrow c_*(A) = \infty$.

$\not\Leftarrow$

Example 1. Let $n = 3$ and $\alpha = 2$. Define A to be a rotation body

$$A := \{x \in \mathbb{R}^3 :\ 0 \leqslant x_1 < \infty,\ x_2^2 + x_3^2 \leqslant \varrho^2(x_1)\},$$

where ϱ is given by one of the following three formulae:

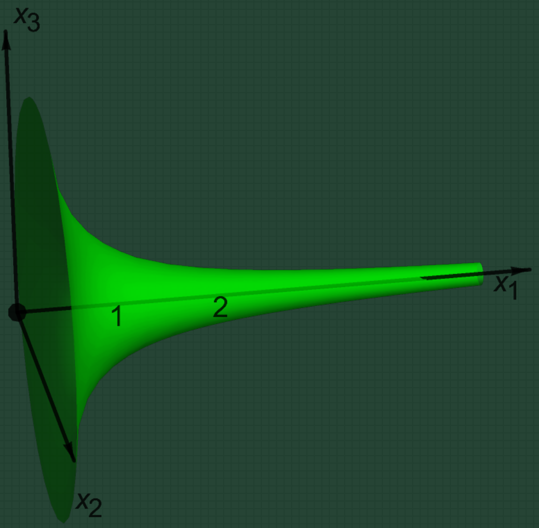
$$\varrho(x_1) = x_1^{-\tau} \quad \text{with} \quad \tau \in [0, \infty), \tag{10}$$

$$\varrho(x_1) = \exp(-x_1^{\tau}) \quad \text{with} \quad \tau \in (0, 1], \tag{11}$$

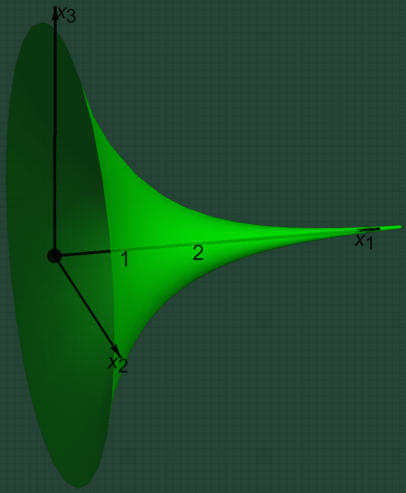
$$\varrho(x_1) = \exp(-x_1^{\tau}) \quad \text{with} \quad \tau \in (1, \infty). \tag{12}$$

Then A is 2-thin at infinity $\iff$ (11) or (12) holds. Moreover,

$$c(A) < \infty \iff \text{(12) takes place.}$$



$$A := \left\{0 \leqslant x_1 < \infty,\ x_2^2 + x_3^2 \leqslant \varrho^2(x_1)\right\} \quad \text{with } \varrho(x_1) = x_1^{-1}$$



$$A := \left\{0 \leqslant x_1 < \infty,\ x_2^2 + x_3^2 \leqslant \varrho^2(x_1)\right\} \quad \text{with } \varrho(x_1) = \exp(-x_1)$$

The concept of inner α-thinness at infinity serves as an efficient tool in applications of balayage theory. This is caused, in particular, by:

Theorem 6. A set A is not inner α-thin at infinity if and only if

$$\zeta^A(\mathbb{R}^n) = \zeta(\mathbb{R}^n) \quad \text{for every } \zeta \in \mathfrak{M}^+.$$

Reminder: In general, $\zeta^A(\mathbb{R}^n) \leqslant \zeta(\mathbb{R}^n)$ for any ζ and any A.

Unless $\alpha < 2$, assume in Theorem 7 that $\overline{A}^c$ is connected.

Theorem 7. If A is inner α-thin at infinity, then

$$\zeta^A(\mathbb{R}^n) < \zeta(\mathbb{R}^n) \quad \text{for every } \zeta \in \mathfrak{M}^+(\overline{A}^c).$$

Theorems 6, 7 were proved in:

• Zorii, N., Harmonic measure, equilibrium measure, and thinness at infinity in the theory of Riesz potentials. Potential Anal. 57 (2022), 447–472, arXiv:2006.12364.

A generalization of Deny's principle of positivity of mass

Preceding theorem. For any $\mu, \nu \in \mathfrak{M}^+$ s. th. $U^\mu \leqslant U^\nu$ on all of $\mathbb{R}^n$,

$$\mu(\mathbb{R}^n) \leqslant \nu(\mathbb{R}^n). \tag{13}$$

See: • Fuglede, B., Zorii, N., Green kernels associated with Riesz kernels, Ann. Acad. Sci. Fenn. Math. 43 (2018), 121–145.

Theorem 8. Given $\mu, \nu \in \mathfrak{M}^+$, assume there is A which is not inner α-thin at infinity and s. th. $U^\mu \leqslant U^\nu$ n.e. on A. Then (13) still holds.

See: • Zorii N., On the role of the point at infinity in Deny's principle of positivity of mass for Riesz potentials, Anal. Math. Phys. 13 (2023), article no. 38, 18 pages, arXiv:2202.12418,

and a recent extension of Th. 8 to α-Green potentials: • Zorii N., Balayage, equilibrium measure, and Deny's principle of positivity of mass for α-Green potentials, Anal. Math. Phys. 15:3 (2025), 20 pp.

An inner Riesz equilibrium measure

$\gamma_A \in \mathfrak{M}^+$ is termed an inner equilibrium measure for a given set A if

$$U^{\gamma_A} = \inf_{\mu \in \Gamma_A} U^\mu \quad \text{on all of } \mathbb{R}^n, \tag{14}$$

where $\quad \Gamma_A := \{\mu \in \mathfrak{M}^+ : \ U^\mu \geqslant 1 \ \text{ n.e. on } A\}$.

Theorem 9. γ_A exists $\iff$ A is α-thin at infinity. Furthermore, then

$$U^{\gamma_A} = 1 \quad \text{on } A^r \quad (\text{whence, n.e. on } A),$$

and so γ_A is the unique solution to the extremal problem (14). Also, γ_A is C-absolutely continuous, supported by $\overline{A}$, and such that

$$U^{\gamma_A} \leqslant 1 \quad \text{on all of } \mathbb{R}^n.$$

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$$U^{\gamma_A} \leqslant 1 \quad \text{on all of } \mathbb{R}^n.$$

In addition, when K ranges over $\mathfrak{C}_A$,

$$\gamma_K \to \gamma_A \quad \text{vaguely}, \quad U^{\gamma_K} \uparrow U^{\gamma_A} \quad \text{pointwise on } \mathbb{R}^n,$$

where $\gamma_K \in \mathcal{E}^+(K)$ is the (classical) equilibrium measure on K, uniquely determined by $U^{\gamma_K} = 1$ n.e. on K. Thus

$$I(\gamma_A) < \infty \iff c_*(A) < \infty.$$

If $I(\gamma_A) < \infty$, then also

$$\gamma_K \to \gamma_A \quad \text{strongly}.$$

Example 1′. Let $n = 3$ and $\alpha = 2$. Define A to be a rotation body

$$A := \{x \in \mathbb{R}^3 : \ 0 \leqslant x_1 < \infty, \ x_2^2 + x_3^2 \leqslant \varrho^2(x_1)\},$$

where ϱ is given by one of the following three formulae:

$$\varrho(x_1) = x_1^{-\tau} \quad \text{with} \quad \tau \in [0, \infty), \tag{15}$$

$$\varrho(x_1) = \exp(-x_1^{\tau}) \quad \text{with} \quad \tau \in (0, 1], \tag{16}$$

$$\varrho(x_1) = \exp(-x_1^{\tau}) \quad \text{with} \quad \tau \in (1, \infty). \tag{17}$$

Then:

$$\gamma_A \text{ exists} \iff A \text{ is 2-thin at infinity} \iff (16) \text{ or } (17) \text{ holds.}$$

Moreover,

$$I(\gamma_A) < \infty \iff c_*(A) < \infty \iff (17) \text{ takes place.}$$

Inner α-thinness and α-ultrathinness of a set at infinity

Reminder: $A \subset \mathbb{R}^n$ is inner α-thin at infinity if for some $y \in \mathbb{R}^n$,

$$\sum_{j \in \mathbb{N}} \frac{c_*(A_j)}{q^{j(n-\alpha)}} < \infty, \tag{18}$$

where $q \in (1, \infty)$ and $A_j := A \cap \left\{x \in \mathbb{R}^n : \; q^j \leqslant |x - y| < q^{j+1}\right\}$.

Theorem 10. If q and A_j are as above, then (compare with (18))

$$\gamma_A \in \mathcal{E}^+ \;\text{ exists} \iff c_*(A) < \infty \iff \sum_{j \in \mathbb{N}} \frac{c_*(A_j)}{q^{2j(n-\alpha)}} < \infty.$$

Such A is said to be inner α-ultrathin at infinity.

On the total mass of α-harmonic measure

Theorem 11. For any $y \in \mathbb{R}^n$ and any $A \subset \mathbb{R}^n$,

$$\varepsilon_y^A(\mathbb{R}^n) = \begin{cases} 1 & \text{if } A \text{ is not } \alpha\text{-thin at infinity,} \\ U^{\gamma_A}(y) & \text{otherwise.} \end{cases}$$

Corollary. Thus, for any $\zeta \in \mathfrak{M}^+$,

$$\zeta^A(\mathbb{R}^n) = \begin{cases} \zeta(\mathbb{R}^n) & \text{if } A \text{ is not } \alpha\text{-thin at infinity,} \\ \int U^{\gamma_A}\, d\zeta & \text{otherwise,} \end{cases}$$

which is seen from Th. 11 by the integral representation formula.

On the relation between inner swept and equilibrium measures

Theorem 12. If $A \subset \mathbb{R}^n$ is inner α-thin at infinity, then

$$\varepsilon_y^{A^*} = (\gamma_A)^* \quad \text{for any } y \in \mathbb{R}^n,$$

where $A^* := A^*_{y,1}$ is the inverse of A with respect to $S_{y,1}$, while $(\gamma_A)^*$ is the Kelvin transform of γ_A, given by

$$d\gamma_A^*(x^*) := |x-y|^{\alpha-n} d\gamma_A(x),$$

x^* being the image of x with respect to $S_{y,1}$.

For Theorems 9–12, see: • Zorii, N., A theory of inner Riesz balayage and its applications, Bull. Pol. Acad. Sci. Math. 68 (2020), 41–67;

• Zorii, N., Harmonic measure, equilibrium measure, and thinness at infinity in the theory of Riesz potentials. Potential Anal. 57 (2022), 447–472.

Applications to the Gauss variational problem

Assume $A =: F$ is closed, and the charges living on the conductor F are also influenced by an external field $f : \mathbb{R}^n \to [-\infty, \infty]$.

The Gauss variational problem: Does there exist $\lambda \in \mathcal{E}^1(F)$ s. th.

$$I_f(\lambda) = \inf_{\mu \in \mathcal{E}^1(F)} I_f(\mu)\,?$$

$I_f(\mu) := \|\mu\|^2 + 2\int f\,d\mu$ is referred to as the Gauss functional.

- Gauss, C.F., Allgemeine Lehrsätze in Beziehung auf die im verkehrten Verhältnisse des Quadrats der Entfernung wirkenden Anziehungs– und Abstoßungs–Kräfte, Werke 5 (1867), 197–244.

- Saff, E.B., Totik, V., Logarithmic Potentials with External Fields, Grundlehren Math. Wiss., 316. Springer, 2024.

- Zorii N., Minimum Riesz energy problems with external fields, J. Math. Anal. Appl. 526 (2023), article no. 127235, 32 pages.
- Zorii N., Inner Riesz pseudo-balayage and its applications to minimum energy problems with external fields, Potential Anal. 60 (2024), 1271–1300, arXiv:2301.00385.
- Zorii N., Minimum energy problems with external fields on locally compact spaces, Constr. Approx. 59 (2024), 385–417.
- Zorii N., On Fuglede's problem on pseudo-balayage for signed Radon measures of infinite energy, Anal. Math. Phys. 15:95 (2025).
- Zorii N., Inner Riesz balayage in minimum energy problems with external fields, Constr. Approx., to appear, arXiv:2306.12788.
- Zorii N., Fractional harmonic measure in minimum Riesz energy problems with external fields, Potential Anal., to appear.
- Zorii N., On an extension of Fuglede's theory of pseudo-balayage and its applications, Expo. Math., to appear.

In the papers listed above, we have shown that the theory of balayage is particularly useful in the Gauss variational problem.

Here, we limit ourselves to the following example of applications.

Given a closed set Q, $y \in Q$ is termed α-ultrairregular if $I(\varepsilon_y^Q) < \infty$, or equivalently if $Q^*_{y,1}$ is α-ultrathin at infinity (cf. Theorem 10).

Let $n = 3$, $\alpha = 2$, $z := (-1, 0, 0)$, and $F := \Delta \cup A$, where

$$\Delta := \bigl\{x \in \mathbb{R}^3 : \quad -1 \leqslant x_1 \leqslant 0, \quad x_2^2 + x_3^2 \leqslant \exp\bigl(-2(x_1 + 1)^{-\beta}\bigr)\bigr\},$$
$$A := \bigl\{x \in \mathbb{R}^3 : \quad 0 \leqslant x_1 < \infty, \quad x_2^2 + x_3^2 \leqslant x_1^{-2}\bigr\}.$$

Then F is not 2-thin at infinity, for so is its subset A,

while z is 2-irregular for Δ, whence for F. This yields respectively

$$\varepsilon_z^F(\mathbb{R}^n) = \varepsilon_z(\mathbb{R}^n) = 1, \quad I(\varepsilon_z^F) < \infty.$$

In the above notations, let moreover $f = -U^{\varepsilon_z}$. Then

$$I_f(\mu) = \|\mu\|^2 - 2\int U^{\varepsilon_z}\,d\mu = \|\mu\|^2 - 2\int U^{\varepsilon_z^F}\,d\mu = \|\mu - \varepsilon_z^F\|^2 - \|\varepsilon_z^F\|^2,$$

the second equality holds, for $U^{\varepsilon_z^F} = U^{\varepsilon_z}$ n.e. on F, whence ν-a.e. for any $\nu \in \mathcal{E}^+(F)$, while the last equality is due to $I(\varepsilon_z^F) < \infty$.

Thus $\inf_{\mu\in\mathcal{E}^1(F)} I_f(\mu)$ is an actual minimum if and only if so is

$$\inf_{\mu\in\mathcal{E}^1(F)} \|\mu - \varepsilon_z^F\|^2.$$

This is indeed so, for $\varepsilon_z^F \in \mathcal{E}^1(F)$, F not being 2-thin at infinity.

Since $\operatorname{Supp}\varepsilon_z^F = \partial F$, and so, in any small neighborhood of z there is some portion of the solution $\varepsilon_z^F \in \mathcal{E}^1(F)$, no blow-up appears between the negative charge $-\varepsilon_z$, creating an external field, and the solution ε_z^F, which seems to contradict our physical intuition.