Variety of Nizhnik models and their symmetry analysis

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In [2], Leonid Nizhnik constructed one of the first (1+2)-dimensional integrable systems of differential equations,

$$w_t = k_1 w_{xxx} + k_2 w_{yyy} + 3(v^1 w)_x + 3(v^2 w)_y, \quad v^1_y = k_1 w_x, \quad v^2_x = k_2 w_y, \tag{1}$$

where $(k_1, k_2) \neq (0,0)$. Later, it was called the Nizhnik system. In fact, this is a family of systems parameterized by two constants, k_1 and k_2. Up to scale and permutation equivalence transformations, there are only two equivalence classes in the family (1). Namely, in the symmetric case $k_1 k_2 \neq 0$, i.e., if both parameters k_1 and k_2 are nonzero, one can set $(k_1, k_2) = (1,1)$. The corresponding system (1) is a spatially-symmetric two-dimensional generalization of the Korteweg–de Vries equation. In the asymmetric case $k_1 k_2 = 0$, i.e., if one of the parameters k_1 and k_2 is zero, one can set $(k_1, k_2) = (1,0)$. Taking the dispersionless limit of systems of the form (1) leads to their dispersionless analogues. Introducing potentials, one can reduce each of the above dispersion and dispersionless systems to a single differential equation. In particular, the asymmetric potential Nizhnik equation is known in the literature as the Boiti–Leon–Manna–Pempinelli equation. We called each of various models related to (1) a Nizhnik model [3, 6]. These models have interesting and even exceptional symmetry properties, which deserve a comprehensive study within the framework of symmetry analysis of differential equations. Classical symmetry analysis of the dispersionless symmetric Nizhnik models has been carried out in [1,4,5].

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