Mean-field dynamics of attractive resource interaction: From uniform to aggregated states

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Modeling systems composed of many components competing for a limited resource is central to sociophysics and mathematical biology [2,7]. A key question is whether such systems converge to a uniform equilibrium (consensus) or an aggregated state (polarization/monopoly).

While repulsive interactions typically model strict competition [3], we focus on *attractive interaction*, which describes cooperation, alliance formation, or capital concentration. This connects to the concept of "conflict" between probability measures [4] and mean-field games [5].

We consider a system on a simplex defined by a probability measure $\mathbf{p} = (p_1, \dots, p_n)$, $\sum_{i=1}^n p_i = 1$. The evolution is governed by the map

$$p_i^{t+1} = \frac{p_i^t(1 + c_i r_i^t)}{z^t}, \quad r_i^t = \frac{1 - p_i^t}{n-1}, \tag{1}$$

where z^t is the normalization factor. The parameters $c_i \in (0, 1]$ represent the "attractiveness" of the i-th component.

Uniform states and consensus. Consider the case where favorable conditions are identical: $c_i = c$ for all i. The dynamics simplify to a nonlinear consensus protocol.

Theorem 1. *If $c_i = c$ for all i, every trajectory with positive initial coordinates converges to the centroid of the simplex*

$$\mathbf{p}^\infty = \lim_{t\to\infty} \mathbf{p}^t = \left(\frac{1}{n}, \dots, \frac{1}{n}\right).$$

Stability analysis. The Jacobian matrix J at the centroid has a specific structure with eigenvalues $\lambda = \frac{n^2-2}{n^2-1}$. Since $|\lambda| < 1$ for $n > 1$, the uniform state is locally asymptotically stable. Boundary states (where some $p_i = 0$) are unstable repellers in this regime. This mirrors results in linear consensus models [6] but arises here from multiplicative interactions.

Aggregated states and bifurcations. Let $M^* = \{k \mid p_k^\infty > 0\}$ be the set of indices for which the coordinates of the limiting vector are strictly positive. Then $\gamma(M^*)$ is the number of elements in the set M^*. The existence and uniqueness of M^* are determined by the iterative procedure (1). Let us denote

$$\Lambda(M^*) = \frac{\gamma(M^*) - 1}{\sum_{k\in M^*} \frac{1}{c_k}}.$$

Theorem 2. *For any set of parameters, the system converges to a unique fixed point $\mathbf{p}^\infty$. For any i, the limit coordinate is*

$$p_i^\infty = \max\left(0, 1 - \frac{\Lambda(M^*)}{c_i}\right). \tag{2}$$

This result implies that a component survives ($p_i^\infty > 0$) if and only if its attractiveness c_i exceeds the systemic threshold Λ.

The transition where a component p_i^∞ vanishes corresponds to a *transcritical bifurcation.* At $c_i = \Lambda$, the stable interior fixed point collides with an unstable boundary fixed point, exchanging stability. This mechanism explains the "exclusion principle" in our model: weak agents are pushed to the boundary ($p_i = 0$) not by repulsion, but by insufficient attraction relative to the mean field. This relates to border-collision bifurcations in piecewise-smooth maps [1].

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Bibliography

[1] Avrutin V., Gardini L., Sushko I., Tramontana F., Continuous and discontinuous piecewise-smooth one-dimensional maps: Invariant sets and bifurcation structures, World Scientific, 2019.

[2] Castellano C., Fortunato S., Loreto V., Statistical physics of social dynamics, *Rev. Mod. Phys.* **81** (2009), 591–646.

[3] Karataieva T., Koshmanenko V., Krawczyk M., Kulakowski K., Mean field model of a game for power, *Physica A* **525** (2019), 535–547.

[4] Koshmanenko V., Theorem of conflicts for a pair of probability measures, *Math. Methods Oper. Res.* **59** (2004), 303–313.

[5] Lasry J.M., Lions P.L., Mean field games, *Jpn. J. Math.* **2** (2007), 229–260.

[6] Olfati-Saber R., Fax J.A., Murray R.M., Consensus and cooperation in networked multi-agent systems, *Proc. IEEE* **95** (2007), 215–233.

[7] Sayama H., Introduction to the modeling and analysis of complex systems, Open SUNY Textbooks, 2015.