Soliton-like solutions of hydrodynamical type equations with variable coefficients

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Differential equations, both ordinary and partial, constitute a powerful instrument of modern applied mathematics. They are extensively employed for the mathematical modeling of a wide range of natural phenomena and engineering processes. The development of adequate mathematical models is, without exaggeration, a genuine art, since in the process of model formulation one must account for a substantial number of properties of the phenomenon or process under investigation, selecting those features that are most essential, while simultaneously avoiding excessive detail. Such over-refinement may lead to an unwarranted increase in model complexity, thereby impeding a meaningful analysis of the resulting mathematical model.

Alongside the development of fundamentally new mathematical models – which at present is a relatively infrequent occurrence – considerable attention is devoted by researchers to the search for natural generalizations of well-known (classical) models. Although these models have already been studied in considerable detail at the current stage of mathematical development, their potential for further application remains quite significant.

Currently, many researchers are studying generalizations of well-known (classical) models. Such generalizations are achieved by introducing variable coefficients, including coefficients that depend not only on the independent variables but also on the unknown function. The latter models are quite complex to study; symmetry analysis methods [1] are rather effective for their investigation. At the same time, these generalizations make it possible to take into account more general properties of phenomena and processes, in particular, for example, in problems of wave propagation, the properties of the medium, which obviously may vary both in time and in space. Therefore, such generalized models, in a certain sense, remain sufficiently adequate for describing phenomena for which these models were originally proposed in a simpler form.

This report will focus on the study of some generalized models of hydrodynamics, namely the variable coefficients (vc) Korteweg–de Vries equation

$$\varepsilon^n u_{xxx} = a(x,t,\varepsilon)u_t + b(x,t,\varepsilon)uu_x, \tag{1}$$

the vc Benjamin–Bona–Mahony equation

$$\varepsilon^2 u_{xxt} = a(x,t,\varepsilon)u_t + b(x,t,\varepsilon)u_x + c(x,t,\varepsilon)uu_x, \tag{2}$$

the vc Burgers equation

$$\varepsilon u_{xx} = a(x,t,\varepsilon)u_t + b(x,t,\varepsilon)uu_x, \tag{3}$$

and the vc modified Camassa–Holm equation

$$a(x,t,\varepsilon)u_t - \varepsilon^2 u_{xxt} + b(x,t,\varepsilon)u^2 u_x = 2\varepsilon^2 u_x u_{xx} + \varepsilon^2 u u_{xxx}, \tag{4}$$

where n is natural.

We assume that the coefficients in equations (1)–(4) can be expressed as asymptotic series in a small parameter, with nonzero main terms, which is necessary to ensure that the corresponding problem is nondegenerate.

In this generalized setting, the explicit form of exact solutions is unknown, since the presence of variable coefficients renders most classical analytical techniques ineffective. However, owing to the existence of a small parameter, asymptotic methods can be successfully employed. Within this framework, we construct approximate solutions that are close, in an appropriate sense, to the wave solutions of the corresponding equations with constant coefficients. In particular, we seek soliton-like solutions of equations (1)–(3), step-like solutions of equation (3), and peakon-like solutions of equation (4).

The methodology applied in our research is based on the nonlinear WKB approach [4] and enables the construction of asymptotic soliton-like solutions for the vc KdV and vc BBM equations, asymptotic soliton- and peakon-like solutions for the vc modified Camassa–Holm equation, as well as asymptotic step-like solutions for the vc Burgers equation. When the variable coefficients are reduced to constants, the constructed solutions recover the classical solitons, peakons, and wave solutions of the corresponding original equations. Accordingly, these solutions may be interpreted as deformations of standard wave-type solutions induced by the presence of variable coefficients. The methodology allows us to find asymptotic one- [6, 8, 12], two- [7] and multy-phase [9, 10] soliton-like solutions. This makes it possible to construct asymptotic multy-phase Σ-solutions to a singularly perturbed equation with variable coefficients [11].

The main idea of the approach is illustrated in the talk using the case of one-phase soliton-like solutions for the vc Korteweg–de Vries equation.

We represent the searched solution as

$$u(x,t,\varepsilon)=\sum_{j=0}^{\infty}\varepsilon^{j}\left[u_j(x,t)+V_j(x,t,\tau)\right], \quad \tau=\frac{x-\varphi(t)}{\varepsilon}, \tag{5}$$

where $u_j(x,t)$ and $V_j(x,t,\tau)$, $j=0,1,\ldots$, are infinitely differentiable functions.

Here, the function $\varphi=\varphi(t)$ is called *a phase function* and is defined as a solution of an ordinary differential equation that is found while constructing solution (5).

The regular part $U_N(x,t,\varepsilon)=\sum_{j=0}^{\infty}\varepsilon^j u_j(x,t)$ of asymptotic solution (5) is considered as background function while the singular part $V_N(x,t,\varepsilon)=\sum_{j=0}^{\infty}\varepsilon^j V_j(x,t,\tau)$ is required to reflect the soliton properties of the asymptotic solution.

This leads us to put on the singular terms $V_j(x,t,\tau)$ appropriate functional contraints. To this end, suitable functional spaces are introduced.

The main results include the development of an algorithm for constructing the regular and singular terms that provide the desired approximate solutions and the justification of the proposed technique, based on accuracy estimates for the constructed asymptotic solutions.

Although the resulting approximate solutions are generally not global – owing to the fact that the associated phase function is governed by a nonlinear differential equation – an appropriate choice of the equation coefficients ensures that the phase function remains globally well defined in time [13]. This property holds for a sufficiently broad class of models with variable coefficients.

The results are illustrated by graphs of the asymptotic approximations for several equations with prescribed coefficients.

The constructed asymptotic solutions can be used for deriving the Rankine–Hugoniot-type conditions having important significance in hydrodynamics [15].

The proposed approach can be extended to construct other classes of solutions, including step-like [2, 3, 14] and peakon-like [5] solutions, following a similar methodology.

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