Soliton-like solutions of hydrodynamical type equations with variable coefficients

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The Korteweg–de Vries (KdV) equation
(1895, Korteweg & de Vries)

$$u_t + uu_x - u_{xxx} = 0$$

The Benjamin–Bona–Mahony (BBM) equation
(1966, Peregrin; 1972, Benjamin, Bona & Mahony)

$$u_t + u_x + uu_x - u_{xxt} = 0$$

The Burgers' equation
(1915, Bateman; 1939, 1940, Burgers)

$$u_t + uu_x - u_{xx} = 0$$

The modified Camassa–Holm (mCH) equation
(1981, Fokas & Fuchssteiner; 1993, Camassa & Holm)

$$u_t - u_{xxt} + 3u^2 u_x = 2u_x u_{xx} + uu_{xxx}$$

Singular perturbed equations

The Korteweg–de Vries (KdV) equation

$$u_t + uu_x - \varepsilon^2 u_{xxx} = 0$$

The Benjamin–Bona–Mahony (BBM) equation

$$u_t + u_x + uu_x - \varepsilon^2 u_{xxt} = 0$$

The Burgers' equation

$$u_t + uu_x - \varepsilon u_{xx} = 0$$

The modified Camassa–Holm (mCH) equation

$$u_t - \varepsilon u_{xxt} + 3u^2 u_x = 2\varepsilon^2 u_x u_{xx} + \varepsilon u u_{xxx}$$

Variable coefficients equations

We study soliton–like solutions to the singular perturbed:
vc Korteweg–de Vries equation;
vc Benjamin–Bona–Mahony equation;
vc modified Camassa–Holm equation
and step-like solutions to vc Burgers' equation.

These equations are direct generalizations of the well–known hydrodynamical equations possessing physically interesting solutions such as solitons, peakons, and other type of wave solutions.

We development a general methodology for constructing asymptotic wave–like solutions.

On particular, we developed a general scheme for finding approximations of any order and studied their accuracy.

The results are illustrated by a number of examples.

The proposed technique can be used for studying wave–like solutions to other equations with variable coefficients and a small dispersion.

Variable coefficients equations

The vc Korteweg–de Vries equation

$$\varepsilon^n u_{xxx} = a(x,t,\varepsilon)u_t + b(x,t,\varepsilon)uu_x, \quad n \in \mathbb{N}$$

The vc Benjamin–Bona–Mahony equation

$$\varepsilon^2 u_{xxt} = a(x,t,\varepsilon)u_t + b(x,t,\varepsilon)u_x + c(x,t,\varepsilon)uu_x$$

The vc Burgers' equation

$$\varepsilon u_{xx} = a(x,t,\varepsilon)u_t + b(x,t,\varepsilon)uu_x$$

The vc modified Camassa–Holm equation

$$a(x,t,\varepsilon)u_t - \varepsilon^2 u_{xxt} + b(x,t,\varepsilon)u^2 u_x = 2\varepsilon^2 u_x u_{xx} + \varepsilon^2 uu_{xxx}$$

In this extended case, the exact form of solutions is not known, as most traditional analytical methods lose their effectiveness due to the presence of variable coefficients.

Due the presence of a small parameter the asymptotic technique can be effectively applied.

The Korteweg-de Vris equation

The KdV equation

$$u_t - 6uu_x + u_{xxx} = 0$$

is well known for its soliton solution

$$u(x,t) = -\frac{a^2}{2}\cosh^{-2}\left(\frac{a}{2}(x - x_0 - a^2 t)\right), \text{ where } a, x_0 \in \mathbf{R};$$

The KdV-equation has a package of one-soliton solutions – the so–called m-soliton solutions:

$$u(x,t) = -2\frac{\partial^2}{\partial x^2}\ln det(E + G),$$

where E is a unique $(m \times m)$–matrix, G is a matrix with elements

$$g_{ij}(x,t) = c_i(t)c_j(t)\frac{\exp\left[-(\varkappa_i + \varkappa_j)x\right]}{\varkappa_i + \varkappa_j},$$

$$c_i(t) = c_i(0)\exp\left(\varkappa_i^3 t\right),\ c_i(0), \varkappa_i \in \mathbb{R},\ i,j = \overline{1,m},\ 0 < \varkappa_1 < \varkappa_2 < \ldots < \varkappa_m.$$

Main problems for the KdV equation

For the KdV equation there are considered the following main problems:

— existence, uniqueness, smoothness (Su C.S., Sjoberg A., Bona J.L., Smith R., Biagioni H.A., Oberguggenberger M.);

— existence solutions with specific features:

– solitons (Gardner C.S., Green J.M., Kruskal M.D., Miura R.M., Hirota R., Zabusky N.J., Marchenko V.O.),

– periodic and finite-gap solutions (Novikov S.P., McLaughlin D.W., Khruslov Je.Y., Kotlyarov V.P., Egorova I.);

— asymptotic analysis (Kruskal M.D., Miura R.M., Lax P.D, Levermore S.D., de Kerf D., Flaschka H., Forest M.G., McLaughlin D.W., Ablowitz M.J., and others)

The KdV equation with a singular pertubation

The KdV equation with a singular perturbation was first studied by Miura R.M. and Kruskal M., 1974, and later by Lax P. and Levermore S.D., 1983.

Miura R.M. and Kruskal M. constructed asymptotic expansion for finite–gap solutions to equation

$$\delta^2 u_{xxx} + 6uu_x + u_t = 0 \qquad (1)$$

(Miura R.M., Kruskal M. Application of **nonlinear WKB–method** to the KdV equation, SIAM J. Appl. Math., (1974), V. 26 (3), P. 376 – 395)

Lax P., Levermore S.D. studied weak limits of solution to equation

$$\delta^2 u_{xxx} + 6uu_x + u_t = 0 \qquad (2)$$

as a small parameter δ tends to zero.

(Lax P., Levermore S.D. The small dispersion limit of the Korteweg–de Vries equation. I – III, Comm. Pure Appl. Math., (1983), V. 36 (3, 5, 6));

WKB method

We study the problem of finding special kind of asymptotic solutions of the Korteweg–de Vries equation with variable coefficients

$$\varepsilon^n u_{xxx} = a(x,t,\varepsilon)u_t + b(x,t,\varepsilon)uu_x, \quad n \in \mathbb{N}, \tag{3}$$

that are similar to soliton–like solutions of KdV equation. Therefore they can be considered as a deformation of the soliton–like waves of this equation.

The constructed asymptotic solutions are called
asymptotic soliton-like solutions.

We apply the *nonlinear Wentzel–Kramers–Brillouin (WKB) method* .

In mathematical physics, the WKB approximation or WKB method is a method for finding approximate solutions to linear DEqs with spatially varying coefficients. It is typically used for a semiclassical calculation in quantum mechanics in which the wave function is recast as an exponential function, semiclassically expanded, and then either the amplitude or the phase is taken to be changing slowly.

The name is an initialism for Wentzel–Kramers–Brillouin. It is also known as the LG or Liouville–Green method. Other often–used letter combinations include JWKB and WKBJ, where the "J" stands for Jeffreys.

Let consider differential equation of the second order

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2}+V(x)\Psi=E\Psi. \tag{4}$$

Here $\Psi=\Psi(x)$ is a wave (complex valued) function, $\hbar$ is the Planck constant, m is the mass of the particle, $V(x)$ is the potential energy, and E is the total energy.

This equation is called the *one-dimensional stationary Schrödinger equation*. It plays an important role in *quantum mechanics*. Its discovery was a significant landmark in the development of quantum mechanics.

It is named after Austrian (later Irish) physicist Erwin Rudolf Josef Alexander Schrödinger (12.08.1887–04.01.1961, Nobel Prize in Physics in 1933), who postulated the equation in 1925 and published it in 1926.

E. Schrödinger is recognized for the Schrödinger equation, that provides a way to calculate the wave function of a system and how it changes dynamically in time.

His research was also related to statistical mechanics and thermodynamics, physics of dielectrics, colour theory, electrodynamics, general relativity, and cosmology, and he tried to construct a unified field theory.

In popular culture, he is best known for his "Schrödinger's cat" thought experiment.

WKB method

An approximate solution of equation (4) can be found using the WKB method, according to which the solution is sought in the form $\Psi(x) = \exp \Phi(x)$. As follows, the function $\Phi(x)$ satisfies the nonlinear differential equation

$$\Phi''(x) + (\Phi'(x))^2 = \frac{2m}{\hbar^2}(V(x) - E), \tag{5}$$

which is reduced to differential equation of the first order for function $V(x) = \Phi'(x) = A(x)e^{iB(x)}$ (the function $A(x)$ is the amplitude of the complex value $\Phi'(x)$, and the function $B(x)$ is its phase).
By introducing new depending functions for the real and the image parts of the function $\Phi'(x)$ according to the formulas $A(x) = \mathrm{Re}\ \Phi'(x)$, $B(x) = \mathrm{Im}\ \Phi'(x)$ equation (5) turns into two differential equations

$$A' + A^2 - B^2 = \frac{2m}{\hbar^2}(V(x) - E), \quad B' + 2AB = 0, \tag{6}$$

the solutions of which are the functions $A(x)$, $B(x)$ sought as an expansion with respect to a parameter $\hbar$:

$$A(x) = \frac{1}{\hbar}\sum_{k=0}^{\infty} \hbar^k A_k(x), \quad B(x) = \frac{1}{\hbar}\sum_{k=0}^{\infty} \hbar^k B_k(x). \tag{7}$$

WKB method

Substituting series (7) into equations (6) after standard calculations, we obtain recurrence relations for their coefficients.

In particular, the main terms $A_0(x)$, $B_0(x)$ and the first terms $A_1(x)$ and $B_1(x)$ in (7) have to satisfy the equations

$$A_0^2(x) - B_0^2(x) = 2m(V(x) - E), \quad A_0(x)B_0(x) = 0, \tag{8}$$

$$A_0' + 2A_0A_1 - 2B_0B_1 = 0, \quad B_0' + 2A_0B_1 + 2A_1B_0 = 0. \tag{9}$$

It is obviously that to solve equations (8), (9) both cases $A_0(x) = 0$ and $B_0(x) = 0$ should be considered.

Taking into account the regularization condition for the wave function according to which $\Psi(x) \to 0$ as $|x| \to +\infty$, we consider only the case of classical region, when $E > V(x)$.

Last condition leads to equality $A_0(x) = 0$. It means that the amplitude of the wave function varies more slowly than its phase. From (8), (9) we have

$$B_0(x) = \pm\sqrt{2m(E - V(x))}, \quad B_1(x) = 0, \quad A_1(x) = -\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}x}\ln\left((E - V(x)\right).$$

WKB method

Thus, the first asymptotic approximation for the function $\Psi(x)$ is given as

$$\Psi(x) = \frac{C_+}{\sqrt[4]{2m(E - V(x))}} \exp\left(\frac{i}{\hbar}\int \sqrt{2m(E - V(x))}\right) + \frac{C_-}{\sqrt[4]{2m(E - V(x))}} \exp\left(-\frac{i}{\hbar}\int \sqrt{2m(E - V(x))}\right), \qquad (10)$$

where C_+, C_- are arbitrary constants.

Applying the WKB method to equation (4), it is usually sufficient to construct only the first two terms of the asymptotic solution.

Formula (10) present the basic solution in the WKB approximation for the stationary Schrödinger equation (4).

It asymptotically approximates the solution for all real arguments $x \in \mathbb{R}$, with except for the neighborhood of turning points, where $V(x) - E = 0$.

In the neighborhood of turning points, the asymptotic solution has a different type of presentation.

It is based on the Taylor series expansion near the turning point.

Nonlinear WKB method

Miura, R.M. and Kruskal, M.D. (1974), Application of nonlinear WKB-method to the Korteweg-de Vries equation, SIAM Appl. Math. **26***(2), pp. 376 – 395,* suggested constructing a solution of a nonlinear differential equation with a singular perturbation in the form of an asymptotic series in a small parameter using the representation:

$$u(x,t,\varepsilon) = U(\theta,x,t;\varepsilon) = U_0(\theta,x,t) + \varepsilon U_1(\theta,x,t) + \cdots \,, \qquad (11)$$

where $(x,t) \in K \times [0;T]$, $K \subset \mathbb{R}$, $T > 0$, and

$$\theta = \theta(x,t,\varepsilon) = \frac{B(x,t,\varepsilon)}{\varepsilon}, \quad B(x,t,\varepsilon) = B_0(x,t) + \varepsilon B_1(x,t) + \cdots \,. \qquad (12)$$

Series in (11) and (12) are formal expansions in ε.

The authors called this technique the nonlinear WKB method. It is clear that form of the solution (11) is more general than (7).

This approach turned out to be quite effective for constructing asymptotic soliton-like solutions of partial differential equations with variable coefficients and a singular perturbation (Korteweg–de Vries Eq., Benjamin–Bona–Mahony Eq., Burgers Eq., modified Camassa–Holm Eq. and others).

Problem under consideration

While modelling the wave processes in inhomogeneous medium with perturbations, there is the Korteweg-de Vries equation with variable coefficients (**vcKdV equation**) and a small parameter

$$\varepsilon^n u_{xxx} = a(x,t,\varepsilon) u_t + b(x,t,\varepsilon) u u_x, \quad n \in \mathbb{N}, \tag{13}$$

where

$$a(x,t,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k a_k(x,t), \quad b(x,t,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k b_k(x,t). \tag{14}$$

Here the functions $a_k(x,t)$, $b_k(x,t) \in C^\infty(\mathbb{R} \times [0;T])$, $k \geq 0$, $T > 0$, and

$$a_0(x,t)\, b_0(x,t) \neq 0 \quad \text{for all} \quad (x,t) \in \mathbb{R} \times [0;T].$$

We are interesting in **asymptotic soliton-like solutions** to equation (3) that are close to soliton solutions.

The KdV-like equation with a small parameter

The vcKdV-like equation with a small parameter

$$u_t + (\rho_1 + 3\rho_2 u)u_x + \varepsilon^2 \rho_3 u_{xxx} + \rho_4 u = 0, \tag{15}$$

where

$\rho_1 = \rho_1(x) = \sqrt{gH(x)}$,

$\rho_2 = \rho_2(x) = \sqrt{gH^{-1}(x)}/2$,

$\rho_3 = \rho_3(x) = \sqrt{gH^5(x)}/6$,

$\rho_4 = \rho_4(x) = \rho_{1x}/2$,

is used for modelling wave processes in shallow water.

Here:
$H(x) > 0$ is a depth of non-perturbed liquid,
g is acceleration of gravity,
ε is a small parameter (Maslov V.P., Omelyanov G.O., 1981).

Preliminary definitions

Definition (asymptotic series (expansion) in Poincaré)

$$f(x,\varepsilon) = \sum_{k=0}^{N} \varepsilon^k f_k(x) + O(\varepsilon^{N+1}), \quad x \in K, \quad \varepsilon \to 0. \qquad (1)$$

Relation (1) is equivalent to

$$\lim_{\varepsilon \to 0} \frac{f(x,\varepsilon) - \sum\limits_{k=0}^{N} \varepsilon^k f_k(x)}{\varepsilon^N} = 0, \quad x \in K.$$

Definition (the Schwartz space $\mathcal{S}(\mathbb{R})$)

Denote by $\mathcal{S}(\mathbb{R})$ the space of quickly decreasing functions $f(x)$ that are infinitely differentiable for all $x \in \mathbb{R}$, and for any integers $m, n \geq 0$ the following condition holds

$$\sup_{x \in \mathbf{R}} \left| x^m \frac{d^n}{dx^n} f(x) \right| < +\infty.$$

Main definitions

Definition (the spaces G and G_0)

Let $G = G(\mathbb{R} \times [0; T] \times \mathbb{R})$ be a space of infinitely differentiable functions $f = f(x, t, \tau)$, $(x, t, \tau) \in \mathbb{R} \times [0; T] \times \mathbb{R}$ such that there are fulfilled the following conditions:

1^0. the relation

$$\lim_{\tau \to +\infty} \tau^n \frac{\partial^p}{\partial x^p} \frac{\partial^q}{\partial t^q} \frac{\partial^r}{\partial \tau^r} f(x, t, \tau) = 0, \quad (x, t) \in K,$$

takes place;

2^0. there exists such a differentiable function $f^-(x, t)$ that on any compact set $K \subset \mathbb{R} \times [0; T]$ condition

$$\lim_{\tau \to -\infty} \tau^n \frac{\partial^p}{\partial x^p} \frac{\partial^q}{\partial t^q} \frac{\partial^r}{\partial \tau^r} \left(f(x, t, \tau) - f^-(x, t)\right) = 0, \quad (x, t) \in K,$$

is true for any non-negative integers n, p, q, r uniformly in $(x, t) \in K$.

Let $G_0 = G_0(\mathbb{R} \times [0; T] \times \mathbb{R}) \subset G$ be a space of functions $f(x, t, \tau) \in G$ when $f^-(x, t) = 0$ in condition 2^0. It means that last assumption implies inclusion $f(x, t, \tau) \in \mathcal{S}(\mathbb{R})$ with respect to the variable τ.

An asymptotic one phase soliton-like function

Definition (an asymptotic one phase soliton-like function)

A function $u = u(x, t, \varepsilon)$, where ε is a small parameter, is called **an asymptotic one phase soliton-like function** if for any integer $N \geq 0$ it can be represented in the following form

$$u(x, t, \varepsilon) = \sum_{j=0}^{N} \varepsilon^j \left[u_j(x, t) + V_j(x, t, \tau)\right] + O(\varepsilon^{N+1}), \quad \tau = \frac{x - \varphi(t)}{\varepsilon}, \qquad (2)$$

where

$\varphi(t) \in C^\infty([0; T])$ is a scalar real-valued function;

$u_j(x, t) \in C^\infty(\mathbb{R} \times [0; T])$, $j = \overline{0, N}$;

$V_0(x, t, \tau) \in G_0$;

$V_j(x, t, \tau) \in G$, $j = \overline{1, N}$.

Here $x - \varphi(t)$ is called **a phase** of the one-phase soliton-like function $u(x, t, \varepsilon)$.

A curve determined by equation $x - \varphi(t) = 0$ is called **a discontinuity curve** for function (2).

Main problem

We study the Korteweg–de Vries equation with variable coefficients (**vcKdV equation**) and a small parameter

$$\varepsilon^n u_{xxx} = a(x,t,\varepsilon)u_t + b(x,t,\varepsilon)uu_x, \quad n \in \mathbb{N}, \tag{3}$$

where

$$a(x,t,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k a_k(x,t), \quad b(x,t,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k b_k(x,t). \tag{4}$$

Here the functions $a_k(x,t)$, $b_k(x,t) \in C^\infty(\mathbb{R} \times [0;T])$, $k \geq 0$, $T > 0$, and

$$a_0(x,t)\, b_0(x,t) \neq 0 \quad \text{for all} \quad (x,t) \in \mathbb{R} \times [0;T].$$

We are looking for **asymptotic soliton-like solutions** of equation (3), that is, solutions close to soliton ones.
The main problem is construction of the asymptotic soliton-like solutions to vc KdV equation with singular perturbation.

It consists of **elaboration of algorithm** for constructing asymptotic solutions and **justification of the algorithm**, that is, establishment of **asymptotic estimates** for constructed asymptotic solutions.

Main steps

The problem is solved in several steps:

I. to specify form of the asymptotic solutions depending on the degree n at the highest derivative;

II. to deduce differential equations for the terms of the asymptotic expansions and to solve them;

III. to find the differential equation for phase function $\varphi(t)$;

IV. to obtain asymptotic estimations for the constructed asymptotic solution:

– in the case of the KdV equation we find the accuracy with which the solution satisfies the equation;

– in the case of the Cauchy problem we estimate difference between the exact and the constructed solutions.

Structure of the asymptotic solution

Representaion of the asymptotic one-phase soliton-like solution depends on the degree of a small parameter at the highest derivative, i.e. on the number n.

The solution is written as

$$u(x,t,\varepsilon)=\sum_{j=0}^{N}\varepsilon^{j}u_j(x,t)+\sum_{j=0}^{N}\varepsilon^{j}V_j(x,t,\tau)+O(\varepsilon^{N+1}),\quad \tau=\frac{x-\varphi(t)}{\varepsilon^{n/2}},\quad (5)$$

if n is odd;

and

$$u(x,t,\varepsilon)=\sum_{j=0}^{k}\varepsilon^{j}u_j(x,t)+\sum_{j=0}^{k}\varepsilon^{j}V_j(x,t,\tau)+\varepsilon^{k}\sum_{j=1}^{2N-2k}\varepsilon^{j/2}u_j(x,t)+$$

$$+\varepsilon^{k}\sum_{j=1}^{2N-2k}\varepsilon^{j/2}V_j(x,t,\tau)+O(\varepsilon^{N+\frac{1}{2}}),\quad \tau=\frac{x-\varphi(t)}{\sqrt{\varepsilon}\;\varepsilon^{k}},\qquad (6)$$

if n is even and $n=2k+1$, where $k\in\mathbb{N}\cup\{0\}$.

The asymptotic one-phase soliton-like solution

We demonstrate basic ideas of constructing asymptotic soliton-like solution for the case $n = 2$.

So, the solution is written as

$$u(x, t, \varepsilon) = \sum_{j=0}^{N} \varepsilon^j u_j(x, t) + \sum_{j=0}^{N} \varepsilon^j V_j(x, t, \tau) + O(\varepsilon^{N+1}), \quad \tau = \frac{x - \varphi(t)}{\varepsilon}. \tag{7}$$

Here

$$U_N(x, t, \varepsilon) = \sum_{j=0}^{N} \varepsilon^j u_j(x, t) \tag{8}$$

is a regular part of asymptotic solution (7). It is a background function.

The function

$$V_N(x, t, \tau, \varepsilon) = \sum_{j=0}^{N} \varepsilon^j V_j(x, t, \tau) \tag{9}$$

is a singular part of asymptotic solution (7) and reflects the soliton properties of the asymptotic solution.

The asymptotic one-phase soliton-like solution

Preliminary remark.

In the case of the KdV equation with constant coefficients

$$u_t - 6uu_x + \varepsilon^2 u_{xxx} = 0 \qquad (10)$$

the asymptotic solution to (10) can be constructed in the form (5), i.e,

$$u(x,t,\varepsilon) = \sum_{j=0}^{N} \varepsilon^j u_j(x,t) + \sum_{j=0}^{N} \varepsilon^j V_j(x,t,\tau) + O(\varepsilon^{N+1}), \quad \tau = \frac{x-\varphi(t)}{\varepsilon},$$

where

$$U_N(x,t,\varepsilon) \equiv 0,$$

$$V_N(x,t,\tau,\varepsilon) \equiv V_0(x,t,\tau) = -\frac{a^2}{2}\cosh^{-2}\left(\frac{a}{2}\tau\right), \quad \tau = \frac{x-a^2 t}{\varepsilon}.$$

Equation (10) has exact solution

$$u(x,t) = -\frac{a^2}{2}\cosh^{-2}\left(\frac{a}{2}\,\frac{x-a^2 t}{\varepsilon}\right).$$

So, asymptotic and exact solutions coincide.

The asymptotic one-phase soliton-like solution

Preliminary remark (continuation).

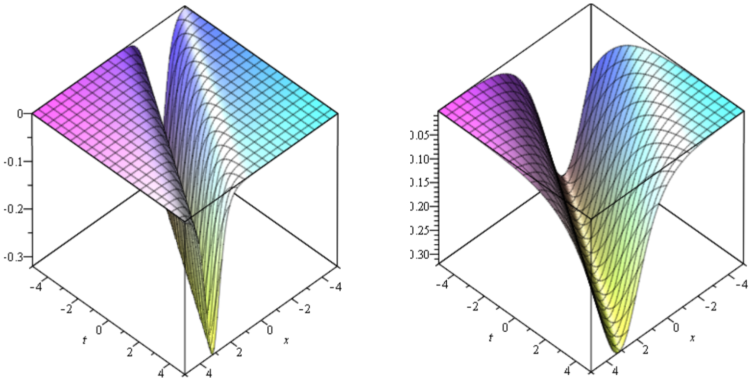


Рис.: The soliton solution of the KdV equation (2.10) as $\varepsilon = 0.25$ (at the left) and $\varepsilon = 0.75$ (at the right).

Differential equations for the regular terms

Terms of the regular part

$$U_N(x, t, \varepsilon) = \sum_{j=0}^{N} \varepsilon^j u_j(x, t)$$

of the asymptotic solution are defined from the following system

$$a_0(x, t)\frac{\partial u_0}{\partial t} + b_0(x, t)u_0\frac{\partial u_0}{\partial x} = 0, \quad (11)$$

$$a_0(x, t)\frac{\partial u_j}{\partial t} + b_0(x, t)u_0\frac{\partial u_j}{\partial x} + b_0(x, t)u_j\frac{\partial u_0}{\partial x} = f_j(x, t), \quad (12)$$

where the functions $f_j(x, t)$, $j = \overline{1, N}$, are recursively defined.

The equations (11) and (12) can be studied through the methods of characteristics.

Regular part of the asymptotic

Since equation (11) is quasilinear, and equation (12) is linear, their solutions can be found, for example, by means of the method of characteristics.

Indeed, for the first regular term, according to the method of characteristic we consider the system of ODEs:

$$\frac{dt}{a_0(x,t)} = \frac{dx}{b_0(x,t)u_0} = \frac{du_0}{0}. \tag{13}$$

The system gives the first regular term in implicit form as follows

$$\Phi\left(u_0(x,t), \psi(x,t,u_0(x,t))\right) = 0, \tag{14}$$

where the function $\Phi(\xi,\eta)$ is arbitrary in the general case, and in the Cauchy problem it is determined by the initial condition. The functions $u_0(x,t)$ and $\psi(x,t,u_0(x,t))$ are first integrals of system (13)

In the similar way we can find the higher regular terms in exact form:

$$\frac{dt}{a_0(x,t)} = \frac{dx}{b_0(x,t)u_0} = \frac{du_j}{f_j - b_0(x,t)u_{0x}u_j}.$$

So, we can assume that these solutions are known.

Differential equations for the singular terms

The terms of the singular part

$$V_N(x,t,\tau,\varepsilon)=\sum_{j=0}^{N}\varepsilon^j V_j(x,t,\tau)$$

of the asymptotic solution are defined as solutions to system of the third order partial differential equations

$$\frac{\partial^3 V_0}{\partial\tau^3}+a_0(x,t)\frac{\partial V_0}{\partial\tau}\varphi'(t)-b_0(x,t)\left(u_0\frac{\partial V_0}{\partial\tau}+V_0\frac{\partial V_0}{\partial\tau}\right)=0, \qquad (15)$$

$$\frac{\partial^3 V_j}{\partial\tau^3}+a_0(x,t)\frac{\partial V_j}{\partial\tau}\varphi'(t)-b_0(x,t)\left(u_0\frac{\partial V_j}{\partial\tau}+\frac{\partial}{\partial\tau}\left(V_0V_j\right)\right)=F_j(x,t,\tau), \quad (16)$$

where functions

$$F_j(x,t,\tau)=F_j(t,V_0(x,t,\tau),\ldots,V_{j-1}(x,t,\tau),u_0(x,t),\ldots,u_j(x,t))$$

are defined recurrently after determining the functions $u_0(x,t)$, $u_1(x,t)$, …, $u_j(x,t)$, $V_0(x,t,\tau)$, $V_1(x,t,\tau)$, …, $V_{j-1}(x,t,\tau)$, $\quad j=\overline{1,N}$.

Equations (15) and (16) are studied in a special way described below in detail.

Prolongation of the singular terms

It's necessary to take into account the following:
1. the solutions to equations (15), (16) must belong to the spaces G_0, G correspondingly;
2. while searching the terms $V_j(x,t,\tau)$, $j=\overline{0,N}$, we have to find a function $\varphi=\varphi(t)$ defining a discontinuity curve $\Gamma=\{(x,t)\in\mathbb{R}\times[0;T] : x=\varphi(t)\}$.

Algorithm of searching the singular terms of the asymptotics:

1. firstly we find $V_0(x,t,\tau)$ on curve Γ, i.e. $v_0(t,\tau)=V_0(x,t,\tau)\Big|_{x=\varphi(t)}$.

Then we prove $v_0(t,\tau)\in G_0$ and put $V_0(x,t,\tau)=v_0(t,\tau)$;

2. later we find the function $v_j(t,\tau)=V_j(x,t,\tau)\Big|_{x=\varphi(t)}$.

If $v_j(t,\tau)\in G_0$ then we put $V_j(x,t,\tau)=v_j(t,\tau)$.

If $v_j(t,\tau)\notin G_0$ then the function is prolonged from the discontinuity curve in a special way.

While finding the function $v_1(t,\tau)$ we get an ordinary differential equation for the phase function $\varphi=\varphi(t)$.

The main term of the singular part

Function $v_0 = v_0(t, \tau) = V_0(x, t, \tau)\Big|_{x=\varphi(t)}$ satisfies nonlinear differential equation

$$\frac{\partial^3 v_0}{\partial \tau^3} + a_0(\varphi, t)\frac{\partial v_0}{\partial \tau}\,\varphi'(t) - b_0(\varphi, t)\left(u_0(\varphi, t)\frac{\partial v_0}{\partial \tau} + v_0\frac{\partial v_0}{\partial \tau}\right) = 0. \qquad (17)$$

Under assumption

$$A(\varphi, t) = -a_0(\varphi, t)\varphi'(t) + b_0(\varphi, t)u_0(\varphi, t) > 0 \qquad (18)$$

equation (17) has a solution in the space G_0 as

$$v_0(t, \tau, \varphi) = -3\,\frac{A(\varphi, t)}{b_0(\varphi, t)}\;\cosh^{-2}\left(\frac{\sqrt{A(\varphi, t)}}{2}(\tau + c_0)\right), \qquad (19)$$

where $c_0 = const$.

Function (19) is similar to soliton solution of the Korteweg–de Vries equation with constant coefficients.

The higher terms of the singular part

Functions $v_j = v_j(t,\tau) = V_j(x,t,\tau)\Big|_{x=\varphi(t)}$, $j=\overline{1,N}$, satisfy linear differential equations

$$\frac{\partial^3 v_j}{\partial \tau^3} + a_0(\varphi,t)\frac{\partial v_j}{\partial \tau}\,\varphi'(t) - b_0(\varphi,t)\left(u_0(\varphi,t)\frac{\partial v_j}{\partial \tau} + \frac{\partial}{\partial \tau}\,(v_0\, v_j)\right) = \mathcal{F}_j(t,\tau). \tag{20}$$

where the right-side functions $\mathcal{F}_j(t,\tau)$, $j=\overline{1,N}$, are defined recurrently.

After integrating equation (20) we go to linear differential equation of the form

$$Lv = f \tag{21}$$

with operator

$$L = \frac{d^2}{d\tau^2} + q(\tau), \quad \tau \in \mathbb{R}.$$

In this case, the following **principal problem** arises:

under what conditions on the potential $q(\tau)$ of the operator L and the right-side function $f(\tau)$ in (21) does this equation have solutions in the space $S(\mathbb{R})$?

The answer is given by the following theorem.

Theorem (Samoilenko Valerii and Samoilenko Yuliia, 2012)

Let the following conditions be fulfilled:

1. $q(\tau) = q_0 + q_1(\tau)$ *with constant* $q_0 < 0$ *and the function* $q_1(\tau) \in \mathcal{S}(\mathbb{R})$*;*

2. the function $f \in \mathcal{S}(\mathbb{R})$*.*

If kernel of the operator $L : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ *is trivial,*
then equation (21) has a solution in the space $\mathcal{S}(\mathbb{R})$
for any function $f \in \mathcal{S}(\mathbb{R})$*.*

Otherwise, if kernel of the operator $L : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ *is not trivial,*
then equation (21) has a solution in the space $\mathcal{S}(\mathbb{R})$ *if and only if*
the function $f \in \mathcal{S}(\mathbb{R})$ *satisfies the orthogonality condition*

$$\int_{-\infty}^{+\infty} f(\tau) v_0(\tau)\, d\tau = 0 \tag{22}$$

for any $v_0 \in ker\ L$*.*

Reference: Samoylenko, V., Samoylenko, Y. *Existence of a solution to the inhomogeneous equation with the one-dimensional Schrodinger operator in the space of quickly decreasing functions.* J Math Sci **187**, 70 – 76 (2012).
https://doi.org/10.1007/s10958-012-1050-6

Lemma 1, existence solution $v_j(t,\tau) \in G$

From Theorem 1 we derive necessary and sufficient conditions for fulfilment of inclusion: $v_j(t,\tau) \in G$.

It means that the solution $v_j(t,\tau)$ of equation (20) belongs to the space G.

These conditions are given as conditions for the right-side functions of equations for $v_j(t,\tau)$ in Lemma 1.

Lemma (1)

Let us suppose $\mathcal{F}_j(t,\tau) \in G_0$, $j = \overline{1,N}$. The solution $v_j(t,\tau)$, $j = \overline{1,N}$, of equation (20) exists in the space G iff the orthogonality condition

$$\int\limits_{-\infty}^{+\infty} \mathcal{F}_j(t,\tau) v_0(t,\tau) d\tau = 0, \quad j = \overline{1,N}, \tag{23}$$

is true.

Remark. By proving Lemma 1, we obtain an important representation of the solution $v_j(t,\tau)$, $j = \overline{1,N}$.

Representation of $v_j(t,\tau)$, $j=\overline{1,N}$. Lemma 2

Solutions $v_j(t,\tau)\in G$, $j=\overline{1,N}$, of equation (20) can be written as

$$v_j(t,\tau)=\nu_j(t)\eta_j(t,\tau)+\psi_j(t,\tau),$$

where $\psi_j(t,\tau)\in G_0$; $\eta_j(t,\tau)\in G$ is a function such that

$$\lim_{\tau\to-\infty}\eta_j(t,\tau)=1;$$

$$\nu_j(t)=[a_0(\varphi,t)\varphi'(t)-b_0(\varphi,t)u_0(\varphi,t)]^{-1}\lim_{\tau\to-\infty}\Phi_j(t,\tau),$$

$$\Phi_j(t,\tau)=\int\limits_{-\infty}^{\tau}\mathcal{F}_j(t,\xi)d\xi+E_j(t),$$

and the function $\Phi_j(t,\tau)$ satisfies condition $\lim\limits_{\tau\to+\infty}\Phi_j(t,\tau)=0,\quad j=\overline{1,N}.$

Lemma (2)

If the function $\mathcal{F}_j(t,\tau)\in G_0$, $j=\overline{1,N}$, and the orthogonality condition (23) takes place, then the function $v_j(t,\tau)\in G_0$, $j=\overline{1,N}$, if and only if

$$\lim_{\tau\to-\infty}\Phi_j(t,\tau)=0,\quad j=\overline{1,N}. \tag{24}$$

Equation for $\varphi = \varphi(t)$

From (23) as $j = 1$ we obtain a non–linear ordinary differential equation of the second order for the phase function $\varphi = \varphi(t)$

$$15a_0(\varphi,t)\, b_0(\varphi,t)\frac{d}{dt}A(\varphi,t) + \Bigg[\bigg(10a_{0x}(\varphi,t)\, b_0(\varphi,t) - 36a_0(\varphi,t)\, b_{0x}(\varphi,t)\bigg)\varphi' + \tag{25}$$

$$+10b_0^2(\varphi,t)u_{0x}(\varphi,t) + 3(b_0^2(\varphi,t))_x u_0(\varphi,t) - 5a_0(\varphi,t)(b_0^2(\varphi,t))_t\Bigg]A(\varphi,t) = 0.$$

Here $A(\varphi,t) = A(\varphi(t),t) = -a_0(\varphi(t),t)\varphi'(t) + b_0(\varphi(t),t)u_0(\varphi(t),t)$.

Remark 1. In general case equation (25) has a local solution.

Remark 2. In particular case equation (25) is simplified, for example, in the case $a_0^5(x) = c_0 b_0^{12}(x)$, where $c_0 \in \mathbb{R}\backslash\{0\}$ is a constant, it is written as

$$(a_0(\varphi))^{2/3}\ \frac{d\varphi}{dt} = c_0.$$

Remark 3. Equation (25) is defined through the main terms of the coefficients of the vcKdV equation (3) and the main term of the regular part of the asymptotics.

Prolongation of the function $v_j(t,\tau)$

Recall that:
if $v_j(t,\tau) \in G_0$ then we put $V_j(x,t,\tau) = v_j(t,\tau)$;

if $v_j(t,\tau) \notin G_0$ then the function $v_j(t,\tau)$ is prolonged from the curve Γ according to the above representation

$$v_j(t,\tau) = \nu_j(t)\eta_j(t,\tau) + \psi_j(t,\tau).$$

Prolongation of the function $v_j(t,\tau)$, $j = \overline{1,N}$, from the discontinuity curve Γ is defined according to the formula

$$V_j(x,t,\tau) = u_j^-(x,t)\eta_j(t,\tau) + \psi_j(t,\tau), \qquad (26)$$

where $u_j^-(x,t)$, $j = \overline{1,N}$, is a solution of the Cauchy problem

$$\Lambda u_j^-(x,t) = f_j^-(x,t), \quad u_j^-(x,t)\big|_\Gamma = \nu_j(t), \quad j = \overline{1,N}, \qquad (27)$$

with the operator $\Lambda = a_0(x,t)\partial_t + b_0(x,t)u_0(x,t)\partial_x + b_0(x,t)u_{0x}(x,t)$.

The functions $f_j^-(x,t)$, $j = \overline{1,N}$, are found after substitution solution of form (26) into vcKdV equation (3) and calculating limit as $\tau \to -\infty$.

Justification of the asymptotics. Theorem 2

Theorem (2)

Let the following conditions be supposed:

1. the functions $a_k(x,t)$, $b_k(x,t) \in C^{(\infty)}(\mathbb{R} \times [0; T])$, $k = \overline{0, N}$, *and* $a_0(x,t) b_0(x,t) \neq 0$;

2. inequality

$$A(\varphi, t) = -a_0(\varphi, t)\varphi'(t) + b_0(\varphi, t) u_0(\varphi, t) > 0$$

takes place for the function $\varphi(t)$, *that is a solution to equation (25) for the phase function;*

3. the functions $\mathcal{F}_j(t, \tau) \in G_0$, $j = \overline{1, N}$, *and the orthogonality condition (23) is true;*

4. the functions $\mathcal{F}_j(t, \tau)$, $j = \overline{1, N}$, *satisfy conditions (24).*

Justification of the asymptotics. Theorem 2

Theorem (2, continuation)

Then the asymptotic one-phase soliton-like solution to the vcKdV equation (3) is written as

$$u_N(x,t,\varepsilon) = Y_N(x,t,\varepsilon) = \sum_{j=0}^{N} \varepsilon^j \left[u_j(x,t) + V_j(t,\tau)\right], \tag{28}$$

where

$$\tau = \frac{x - \varphi(t)}{\varepsilon}.$$

In addition, function (28) satisfies (3) on the set $\mathbb{R} \times [0;T]$ *with accuracy* $O(\varepsilon^N)$.

As $\tau \to \pm\infty$ *function (28) satisfies (3) with accuracy* $O(\varepsilon^{N+1})$.

Justification of the asymptotics. Theorem 3

Theorem (3)

Let conditions 1 – 3 of Theorem 2 be true and

the Cauchy problem (27) has a solution on the set $\{(x,t) \in \mathbb{R} \times [0;T] : x - \varphi(t) \leq 0\}$ *(condition 4′).*

Then the asymptotic one-phase soliton-like solution can be written as

$$u_N(x,t,\varepsilon) = Y_N(x,t,\varepsilon) = \sum_{j=0}^{N} \varepsilon^j \left[u_j(x,t) + V_j(x,t,\tau)\right], \qquad (29)$$

where

$$\tau = \frac{x - \varphi(t)}{\varepsilon}.$$

In addition, function (29) satisfies equation (3) with accuracy $O(\varepsilon^N)$ *on the set* $\mathbb{R} \times [0;T]$*.*
As $\tau \to \pm\infty$ *solution (29) satisfies (3) with accuracy* $O(\varepsilon^{N+1})$*.*

Example 1: KdV, $n = 2$

Power of singularity $n = 2$

Example

$$\varepsilon^2 u_{xxx} = -\left(x^2 + 1\right)^{3/2} u_t + \left(x^2 + 1\right)^{5/8} u u_x. \qquad (1)$$

Using the algorithm described above let us construct the first order approximation for the soliton-like solution of the equation. Note this equation is a special case of the vc-KdV equation when its coefficients satisfy the relation

$$a_0^5(x) = -b_0^{12}(x),$$

due to which differential equation for the phase function $\varphi = \varphi(t)$ is significantly simplified.
We consider zero background case, i.e. $U_N(x, t, \varepsilon) \equiv 0$.

The phase function $\varphi(t)$ satisfies the equation

$$\left(\varphi^2 + 1\right) \frac{d\varphi}{dt} = 1. \qquad (2)$$

Example 1: KdV, $n = 2$

Under the initial condition $\varphi(0) = 0$ we calculate its solution as

$$\varphi(t) = \sqrt[3]{\frac{3}{2}t + \sqrt{1 + \frac{9}{4}t^2}} + \sqrt[3]{\frac{3}{2}t - \sqrt{1 + \frac{9}{4}t^2}}. \qquad (3)$$

The function (3) is defined for all t.
Condition $A(\varphi(t), t) = \sqrt{\varphi^2(t) + 1} > 0$ holds for all $t \in \mathbb{R}$.
The main term of the singular part of the asymptotic solution is defined as a solution of differential equation of the form

$$\frac{\partial^3 v_0}{\partial \tau^3} - \sqrt{\varphi^2(t) + 1}\frac{\partial v_0}{\partial \tau} - \left(\varphi^2(t) + 1\right)^{5/8} v_0 \frac{\partial v_0}{\partial \tau} = 0.$$

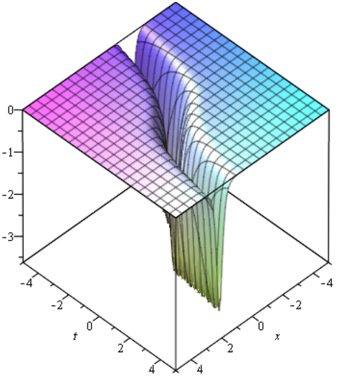
Its solution is as follows

$$v_0(t, \tau) = -3\left(\varphi^2(t) + 1\right)^{-1/8} \cosh^{-2}(\varkappa(t)\tau), \quad \varkappa(t) = \frac{1}{2}\left(\varphi^2(t) + 1\right)^{1/4},$$

and belongs to the space G_0. Hence, we can put

Example 1: KdV, $n = 2$

$$V_0(x,t,\tau) = v_0(t,\tau) = -3\left(\varphi^2(t)+1\right)^{-1/8}\cosh^{-2}\left(\varkappa(t)\tau\right). \qquad (4)$$



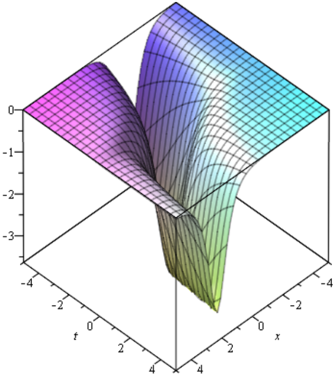


Рис.: The main term of the asymptotic soliton-like solution of the vcKdV equation (8) as $\varepsilon = 0.25$ (at the left) and $\varepsilon = 0.75$ (at the right).

Example 1: KdV, $n = 2$

To find $v_1(t,\tau)$ we calculate

$$\Phi_1(t,\tau) = \frac{\varphi(t)}{(\varphi^2(t)+1)^{5/8}} \left[\frac{21}{2}\tau + \frac{45}{8}\tau \ \cosh^{-2}(\varkappa(t)\tau) - \right.$$

$$\left. - \frac{15}{4}(\varphi^2(t)+1)^{-1/4} \tanh(\varkappa(t)\tau) \right] \cosh^{-2}(\varkappa(t)\tau).$$

The function $\Phi_1(t,\tau) \in G_0$.
Accordingly Lemma 2 the first singular term on the discontinuity curve Γ belongs to the space G_0. Hence, we can put $V_1(x,t,\tau) = v_1(t,\tau)$, where $v_1(t,\tau)$ is a solution of equation

$$\frac{\partial^3 v_1}{\partial \tau^3} - \sqrt{\varphi^2(t)+1}\,\frac{\partial v_1}{\partial \tau} - \left(\varphi^2(t)+1\right)^{5/8} \frac{\partial}{\partial \tau}(v_0 v_1) = \mathcal{F}_1(t,\tau), \qquad (5)$$

where

$$\mathcal{F}_1(t,\tau) = -\sqrt{\varphi^2(t)+1}\,\frac{\partial v_0}{\partial t} + \frac{3}{2}\frac{\tau}{\sqrt{\varphi^2(t)+1}}\frac{\partial v_0}{\partial \tau} + \frac{5}{8}\frac{\tau}{(\varphi^2(t)+1)^{3/8}} v_0 \frac{\partial v_0}{\partial \tau}.$$

Example 1: KdV, $n = 2$

By integrating differential equation (5), we obtain the first singular term as

$$V_1(x,t,\tau) = v_1(t,\tau) = \frac{15}{(\varphi^2(t)+1)^{9/8}} \frac{\tau}{2} \cosh^{-4}(\varkappa(t)\tau) \times$$
$$\times \left[\frac{37}{4} \left(\cosh^2(\varkappa(t)\tau) - \cosh^{-2}(\varkappa(t)\tau) \right) - \frac{1}{2} \right] +$$
$$+ \frac{3\varphi(t)}{(\varphi^2(t)+1)^{11/8}} \tanh(\varkappa(t)\tau) \cosh^{-2}(\varkappa(t)\tau) \times \tag{6}$$
$$\times \left[\frac{3}{8} + \frac{155}{8} \cosh^{-2}(\varkappa(t)\tau) + \frac{1}{32} \cosh^{-4}(\varkappa(t)\tau) + \frac{165}{2} \ln|\cosh(\varkappa(t)\tau)| \right] -$$
$$- \frac{3\varphi(t)}{(\varphi^2(t)+1)^{7/8}} \left[\frac{525}{32} \tau^2 \cosh^{-4}(\varkappa(t)\tau) + \frac{5}{8} \right] \tanh(\varkappa(t)\tau) \cosh^{-2}(\varkappa(t)\tau).$$

Thus, the first order approximation of the asymptotic soliton-like solution of the vcKdV equation (8) can be represented as

$$Y_1(x,t,\varepsilon) = V_0(x,t,\tau) + \varepsilon V_1(x,t,\tau), \tag{7}$$

where the functions $V_0(x,t,\tau)$ and $V_1(x,t,\tau)$ are given by formulae (4), (6), and $\tau = (x - \varphi(t))/\varepsilon$.

Example 1: KdV, $n = 2$

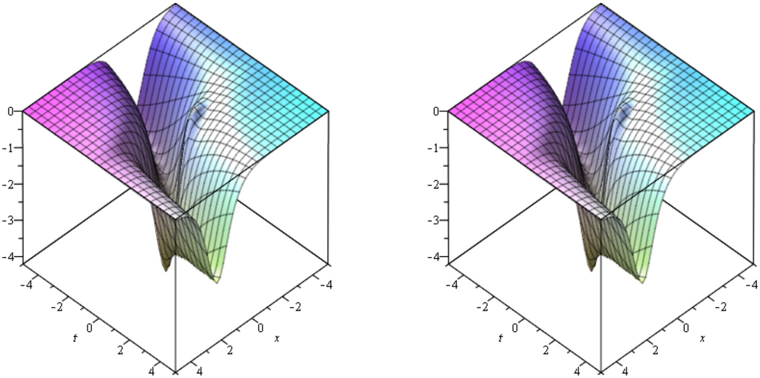


Рис.: The first order approximation for the asymptotic soliton-like solution of the vcKdV equation (8) as $\varepsilon = 0.25$ (at the left) and $\varepsilon = 0.75$ (at the right).

Function (7) is quickly decreasing function with respect to variable τ. Its graphs at $\varepsilon = 0.25$ and $\varepsilon = 0.75$ are demonstrated on the Fig. 3.

Example 2: KdV, $n = 1$

Power of singularity $n = 1$

Example

$$\varepsilon u_{xxx} = -(x^2 + 1)^{3/2} u_t + u u_x. \tag{8}$$

Consider zero background case, i.e. $U_N(x, t, \varepsilon) \equiv 0$.

The phase function $\varphi(t)$ satisfies the differential equation

$$(\varphi^2 + 1)^{5/2} \frac{d\varphi}{dt} = \gamma, \quad \gamma \in \mathbf{R}. \tag{9}$$

Lemma

For any positive γ the Cauchy problem for differential equation (9) under initial condition $\varphi(0) = 0$ has solution which is implicitly given by formula

$$\varphi \sqrt{\varphi^2 + 1} \left(8\varphi^4 + 26\varphi^2 + 33\right) + 15 \ln |\varphi + \sqrt{\varphi^2 + 1}| = 48\gamma t. \tag{10}$$

This solution is defined for all $t \in \mathbf{R}$.

Example 2: KdV, $n = 1$

The main term of the singular part

$$V_0(x,t,\tau) = v_0(t,\tau) = -\frac{3\cosh^{-2}\vartheta(t,\tau)}{\varphi^2+1}, \tag{11}$$

where

$$\vartheta(t,\tau) = \frac{\sqrt{\gamma}\,\tau}{2\sqrt{\varphi^2+1}}, \quad \tau = \frac{x-\varphi}{\sqrt{\varepsilon}}, \quad \varphi = \varphi(t), \quad (t,\tau) \in \mathbf{R}^2.$$

To find the function $v_1(t,\tau)$ we calculate

$$\Phi_1(t,\tau) = \frac{12\varphi}{(\varphi^2+1)^3}\left[\sqrt{\varphi^2+1}\,(\tanh\vartheta(t,\tau)-1) - \tau\cosh^{-2}\vartheta(t,\tau)\right].$$

The function $\Phi_1(t,\tau) \notin G_0$, but $\Phi_1(t,\tau) \in G$.

Example 2: KdV, $n = 1$

The first singular term on the discontinuity curve

$$v_1(t,\tau) = \frac{3\varphi}{(\varphi^2+1)^2}\Bigg[\Bigg[\left(36 - 10\sqrt{\varphi^2+1}\right)\tau \;+ \Big((20\tau+12)\sqrt{\varphi^2+1} -$$

$$-10\tau\Big) \times \cosh^{-2}\vartheta(t,\tau) - \left(30 + 10\sqrt{\varphi^2+1}\right)\tau\cosh^{-4}\vartheta(t,\tau) -$$

$$-\Bigg(5\sqrt{\varphi^2+1} + \frac{1}{\sqrt{\varphi^2+1}} - 35\sqrt{\varphi^2+1}\cosh^{-2}\vartheta(t,\tau) +$$

$$+\frac{105\,\tau^2}{4\sqrt{\varphi^2+1}}\cosh^{-4}\vartheta(t,\tau) + 140\sqrt{\varphi^2+1}\;\ln\cosh\vartheta(t,\tau) - 3\tau\Bigg) \times$$

$$\times\tanh\vartheta(t,\tau)\Bigg]\cosh^{-2}\vartheta(t,\tau) - 4\sqrt{\varphi^2+1}\left(\tanh\vartheta(t,\tau) - 1\right)\Bigg]. \quad (12)$$

Example 2: KdV, $n=1$

Thus, the function $v_1(t,\tau)$ is represented as $v_1(t,\tau) = \nu_1(t)\eta_1(t,\tau) + \psi_1(t,\tau)$, where

$$\nu_1(t) = -(\varphi^2+1) \lim_{\tau\to-\infty} \Phi_1(t,\tau) = 24\,\frac{\varphi\sqrt{\varphi^2+1}}{(\varphi^2+1)^2}, \tag{13}$$

$$\eta_1(t,\tau) = -\frac{1}{2}\tanh\vartheta(t,\tau) + \frac{1}{2}, \quad \psi_1(t,\tau) = v_1(t,\tau) - \nu_1(t)\eta_1(t,\tau). \tag{14}$$

To prolong the function $v_1(t,\tau)$ from the discontinuity curve Γ we solve the Cauchy problem

$$-(x^2+1)^{-3/2}\,\frac{\partial}{\partial t}\,u_1^-(x,t) = 0, \quad u_1^-(x,t)\Big|_\Gamma = \nu_1(t) \tag{15}$$

and obtain its solution as follows

$$u_1^-(x,t) = 24x(x^2+1)^{-3/2}. \tag{16}$$

The first term of the singular part is found in explicit form

$$V_1(x,t,\tau) = u_1^-(x,t)\eta_1(t,\tau) + \psi_1(t,\tau). \tag{17}$$

Example 3: vc Burgers' equation

Let us consider the Burgers equation with specified variable coefficients:

Example

$$\varepsilon u_{xx} = \left(t^2 + 1 + \varepsilon (x^2+1)^2\right) u_t + \left(1 + \varepsilon \, \frac{(x^2+1)^2}{t^2+1}\right) u u_x \qquad (18)$$

and construct the first approximation for its asymptotic solution for the case of zero background.

The first coefficients of asymptotic series in (18) are written as

$$a_0(x,t) = t^2 + 1, \quad a_1(x,t) = (x^2+1)^2, \qquad (19)$$

$$b_0(x,t) = 1, \quad b_1(x,t) = \frac{(x^2+1)^2}{t^2+1}. \qquad (20)$$

Taking into account (19), (20) and zero background conditions

$$u_0(x,t) = u_1(x,t) = 0,$$

from (25) we get equation for the function $\varphi = \varphi(t)$ in the form:

$$(t^2+1)\varphi' = \rho, \quad \rho \neq 0.$$

Solution of this equation with initial condition $\varphi(0) = 0$ is global (is defined for all $t \in \mathbb{R}$) and is written as $\varphi(t) = \rho \arctan t$.

Example 3: vc Burgers' equation

Then we find $A = A(t) = \rho$, $\beta = \rho/2$.

You can easily make sure that all conditions of Theorem 1 and Theorem 2 are fulfilled.

The main term of the singular part of the asymptotics for equation (18) is given with the following formula

$$V_0(x,t,\varepsilon) = 1 - \tanh\left(\rho\,\frac{x-\rho\arctan t}{2\varepsilon}\right), \tag{21}$$

and the first term of the singular part of the asymptotics is written as

$$V_1(x,t,\varepsilon) = \left[c - \rho^2\,\frac{\left(1+\rho^2\arctan^2 t\right)^2}{t^2+1}\,\frac{x-\rho\arctan t}{2\varepsilon}\right] \times \cosh^{-2}\left(\rho\,\frac{x-\rho\arctan t}{2\varepsilon}\right). \tag{22}$$

It is easy to see that $V_0(x,t,\varepsilon) \in G$, $V_1(x,t,\varepsilon) \in G_0$.

Example 3: vc Burgers' equation

The function

$$Y_1(x,t,\varepsilon) = 1 - \tanh\left(\frac{x-\rho\arctan t}{2\varepsilon}\right) + \varepsilon\times$$
$$\left[c - \rho^2\,\frac{\left(1+\rho^2\arctan^2 t\right)^2}{t^2+1}\,\frac{x-\rho\arctan t}{2\varepsilon}\right]\times\cosh^{-2}\left(\rho\,\frac{x-\rho\arctan t}{2\varepsilon}\right) \quad (23)$$

is the first asymptotic step-like approximation for solution of equation (18).

According to Theorem 2 the constructed asymptotic solution satisfies the equation with an asymptotical accuracy $O(\varepsilon)$.

Moreover, the first approximation satisfies the equation with an accuracy $O(\varepsilon^2)$ as $\tau \to \pm\infty$.

Graphs of functions (21), (22) and (23) are given for values $\rho = 1$, $c = 0$, $\varepsilon = 0,9$, $\varepsilon = 0,45$ and $\varepsilon = 0,15$.

Figures as $\varepsilon = 0.9$ and $\varepsilon = 0.45$

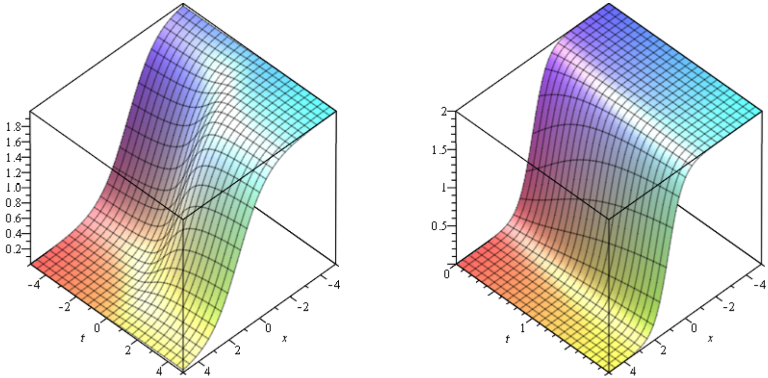


Fig. 1: The main term of the asymptotic solution $V_0(x, t, \varepsilon)$ as $\varepsilon = 0.9$ (at the left), and $\varepsilon = 0.45$ (at the right)

Figures as $\varepsilon = 0.9$ and $\varepsilon = 0.15$

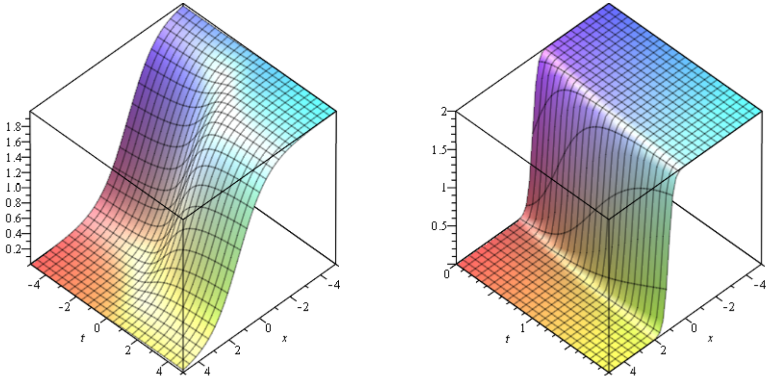


Fig. 2: The main term of the asymptotic solution $V_0(x, t, \varepsilon)$ as $\varepsilon = 0.9$ (at the left) and $\varepsilon = 0.15$ (at the right).

Figures as $\varepsilon = 0.9$ and $\varepsilon = 0.45$

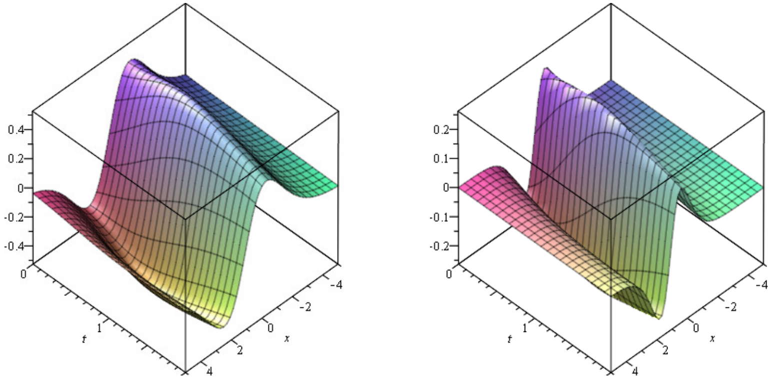
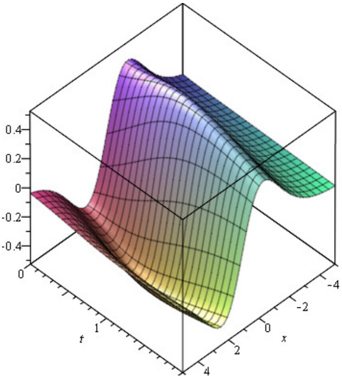


Fig. 3: The first term $V_1(x, t, \varepsilon)$ of asymptotic solution as $\varepsilon = 0.9$ (at the left) and $\varepsilon = 0.45$ (at the right).

Figures as $\varepsilon = 0.9$ and $\varepsilon = 0.15$



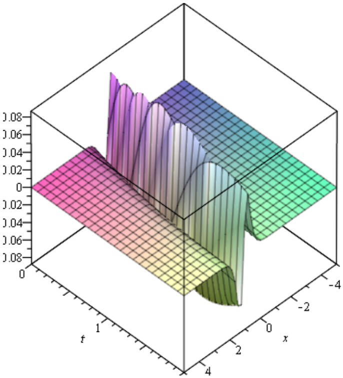


Fig. 4: The first term $V_1(x, t, \varepsilon)$ of asymptotic solution as $\varepsilon = 0.9$ (at the left) and $\varepsilon = 0.15$ (at the right).

Thank you very much
for your attention !
Have a nice day !