Why we have to like SYMMETRY

Kyiv-2025

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Recollections

The recollections are the big privilege of old people, and let me start with them.

I was educated as a physicist with the speciality ”Theory of Elementary Particles” in Taras Shevchenko University. Among my teachers there was a number of outstanding scientists. In particular, the special point of quantum mechanics called ”The Dirac picture” was presented for me by three Academicians: Pecar, Parasiuk and Sitenko.

It was a good time for the theoretical physics. The specialists in particle physics were requested in many places, in particular, in the new organised Bogoliubov Institute for Theoretical Physics.

The study of the elementary particle physics was headed by Ostap Stepanovych Parasiuk who was a big expert in many fields of mathematics and mathematical physics. This fact is well known, but I have the special proof for it. Once Izrael Moiseevych Gelfand asked me about my teachers. When I mentioned Parasiuk among them, Gelfands reaction was immediate: ”Owe, I know him very well, and in any case when I mit him I obtain a very interesting and absolutely new for me information about some field of mathematics!”

Ostap Stepanovych was very and very kind person. Being the Academic Secretary of the Physical Department of the Academy of Science he called us (his students!) to visit the department and discuss with him the Physics because the administrative work is very boring! It had introduced me to Wilhelm Illich Fushchich who became my chief for many and many years. In addition to the four books we published more than forty research papers.

After the University I was supposed to make the military service for two years. During this service I have prepared my the first journal publication. It is a pity that I cannot represent you the the reaction of my commander when I present him the paper preprint! Yes, it was approving, but absolutely obscene!

Since 1971 till now I am employed in the Institute of Mathematics of the National Academy of Sciences of Ukraine. I will not comment this long and nice period, restricting myself to the following statement: I was absolutely happy with my collaborators, and this rule has only one exception. In particular I had a rather useful and fruitful collaboration with foreign scientists, the number of the states visited by me is equal to sixteen, and in many cases they were multiple visits.

It was my pleasure to be the editor of journal SIGMA created by my young collaborators Viacheslav Boyko and Alexander Zhalij.

SYMMETRY was the main subject of my research. My the first contact with symmetries in mathematical physics - the Landay and Livshyts book ”Mechanics”, where you can find the derivation of Newton equations using the Lagrangian formalism, but the lagrangian is discovered using the hypothesis of its invariance w.r.t. the Euclide group. A bit later I was happy to derive the Maxwell and some other relativistic equations using the only conditions: relativistic invariance and the first order in derivations, refer to the Fushchich and myself book ”Symmetries of Maxwell equations”.

Let me remind that the creation of the special relativity in fact based on the symmetries of Maxwell equations. It was Lorentz who had discovered these symmetries. Then Poincare had corrected the invariance transformations of these equations and named them ”Lorentz transformations”. Notice that in fact the Lorentz transformations were discovered before the Lorentz by Prof. Voigt who works in the same University as Lorentz! And it has been done making the search of symmetries of coupled systems of wave equations.

Symmetry has sometimes mysterious effectiveness in mathematical physics. A perfect example of such mystery is the Dirac equation which was derived using the following supposition: to be the first order in derivatives and to be consistent with Klein-Gordon equation. This equation looks as follows:

$$(\gamma^\mu p_\mu - m)\Psi = 0 \tag{1}$$

where Ψ is a four component function, $p_\mu = -\mathrm{i}\frac{\partial}{\partial x_\mu}$, and summation is imposed with respect the repeating indices μ over the values $0, 1, 2, 3$. The symbols γ^μ denote the 4×4 matrices satisfying the the following algebra:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \tag{2}$$

$g^{00} = -g^{11} = - - g^{22} = -g^{33} = 1$, the remaining components are zero.

Equation (1) describes the free relativistic particle with spin 1/2 (say, electron). The mystery consists in the following fact: whenever we change in (1) $p_\mu \to p_\mu - eA_\mu$ where A_μ is the vector potential of the electromagnetic field and e is the electric charge, this equation perfectly describes the movement of the charged particle in an external electromagnetic field. In particular, it describes the so called Pauli, spin-orbit and Darwin interactions and predict absolutely correct coupling constants for these interactions. Notice that in the nonrelativistic quantum mechanics these (rather small) constants were introduced ”by hands”, and it was the reason to interpret the mentioned interactions as a relativistic effects.

This point has been overlooked after the paper of Levi Leblond who presented the nonrelativistic (but Galilei invariant) equation which keeps some properties of the Dirac one and also predicts the correct form of the Pauli interaction. However, the spin-orbit and Darwin couplings continued be interpreted as purely relativistic ones.

It was a challenge for Wilhelm Fushchich and me to verify if it is do the correct interpretation. One more challenge is that physicists prove the existence of elementary particles whose spin is more large than 1/2. I was an old business to deduce motion equations for such particles, however it happens that all they had the principal defect since in spite of their relativistic invariance predict the particle motion with the velocity higher than the velocity of light.

Just the search for the relativistic wave equations was the the subject of our the first paper with Wilhelm Illich and his the first postgraduate student Anatoly Grishchenko, refer to

On relativistic equations of motion without"redundant" components VI Fushchich, AL Grishchenko, AG Nikitin Theoretical and Mathematical Physics 8 (2), 766-775 (1971)

Surely I will not discuss the paper content but let me present the nice relativistic invariance condition for the Hamiltonian H of particle with arbitrary spin:

$$[[H, x_i], [H, x_j]] = 4S_{ij}$$

where S_{ij} are the matrices realising irreducible representation D(s) of the rotation group.

A good version of the motion equation for particle of spin 3/2 was created a bit later and published in

Relativistic wave equations for interacting, massive particles with arbitrary half-integer spins J Niederle, AG Nikitin Physical Review D 64 (12), 125013 (2001);

The relativistic Coulomb problem for particles with arbitrary half-integer spin J Niederle, AG Nikitin Journal of Physics A: Mathematical and General 39, 10931 (2006)

The next challenge were Galilei invariant wave equations. We classify an extended class of such equations which look like the Dirac one but instead of Dirac matrices include another ones. Like the Dirac equation, they give the correct description of the spin-orbit and Darwin couplings and this fact means that these couplings are not purely relativistic effects and are compatible with the Galilei symmetry. The Galilei invariant equations are represented in our books, see also

Galilei invariant theories: I. Constructions of indecomposable finite-dimensional representations of the homogeneous Galilei group: directly and via contractions M De Montigny, J Niederle, AG Nikitin Journal of Physics A: Mathematical and General 39, 9365 (2006)

The next research field which I exploited were symmetries of partial differential equations. Since practically all my collaborators deal with the Lie symmetries, I was supposed to do something in this area. In particular, I had classified symmetries of systems of coupled quasi linear diffusion equations. It was a very hard job, since the number of such systems with inequivalent symmetries appears to be very large (around 300). I have published ten papers devoted to this subject, which collect more than 500 citations, thus this classification was hard but useful business.

My favorite paper devoted to diffusion equations was written in collaboration with Academician Anton Naumovets. One day Anatoliy Michailovych Samoilenko asked me to visit the Naumovets office since the latter would like to contact with me. Surely I went as soon as possible. Anton Grygorovych sad me that needs a help with creation of the model equations for some diffusion systems which were studied experimentally in his department in the Institute of Physics. When I asked him why he chose exactly me he explain that in accordance with the academic reports of Institute of Mathematics I can be treated as an expert in the requested field.

I was very surprised and set a crazy question: You would like to state that somebody reds these useless documents? The Anton Grygorovych reaction was very violent: ”It is me who do this job, and do very diligrntly!” Let me omit the details of the discussion which followed.

Everything which I obtained to create the requested model equation where the very extended tables with experimental data. Nevertheless, thanks to numerous discussions some of which were realised in the experimental laboratory we created a sufficiently good model. Moreover, we present the effective method for constructing such models which was called ERFEX (error function expansions) method. It was done with the kind help of Stanislav Spichak and the experimentalist Prof. Vedula. The exact reference is

Symmetries and modelling functions for diffusion processes AG Nikitin, SV Spichak, YS Vedula, AG Naumovets Journal of Physics D: Applied Physics 42, 055301

It looks like that the ERFEX method was successfully used by other researchers, at least it had been declared by some of them. Let me inform you that the main streem of this methods is the multiple use of Lie symmetries for optimisation of the choice of dependent and independent variables.

It happens that much more efforts had been pay by myself for investigation of so called higher symmetries. A perfect example of such symmetry is the Fock hidden symmetry of the Hydrogen atom. This physical system is described by the Schrödinger equation including the Coulomb potential:

$$H\Psi = \left(P_1^2 + P_2^2 + P_3^2 + \frac{\alpha}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \right) \Psi = E\Psi$$

where $P_a = -\mathrm{i}\frac{\partial}{\partial x_a}$.

The presented equation has the transparent symmetry with respect to the rotation group. In addition, Hamiltonian H commutes with the second order differential operators

$$R_a = L_{ab}P_b + \frac{\alpha x_a}{x_1^2 + x_2^2 + x_3^2} \tag{3}$$

where $L_{ab} = x_a P_b - x_b P_a$ are the orbital momenta operators.

Equation (3) represents the Laplace=Runge-Lentz vector. It happens that operators L_{ab} and R_a form algebra o(4) on the set of solutions of the considered Schrödinger equation.

I was laky to find new quantum mechanical systems admitting generalized LRL vectors. In contrast with the Hydrogen atom these systems describe particles with non-trivial spin. Among these systems is a relativistic one, described by Dirac equation. All these systems are three dimensional And it was my pleasure that Christian Quesne had verified my results and generalised them to the case of arbitrary dimensional spaces, refer to

Quesne, C. (2025). Algebraic approach to ad-dimensional matrix Hamiltonian with so (d+ 1) symmetry. Journal of Physics A: Mathematical and Theoretical.(2025);

New exactly solvable systems with Fock symmetry AG Nikitin Journal of Physics A: Mathematical and Theoretical 45 (48), 485204 (2012)

I have no time to represent you my other results, which are devoted to supersymmetries, parasupersymmetries , and so on. But let me mention just the results obtained during the war.They were directed to Hamiltonian systems some of which have the following global properties

Exact solvability: all energy levels can be calculated algebraically; and the corresponding wavefunctions are polynomials multiplied by an overall gauge factor.

Integrability: the existence of $n-1$ integrals of motion commuting with the Hamiltonian and amongst each other.

Superintegrability: more integrals of motion than degrees of freedom.

Maximal superintegrability: $2n-1$ integrals of motion, including the Hamiltonian. Only n of them can (and have to) commute amongst them.

QM systems and integrals of motion

The general problem: to classify integrable systems with position dependent mass.

Hamiltonian with position dependent mass:

$$\hat{H} = p_a f(\mathbf{x}) p_a - V(\mathbf{x}) = -\partial_a f(\mathbf{x}) \partial_a - V(\mathbf{x}). \qquad (4)$$

Here $\mathbf{x} = (x^1, x^2, x^3)$, $p_a = -i\partial_a$, $V(\mathbf{x})$ and $f(\mathbf{x}) = \frac{1}{m(\mathbf{x})}$ are arbitrary functions associated with the effective potential and inverse effective PDM, and summation from 1 to 3 is imposed over the repeating index a.

QM systems with position dependent mass

SE with position dependent mass are requested for description of various condensed-matter systems such as:

semiconductors,

quantum liquids and metal clusters,

quantum wells, wires and dots,

supper-lattice band structures,

and many, many others.

Surely it is interesting to search for symmetries of these systems. In addition to their physical consistence it is an interesting mathematical problem which has the direct relation to the classification of inequivalent curved spaces. However for a long time even their Lie symmetries were unknown, and there where known only some particular results connected to higher symmetries. The completed classification results for the second order integrals of motion were known for two dimensional space only. For the 3d case we had the classification of maximally superintegrable systems which form only small subclass of systems admitting second order integrals of motion. And it was a challenge for me to solve the problem of classify complete classification of the 3d systems admitting such integrals. A subproblem of this problem is the classification of Lie symmetries which are requested for description of equivalence relations.

Second order integrals of motion can be represented as follows:

$$Q = \partial_a \mu^{ab} \partial_b + \eta \tag{5}$$

where $\mu^{ab} = \mu^{ba}$, ξ^a and η are functions of $\mathbf{x}$. Equating to zero the commutator of (5) with the Hamiltonian we come to the following determining equations:

$$\begin{aligned} 5\left(\mu_c^{ab} + \mu_b^{ac} + \mu_a^{bc}\right) = \delta^{ab}\left(\mu_c^{nn} + 2\mu_n^{cn}\right) \\ + \delta^{bc}\left(\mu_a^{nn} + 2\mu_n^{an}\right) + \delta^{ac}\left(\mu_b^{nn} + 2\mu_n^{bn}\right), \end{aligned} \tag{6}$$

(conformal Killing tensor), and

$$\left(\mu_a^{nn} + 2\mu_n^{na}\right) f - 5\mu^{an} f_n = 0, \tag{7}$$

$$\mu^{ab} V_b - f\eta_a = 0 \tag{8}$$

where again $V_a = \frac{\partial V}{\partial x_a}$ etc.

A particular solution of equations for the conformal Killing tensor is $\mu^{ab} = \mu_0^{ab}$ where

$$\mu_0^{ab} = \delta^{ab} g(\mathbf{r}) \qquad (9)$$

with arbitrary function $g(\mathbf{r})$.

Whenever tensor μ_0^{ab} is nontrivial, the determining equations (6) and (7) represent the coupled system of three *nonlinear* partial differential equation equations for two unknowns $g(\mathbf{x})$ and $f(\mathbf{x})$.

Fortunately, this system can be linearizing by introduction of the new dependent variables

$$M = \frac{1}{f}, \; N = \frac{g}{f} \tag{10}$$

which reduces (6) to the following form:

$$(\mu_a^{nn} + 2\mu_n^{na})\, M + 5(\mu^{an} M_n + N_a) = 0. \tag{11}$$

Liner but in fact very complicated since the conformal Killing tensor includes 35 arbitrary parameters.

Generic equivalence relations: 3d conformal group + the discrete transformation of total inversion + ”the coupling constants isomorphism”

The 3d conformal group is locally isomorphic to the Lorentz group in (1+4) dimensional space. The coupling constants isomorphism is a very specific symmetry which I will present you.

Consider the PDM Schrödinger equation

$$H\Psi = E\Psi \tag{12}$$

where

$$H = -\frac{1}{\sqrt{M(x)}}(\partial_1^2 + \partial_2^2 + \partial_3^2)\frac{1}{\sqrt{M(x)}} + \alpha \cdot V. \tag{13}$$

where V is potential and α is coupling constant.

Multiplying the hamiltonian from the left by $1/\sqrt{V}$ and wave function Ψ by V we transform equation (12) to the following form:

$$\hat{H}\tilde{\Psi} = -\alpha\tilde{\Psi} \tag{14}$$

where

$$\hat{H} = -\frac{1}{\sqrt{M(x)V}}(\partial_1^2 + \partial_2^2 + \partial_3^2)\frac{1}{\sqrt{M(x)V}} - \frac{E}{V}. \qquad (15)$$

Surely equations (12) and (14) are equivalent. And there are very interesting games with this equivalence.
(Examples)
Notice that the eigenvalue E and coupling constant α change their roles.
The considered classification problem is very and very complicated. In particular it includes the subproblem: to find all inequivalent enveloping algebras of algebra so(1,4) formed by second order polynomials in generators of this algebra. No immediate direct solution, but it is possible to search for it step by step.

The strategy: to separate the problem to special subproblems which can be solved and have their own interest. To do it we can suppose that the PDM system admits some Lie symmetry.

It was shown by Zasadko and myself that the PDM Schroedinger equation can admit six, four, three, two or one parametric Lie symmetry groups. In addition, there are also such equations which have no Lie symmetry.

In other words, there are six well defined classes of such equations which admit n-parametric Lie groups with $n = 6, 4, 3, 2, 1$ or do not have any Lie symmetry. And it is a natural idea to search for second order integrals of motion consequently for all these classes.

The cases $n = 6$ and 4 are rather simple since the related PDM systems do not include arbitrary elements.

The case n=3 include three inequivalent possibilities: the symmetry groups are the rotation group O(3), the Lorentz group O(1,2) or Euclide group E(2). In all these the related determining equations are added by additional constrains which make them solvable.

Superintegrable quantum mechanical systems with position dependent masses invariant with respect to three parametric Lie groups AG Nikitin Journal of Mathematical Physics 64 (11) 2023)

The systems invariant with respect to rotations were classified by me before the war. They include:

Ten systems admitting vector second order integrals of motion;

ten systems with tensor integrals of motion.

All these systems are maximally superintegrable, supersymmetric and exactly solvable.

Two-fold shape invariance.

The systems admitting two parametric Lie groups have been classified in paper

Journal of Physics A: Mathematical and Theoretical 56 (39), 395203 (2023). Up to equivalence there are six such groups and for any of them the related determining equations are exactly solvable.

The next and the most important step is to classify the PDM systems admitting second order integrals of motion and one parametric Lie groups. Again there are six of such groups including up to equivalence these groups are reduced to dilatations, shifts along the fixed coordinate axis, rotations around this axis and some specific combinations of the mentioned transformations. We will conventionally call them the natural and exotic symmetries respectively.

The systems admitting the natural symmetries had been classified in my papers published in

*Journal of Physics A: Mathematical and Theoretical 57 (2), 265201 (2024)*and

Journal of Physics A: Mathematical and Theoretical, Volume 58 58 (14), 145201 (2025)

The systems invariant with respect to the exotic one parametric Lie groups have been classified also. I have prepared the paper for publication but will present it next year.

The final step is to classify the systems which do not have Lie symmetry. Thanks to more strong equivalence group this problems also has good chances to be completely solved, but now I know only many examples whose list is not complete jet. However, we again can fix the completed classes of equations admitting *DISCRETE*symmetries.

Very high order integrals of motion for standard Schrödinger equation

The problem: to classify Schrödinder equations admitting second order integrals of motion.

A. Makarov, J. Smorodinsky, Kh. Valiev and P. Winternitz, Nuovo Cim. A **52**, 1061-1084 (1967);

N. Evans, J. Math. Phys. **32**, 3369 (1991).

P. Winternitz and L. Yurdusen, J.Math.Phys., **47**, 103509 (2006),

A. G. Nikitin, J. Math. Phys. 54, 123506 (2013); J. Math. Phys. 53, 122103 (2012)

The modern trend: the systems with third-, fourth-, and even arbitrary order integrability. Let us make small support for this problems.
Consider the generic 2d Hamltonian

$$H = P_1^2 + P_2^2 + V. \qquad (16)$$

where $P_a = -\mathrm{i}\frac{\partial}{\partial_a}$, $V = V(x_1, x_2)$, $a = 1, 2$.
As it was shown already by Smorodinsky and Winternitz Hamiltonian (16) commutes with the following second order differential operators

$$Q_1 = L_3^2 + \frac{\mu}{sin(\varphi)^2} + \frac{\nu}{cos(\varphi)^2}, \qquad (17)$$

$$Q_2 = P_1^2 - P_2^2 - \frac{\mu}{x_1^2} + \frac{\nu}{x_2^2} \qquad (18)$$

provided the potential V has the following form:

$$V = \omega r^2 + \frac{\mu}{x_1^2} + \frac{\nu}{x_2^2} \qquad (19)$$

A very interesting generalization of (18) is the Trembly-Turbiner-Winternitz potential which looks as follows

$$V = \omega r^2 + \frac{\mu k^2}{r^2 \sin(k\varphi)^2} + \frac{\nu k^2}{r^2 \cos(k\varphi)^2} \tag{20}$$

where k is integer, k and $\omega > 0$, and $\mu, \nu > \frac{1}{4k}$.

There are many reasons to say that this potential is interesting. In particular it is requested by some realistic physical systems. But for as the main point is that it has numerous transparent and also hidden symmetries.
It had been shown by Trembly, Turbiner and Winternitz that the Schrödinger equation with such potential admits integrals of motion which are differential operators of order 2k.

Such integrals were presented in paper *F. Tremblay, A.V. Turbiner and P. Winternitz, An infinite family of solvable and integrable quantum systems on a plane, Journal of Phys. A42 (2009) 242001* in explicit form but only for the cases $n = 2, 3, 4$ without any prove they do commute with the mentioned Hamiltonian. In particular for k=2 this integral looks as follows:

$$
\begin{aligned}
Q &= (P_1^2 - P_2^2 - \omega^2(x_1^2 - x_2^2))^2 \\
&+ \mu \left\{ \frac{(x_1^2 - x_2^2)}{x_1^2 x_2^2}, (P_1^2 - P_2^2) \right\} \\
&- 4\nu \left\{ \frac{(x_1^2 + x_2^2)}{(x_1^2 - x_2^2)}, (P_1^2 + P_2^2) \right\} - 16\nu \left\{ \frac{x_1 x_2}{(x_1^2 - x_2^2)^2}, P_1 P_2 \right\} \\
&- \frac{2\mu\omega^2(x_1^4 + x_2^4)}{x_1^2 x_2^2} + \frac{16\nu^2}{(x_1^2 - x_2^2)^2} + \frac{\mu^2(x_1^2 - x_2^2)}{x_1^4 x_2^4} + \frac{8\mu\nu}{x_1^2 x_2^2}.
\end{aligned}
\tag{21}
$$

A bit cumbersome expression. In the cases $k = 4$ and $k = 4$ the related expression are absolutely huge and request seven journal pages.

It is hard to believe that such nice models generate such un nice symmetries. It is hard to believe also that at least one of researchers cited the TTW paper verified the commutativity of the presented integrals with the Hamiltonian.

Since I have some experience in calculating the higher symmetries I decide to make such verification. The determining equations for such symmetries can be easily obtained by calculating the potential integral of motion with the Hamiltonian, and they were presented already in our book.

The generic form of the fourth order integral of motion:

$$Q = \{\{\{\{\mu^{abcd}, P_a,\}P_b,\}P_c,\}P_d\} + \{\{\mu^{ab}, P_a,\}, P_b\} + F$$

where μ^{abcd}, μ^{ab} and F are unknown functions.

No odd terms! (Theorem)
The determining equations look as follows:

$$\mu_k^{abcd} + \mu_d^{kabc} + \mu_c^{dkab} + \mu_b^{cdka} + \mu_a^{bcdk} = 0, \qquad (22)$$

$$\begin{aligned}
&4\mu^{1111}V_1 + \mu^{1112}V_2 - 2\nu_1^{11} = 0,\\
&4\mu^{2222}V_2 + \mu^{1222}V_1 - 2\nu_2^{22} = 0,\\
&3\mu^{1112}V_1 + 2\mu^{1122}V_2 - 2\nu_2^{11} - 2\nu_1^{12} = 0,\\
&3\mu^{1222}V_2 + 2\mu^{1122}V_1 - 2\nu_1^{22} - 2\nu_2^{12} = 0,
\end{aligned} \qquad (23)$$

$$\begin{aligned}
&V_1 - \mu_2^{11} - \mu_1^{12} = 0,\\
&V_1 - \mu_1^{22} - \mu_2^{11} = 0,\\
&2\mu^{11}V_1 + \mu^{12}V_2 - 2F_1 = 0,\\
&\mu^{12}V_1 + 2\mu^{22}V_1 - 2F_2 = 0
\end{aligned} \qquad (24)$$

The system is rather large but overdetermined, and whenever the potential V is fixed it is relatively easy integrated. For the TTW potential these solutions are:

$$\mu^{a12b} = 1, \ \mu^{11} = \frac{\mu}{x_2^2}, \ \mu^{22} = \frac{\mu}{x_1^2}, \ \mu^{12} = -\frac{8\nu x_1 x_2}{(x_2^2 - x_2^2)^2}, \\ F = (\omega x_1 x_2)^2 - \frac{8\nu\omega}{(x_2^2 - x_2^2)^2} + \frac{\nu^2}{x_1^2 x_2^2} - 16\frac{\mu^2 x_1^2 x_2^2}{(x_1^2 - x_2^2)^4}. \tag{25}$$

and the related integral of motion looks as:

$$\hat{Q} = (P_1^2 - \omega x_1^2 + \frac{\mu}{x_1^2})(P_2^2 - \omega x_2^2 + \frac{\mu}{x_2^2}) \\ +(P_1 P_2 - \omega x_1 x_2 - 4\sin(2\varphi)V)^2$$

Much more simple and transparent than in (21). We have a polynomial in the integrals of motion for the Smorodinsky-Winternitz system added by the last term. The relation with (21)

$$\hat{Q} = \frac{1}{4}(Q - H^2)$$

A system with two-fold shape invariance

Consider the following hamiltonian

$$H = \frac{1}{x} p^2 \frac{1}{x} + \frac{\alpha}{x^2}$$

Radial equation

$$-\frac{1}{x^2}\frac{\partial^2 \phi_{lm}}{\partial x^2} + \left(\frac{l(l+1)}{x^4} + \frac{\alpha}{x^2}\right)\phi_{lm} = E\phi_{lm}.$$

A system with two-fold shape invariance

Introducing new independent variable $y = \frac{x^2}{2}$ we obtain

$$\mathcal{H}_\mu \Phi_{\mu m} \equiv \left(-\frac{\partial^2}{\partial y^2} + \frac{\mu(\mu+1)}{y^2} + \frac{\nu^2}{y} \right) \Phi_{\mu m} = E \Phi_{\mu m}$$

where

$$\mu = \frac{l}{2} - \frac{1}{4}, \quad l = 0, 1, 2, \ldots.$$

Up to the meaning of μ – the radial equation for the Hydrogen atom (HA)

A system with two-fold shape invariance

Hamiltonian $\mathcal{H}_\mu$ can be factorized:

$$\mathcal{H}_\mu = a_\mu^+ a_\mu^- + c_\mu$$

where

$$a_\mu^- = \frac{\partial}{\partial y} + \quad a_\mu^+ = -\frac{\partial}{\partial y} + W_\mu, \quad c_\mu = \frac{\nu^4}{4(\mu+1)^2},$$

and $W_\mu = \frac{\nu^2}{2(\mu+1)} - \frac{\mu+1}{y}$ is a superpotential.

A system with two-fold shape invariance

Hamiltonian $\mathcal{H}_\mu$ is shape invariant since
$\mathcal{H}_\mu^+ = a_\mu^- a_\mu^+ = \mathcal{H}_{\mu+1} + c_\mu - c_{\mu+1}$.
The ground state Φ_{lm}^0 solves the first order equation $a_\mu^- \Phi_{lm}^0 = 0$ and is given by the following formula:

$$\Phi_\mu^0 = C_0 y^{\mu+1} e^{-\frac{\nu^2 y}{(\mu+1)}}$$

The n^{th} exited state and the corresponding eigenvalue E_n are

$$\Phi_\mu^n = a_\mu^+ a_{\mu+1}^+ ... a_{\mu+n-1}^+ \Phi_{\mu+n}^0,$$

$$E_n = -\frac{\nu^4}{4(\mu+n+1)^2} = -\frac{\alpha^2}{(4n+2l+3)^2}$$

A system with two-fold shape invariance

Alternatively, the considered radial equation

$$-\frac{1}{x^2}\frac{\partial^2\phi_{lm}}{\partial x^2}+\left(\frac{l(l+1)}{x^4}+\frac{\alpha}{x^2}\right)\phi_{lm}=E\phi_{lm}$$

can be solved using the following trick.
Multiplying it by x^2 we come to the following equation:

$$\mathcal{H}_l\phi_{lm}\equiv\left(-\frac{\partial^2}{\partial x^2}+\frac{l(l+1)}{x^2}+\frac{\omega^2}{4}x^2\right)\phi_{lm}=\mathcal{E}\phi_{lm}$$

where we denote $-E=\frac{\omega^2}{4}$ and $-\alpha=\mathcal{E}$.

A system with two-fold shape invariance

The obtained equation is also shape invariant, but needs ANOTHER SUPERPOTENTIAL

$$W = \frac{\omega x}{2} - \frac{l(l+1)}{x}$$

Eigenvalues:

$$\mathcal{E} = \omega(2n + l + 3/2)$$

also can be found algebraically and are in perfect accordance with the previous results.

Systems with two arbitrary parameters

Hamiltonian

$$H = \frac{(x^4+1)^2}{x^4-2\kappa x^2-1}p^2 + \frac{\alpha x^2}{x^4-2\kappa x^2-1}$$

and radial equations:

$$\left(-\frac{(x^4+1)^2}{x^4-2\kappa x^2-1}\left(\frac{\partial^2}{\partial x^2}-\frac{l(l+1)}{x^2}\right)+\frac{\alpha x^2}{x^4-2\kappa x^2-1}\right)\phi_{lm} = E\phi_{lm}.$$

Eigenvalues:

$$E_n = (2l+3+4n)^2\left(\kappa - \sqrt{\kappa^2+1+\frac{\alpha-4}{(2l+3+4n)^2}}\right).$$

We classify PDM Schrödinger equations which admit first and second order integrals of motion.

All rotationally invariant systems admitting second order integrals of motion are both superintegrable, supersymmetric and exactly solvable.

The phenomenon of the two-fold shape invariance is indicated.

The incompleteness of the Boyer classification of linear Schrödinger equations is proved.

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Consider a multidimensional hamiltonian:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V} \tag{26}$$

where $\hat{p}^2 = p_1^2 + p_2^2 + ... + p_d^2$. Suppose H is invariant with respect to the rotation group in d dimensions whose generators have the following standard form:

$$J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu}$$

where $S_{\mu\nu}$ are matrices satisfying the familiar $so(d)$ commutation relations

$$[S_{\mu\nu}, S_{\lambda\sigma}] = \mathrm{i}(\delta_{\mu\lambda}S_{\nu\sigma} + \delta_{\nu\sigma}S_{\mu\lambda} - \delta_{\mu\sigma}S_{\nu\lambda} - \delta_{\nu\lambda}S_{\mu\sigma}) \tag{27}$$

By definition H commutes with $J_{\mu\nu}$, then

$$[V, J_{\mu\nu}] = 0. \quad (28)$$

We ask for additional integrals of motion $K_\mu,\ \mu = 1, 2, ..., d$ of the following generic form:

$$K_\mu = \frac{1}{2m}(p_\nu J_{\mu\nu} + J_{\mu\nu} p_\nu) + x_\mu V. \quad (29)$$

K_μ commutes with H iff

$$x_\nu \nabla_\nu V + V = 0, \quad (30)$$

$$S_{\mu\nu} \nabla_\nu V + \nabla_\nu V S_{\mu\nu} = 0 \quad (31)$$

where $\nabla_\nu = \frac{\partial}{\partial x_\nu}$ and summation from 1 to d is imposed over the repeating index ν.

If conditions (28), (30) and (31) are fulfilled then operators $J_{\mu\nu}$ and K_μ satisfy the following relations:

$$[J_{\mu\nu}, H] = [K_\mu, H] = 0, \tag{32}$$

$$\begin{aligned} [K_\mu, J_{\nu\lambda}] &= i(\delta_{\mu\lambda} K_\nu - \delta_{\mu\nu} \hat{K}_\lambda), \\ [K_\mu, K_\nu] &= -\frac{2\mathrm{i}}{m} J_{\mu\nu} H, \end{aligned} \tag{33}$$

$$[J_{\mu\nu}, J_{\lambda\sigma}] = \mathrm{i}(\delta_{\mu\lambda} J_{\nu\sigma} + \delta_{\nu\sigma} J_{\mu\lambda} - \delta_{\mu\sigma} J_{\nu\lambda} - \delta_{\nu\lambda} J_{\mu\sigma}). \tag{34}$$

Hidden symmetry w.r.t. SO(d+1) for bound states.

Scalar systems

Let matrices $S_{\mu\nu}$ are trivial. The corresponding potential should satisfy:

$$[\hat{V}, L_{\mu\nu}] = 0,$$
$$x_\nu \nabla_\nu \hat{V} = -\hat{V}$$

where where $L_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$. The general solution:

$$V = -\frac{\alpha}{r} \tag{35}$$

where α is a constant.

The corresponding Schrödinger equation is superintegrable and admits a d-dimensional analogue of the LRL vector:

Thus we recover the known result of Sudarshan, Mukunda and ORaifeartaigh (1965) concerning the generalization of the LRL vector in d dimensions. Moreover, we also present a formal proof that the only scalar potential which is compatible with the d-dimensional LRL vector is the d-dimensional Coulomb potential.

Suppose than when so(d)$\rightarrow$so(3) we obtain a direct sum of irreducible representations $D\left(\frac{1}{2}\right)$. This means that eigenvalues of matrices $S_{\mu\nu}$ are equal to $\pm 1/2$, and

$$S_{\mu\nu} = \frac{1}{4}\left(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu\right) \qquad (36)$$

with γ_μ satisfying

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}. \qquad (37)$$

The dimension of irreducible matrices γ_μ is equal to $2^{\left[\frac{d}{2}\right]}$ where $\left[\frac{d}{2}\right]$ is the entire part of $\frac{d}{2}$.

The general solution for potential:

$$\hat{V} = \frac{\alpha}{r}\gamma_a n_a. \tag{38}$$

Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{\alpha}{r^2}\gamma_a x_a \tag{39}$$

Eigenvalues:

$$E = -\frac{m\alpha^2}{2(n + j + \frac{d-1}{2})^2} \tag{40}$$

with $n = 0, 1, ...$ and $j = \frac{1}{2}, \frac{3}{2}, ...$

Spin one

Consider a bosonic d-dimensional system admitting generalized LRL vector with $S_{\mu\nu} \in$ IR D(1,0,0,...0) of algebra so(d), where the symbols in brackets are the Gelfand-Tsetlin numbers. Up to equivalence, their entries $(S_{\mu\nu})_{ab}$ can be represented in the following form:

$$(S_{\mu\nu})_{ab} = \mathrm{i}(\delta_{\mu a}\delta_{\nu b} - \delta_{\nu a}\delta_{\mu b}). \qquad (41)$$

Exact form of the potential:

$$V = \frac{\alpha}{(d-2)r}\left((d-1)(d-4) + 2S_{\mu\nu}n_\nu S_{\mu\lambda}n_\lambda\right),$$
$$d \neq 2. \qquad (42)$$

Using realization (41) for matrices $S_{\mu\nu}$ it is possible to find entries $V_{\mu\nu}$ of matrix potential (42) in the following form:

$$V_{\mu\nu} = \frac{\alpha}{2r}\left((d-3)\delta_{\mu\nu} + 2n_\mu n_\nu\right). \qquad (43)$$

Hamiltonian spectrum:

$$E = -\frac{m\alpha^2}{2k^2} \qquad (44)$$

where $k = \frac{2n+2l+d-1}{d-1}$.
The corresponding eigenvectors are expressed via confluent hypergeometric functions.

Conclusion

- We find all non-equivalent PDM Schrödinger equations which admit at least one first order integral of motion. Among them are superintegrble and so exactly solvable systems.
- d dimension QM systems with arbitrary spin, which admit a hidden symmetry w.r.t. group $O(d+1)$, are classified. This symmetry is generated by quantum analogues of LRL vector.

- All systems with s = 0, 1/2 for any d appear to be shape invariant. The same is true for 2d systems with arbitrary spin. Thus we can (and it has been done) solve these systems exactly using tools of SUSY QM.

- Physical interpretation for 2d and 3d systems: for spin $s = 1/2$ - a neutral particle with non-trivial dipole momentum; for $s = 1$ - neutral particle with trivial dipole, but non-trivial quadrupole momentum, in general -a system with a multipole momentum.

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