Admissible transformations and point symmetries of linear Schrödinger equations with complex-valued time-independent potentials in space dimension one

Jean de Dieu MANIRAGUHA [1], *Roman O. POPOVYCH* [2 3]

[1] *College of Science and Technology, University of Rwanda, Kigali, Rwanda*
E-mail: *jadomanir99@gmail.com*

[2] *Mathematical Institute, Silesian University in Opava, Opava, Czech Republic*

[3] *Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine*
E-mail: *rop@imath.kiev.ua*

The study of Lie symmetries for Schrödinger equations was started in the 1970s with the linear case, and it has been continued up to now for more complicated classes of linear Schrödinger equations, see [1, 2, 6] and references therein. In the paper [1], the notion of uniformly semi-normalized classes of differential equations was introduced and the algebraic method of group classification for such classes was suggested. Then this method was used for the complete group classification of the class $\mathcal F$ of (1+1)-dimensional linear Schrödinger equations with time-dependent complex-valued potentials, which are of the general form

$$i\psi_t+\psi_{xx}+V(t,x)\psi=0, \tag{1}$$

where ψ is an unknown complex-valued function of two real independent variables (t,x) and V is an arbitrary smooth complex-valued potential also depending on (t,x). The equivalence groupoid $\mathcal G^\sim_{\mathcal F}$ and the equivalence group $G^\sim_{\mathcal F}$ of the class $\mathcal F$ were computed, and it was thus shown that this class is uniformly semi-normalized with respect to the linear superposition of solutions. Hence, the group classification of $\mathcal F$ reduces to the classification of specific low-dimensional subalgebras of the associated equivalence algebra. The subclass $\mathcal F_{\mathbb R}$ of linear Schrödinger equations with real-valued potentials was considered in the same framework based on the study of the entire class $\mathcal F$. The results of [1] on semi-normalized classes of systems of differential equations were developed in [2] and then applied therein to study transformational properties of multidimensional linear Schrödinger equations with time-dependent complex-valued potentials and to completely solve the group classification problem for such equations in space dimension two.

In the present work, we consider the subclasses $\mathcal F'$ and $\mathcal F'_{\mathbb R}$ of the class $\mathcal F$ that consist of the equations of the form (1) with time-independent complex- and real-valued potentials, respectively. Based on the description of $\mathcal G^\sim_{\mathcal F}$, we construct the equivalence groups $G^\sim_{\mathcal F'}$ and $G^\sim_{\mathcal F'_{\mathbb R}}$ of the above subclasses and describe their equivalence groupoids via classifying the admissible transformations within these subclasses. We exhaustively solve the group classification problems for them up to general point equivalence and up to the equivalences generated by the corresponding equivalence groups. We also describe the point symmetry pseudogroups of the linear Schrödinger equations from the class $\mathcal F'$ that appear as $G^\sim_{\mathcal F'}$-inequivalent essential Lie-symmetry extensions in the course of group classification of this class.

Theorem 1. *(i) The equivalence group $G^\sim_{\mathcal F'}$ of the class $\mathcal F'$ consists of the point transformations in the space with the coordinates (t,x,ψ,ψ^*,V,V^*) whose (t,x,V)-components are of the form*

$$\tilde t=\lambda_1t+\lambda_0,\quad \tilde x=\varepsilon|\lambda_1|^{1/2}x+\nu,\quad \tilde\psi=\mathrm e^{i\lambda_3t+i\lambda_2+\lambda_5t+\lambda_4}\hat\psi,\quad \tilde V=\frac{\hat V}{|\lambda_1|}+\frac{\lambda_3-i\lambda_5}{\lambda_1},$$

where λ_0, …, λ_5 and ν are real constants with $\lambda_1\neq0$, and $\varepsilon:=\pm1$.

(ii) The equivalence group $G^\sim_{\mathcal F'_{\mathbb R}}$ of the class $\mathcal F'_{\mathbb R}$ is singled out from the group $G^\sim_{\mathcal F'}$ by the constraint $\lambda_5=0$.

Here and in what follows, the hat over a complex value in a transformation denotes the same value or its complex conjugate if the derivative of the t-component of this transformation or, equivalently, the constant ε' is positive or negative, respectively.

Theorem 2. *(i) A minimal self-consistent generating (up to the $G^\sim_{\mathcal F'}$-equivalence and the linear superposition of solutions) set of admissible transformations for the class $\mathcal F'$ is the union of the following families of admissible transformations $(V,\Phi,\tilde V)$:*

$$\mathcal T_{1\mu}:=\left(\frac{\mu}{x^2}-x^2,\,\Phi_1,\,\frac{\mu}{\tilde x^2}\right),\quad \Phi_1\colon\ \tilde t=\frac12\tan 2t,\ \tilde x=\frac{x}{\cos 2t},\ \tilde\psi=|\cos 2t|^{1/2}{\rm e}^{i\tan(2t)x^2/2}\psi,$$

$$\mathcal T_{2\mu}:=\left(\frac{\mu}{x^2}+x^2,\,\Phi_2,\,\frac{\mu}{\tilde x^2}\right),\quad \Phi_2\colon\ \tilde t=\frac14{\rm e}^{4t},\ \tilde x={\rm e}^{2t}x,\ \tilde\psi={\rm e}^{ix^2/2-t}\psi,$$

$$\mathcal T_{3\alpha}:=(i\alpha x+x,\,\Phi_{3\alpha},\,i\alpha\tilde x),\quad \Phi_{3\alpha}\colon\ \tilde t=t,\ \tilde x=x-t^2,\ \tilde\psi={\rm e}^{-itx+(i+\alpha)t^3/3}\psi,$$

$$\mathcal T_{4\alpha\kappa}:=\left(i\alpha x-x^2,\,\Phi_{4\alpha\kappa},\,i\alpha\tilde x-\tilde x^2\right),$$
$$\Phi_{4\alpha\kappa}\colon\ \tilde t=t,\ \tilde x=x+2\kappa\cos 2t,\ \tilde\psi={\rm e}^{\kappa\sin 2t\,(-2ix-2i\kappa\cos 2t-\alpha)}\psi,$$

$$\mathcal T_{5\alpha\kappa\nu}:=\left(i\alpha x+x^2,\,\Phi_{5\alpha\kappa\nu},\,i\alpha\tilde x+\tilde x^2\right),$$
$$\Phi_{5\alpha\kappa\nu}\colon\ \tilde t=t,\ \tilde x=x+2\kappa{\rm e}^{2t}+2\nu{\rm e}^{-2t},\ \tilde\psi={\rm e}^{(\kappa{\rm e}^{2t}-\nu{\rm e}^{-2t})(2ix+2i\kappa{\rm e}^{2t}+2i\nu{\rm e}^{-2t}-\alpha)}\psi,$$

where $\mu\in\mathbb C$ and $\alpha,\kappa,\nu\in\mathbb R$ with $\alpha\kappa\ne0$ in the fourth family, $(\alpha\kappa,\alpha\nu)\ne(0,0)$ in the fifth family, and, modulo the $G^\sim_{\mathcal F'}$-equivalence, $\operatorname{Im}\mu\geqslant0$, $\alpha\geqslant0$ in the third family, $\alpha>0$ and $\kappa>0$ in the fourth family, and $\alpha>0$ and ($\kappa>0$ or $\kappa=0$ and $\nu>0$ in the fifth family.

(ii) A minimal self-consistent generating (up to the $G^\sim(\mathcal F'_{\mathbb R})$-equivalence and the linear superposition of solutions) set of admissible transformations for the class $\mathcal F'_{\mathbb R}$ is the union of the following families of admissible transformations $\{\mathcal T_{1\mu}\}$, $\{\mathcal T_{2\mu}\}$, $\{\mathcal T_{30}\}$, $\{\mathcal T_{40\kappa}\}$ and $\{\mathcal T_{50\kappa\nu}\}$, where $\mu,\kappa,\nu\in\mathbb R$, $\kappa\ne0$ in the fourth family, $(\kappa,\nu)\ne(0,0)$ in the fifth family.

Since the classes $\mathcal F'$ and $\mathcal F'_{\mathbb R}$ are not normalized, to exhaustively solve the group classification problems for these classes up to the general point equivalence and up to the equivalences generated by the corresponding equivalence groups, we use results of [1] and Theorems 1 and 2.

Corollary 3. *(i) Complete lists of $G^\sim_{\mathcal F'}$- and $\mathcal G^\sim_{\mathcal F'}$-inequivalent essential Lie-symmetry extensions in the class $\mathcal F'$ are exhausted by the cases and Cases 1, 2, 3a, 4a and 5a of Table 1, respectively.*

(ii) Complete lists of $G^\sim_{\mathcal F'}$- and $\mathcal G^\sim_{\mathcal F'}$-inequivalent essential Lie-symmetry extensions in the class $\mathcal F'_{\mathbb R}$ are exhausted by Cases 0, 4a, 4b, 4c, and 5a–5d and by Cases 0, 4a and 5a of Table 1, where $V(x)$ is an arbitrary real-valued potential, and $\mu\in\mathbb R_{\ne0}$.

For each of the essential Lie-symmetry extensions within the class $\mathcal F'$ that are listed in Table 1, we compute the corresponding point symmetry pseudogroup.

Theorem 4. *The point symmetry pseudogroups of the (1+1)-dimensional linear Schrödinger equations of the form (1) with potentials of Cases 1, 2 and 3a of Table 1 are respectively constituted by the following point transformations:*

$$V=i\alpha x-x^2\colon\quad \tilde t=\varepsilon t+\lambda,\quad \tilde x=\varepsilon x+2\kappa\cos(2t+\nu),$$
$$\tilde\psi=\sigma\exp\big(\kappa\sin(2t+\nu)(-2ix-2i\varepsilon\kappa\cos(2t+\nu)-\varepsilon\alpha)\big)(\hat\psi+\hat\Lambda),$$

$$V=i\alpha x+x^2\colon\quad \tilde t=\varepsilon t+\lambda,\quad \tilde x=\varepsilon x+2\kappa{\rm e}^{2t}+2\nu{\rm e}^{-2t},$$
$$\tilde\psi=\sigma\exp\big((\kappa{\rm e}^{2t}-\nu{\rm e}^{-2t})(2ix+2i\varepsilon\kappa{\rm e}^{2t}+2i\varepsilon\nu{\rm e}^{-2t}-\varepsilon\alpha)\big)(\hat\psi+\hat\Lambda),$$

$$V=ix\colon\quad \tilde t=\varepsilon t+\lambda,\quad \tilde x=\varepsilon x+2\kappa t+\nu,\quad \tilde\psi=\sigma\exp\big(i\kappa x+\varepsilon i\kappa^2t-\varepsilon\kappa t^2-\varepsilon\nu t\big)(\hat\psi+\hat\Lambda),$$

where α, λ, κ and ν are arbitrary real constants with $\alpha\ne0$, σ is an arbitrary nonzero complex constant, $\varepsilon=\pm1$, and $\Lambda=\Lambda(t,x)$ is an arbitrary solution of the corresponding equation.

Table 1. Group classification of the class $\mathcal{F}'$. $\mu\in\mathbb{C}_{\neq0}$ with $\operatorname{Im}\mu\geqslant0$, $\alpha\in\mathbb{R}_{>0}$.

no.	V	Basis of $\mathfrak{g}^{\rm ess}_V$
0	$V(x)$	$M,\ I,\ D(1)$
1	$i\alpha x-x^2$	$M,\ I,\ D(1),\ P(\cos2t)-\frac12\alpha\sin2t\,I,\ P(\sin2t)+\frac12\alpha\cos2t\,I$
2	$i\alpha x+x^2$	$M,\ I,\ D(1),\ P({\rm e}^{2t})-\frac12\alpha{\rm e}^{2t}\,I,\ P({\rm e}^{-2t})+\frac12\alpha{\rm e}^{-2t}\,I$
$3a$	ix	$M,\ I,\ D(1),\ P(1)-tI,\ P(t)-\frac12t^2I$
$3b$	$i\alpha x+x$	$M,\ I,\ D(1),\ P(1)+tM-\alpha tI,\ P(t)+\frac12t^2M-\frac12\alpha t^2I$
$4a$	μx^{-2}	$M,\ I,\ D(1),\ D(t),\ D(t^2)-\frac12tI$
$4b$	$\mu x^{-2}-x^2$	$M,\ I,\ D(1),\ D(\cos4t)+\sin4t\,I,\ D(\sin4t)-\cos4t\,I$
$4c$	$\mu x^{-2}+x^2$	$M,\ I,\ D(1),\ D({\rm e}^{-4t})+{\rm e}^{-4t}I,\ D({\rm e}^{4t})-{\rm e}^{4t}I$
$5a$	0	$M,\ I,\ D(1),\ D(t),\ D(t^2)-\frac12tI,\ P(1),\ P(t)$
$5b$	$-x^2$	$M,\ I,\ D(1),\ D(\cos4t)+\sin4t\,I,\ D(\sin4t)-\cos4t\,I,\ P(\cos2t),\ P(\sin2t)$
$5c$	x^2	$M,\ I,\ D(1),\ D({\rm e}^{-4t})+{\rm e}^{-4t}I,\ D({\rm e}^{4t})-{\rm e}^{4t}I,\ P({\rm e}^{-2t}),\ P({\rm e}^{2t})$
$5d$	x	$M,\ I,\ D(1),\ D(t)+\frac32P(t^2)+\frac12t^3M,\ D(t^2)+P(t^3)+\frac14t^4M-\frac12tI,$ $P(1)+tM,\ P(t)+\frac12t^2M$

Corollary 5. *The point symmetry pseudogroup $G_{i\alpha x+x}$ of the (1+1)-dimensional linear Schrödinger equation of the form* (1) *with the potential $V(x)=i\alpha x+x$, where $\alpha\in\mathbb{R}_{\neq0}$, consists of the point transformations*

$$\tilde t=\varepsilon t+\lambda,\quad \tilde x=\varepsilon x+(1-\varepsilon)t^2+2(\kappa+\varepsilon\lambda)t+\lambda^2+\nu,$$

$$\tilde\psi=\sigma\exp\big(i(\varepsilon t+\lambda)(\varepsilon x-\varepsilon t^2+2\kappa t+\nu)+i\kappa(x-t^2)-\varepsilon t(ix+\kappa\alpha t+\alpha\nu-i\kappa^2)$$
$$+\tfrac13(2i-\alpha)(\varepsilon t+\lambda)^3+\tfrac13(i\varepsilon+\alpha)t^3\big)(\hat\psi+\hat\Lambda),$$

where $\varepsilon=\pm1$, κ, λ and ν are arbitrary real constants, σ is an arbitrary nonzero complex constant, and $\Lambda=\Lambda(t,x)$ is an arbitrary solution of this equation.

For each of the above potentials, $V(x)=i\alpha x+\delta x^2$ with $\delta\in\{-1,0,1\}$ and $V(x)=i\alpha x+x$, the pseudogroup G_V splits over $G^{\rm lin}_V$, $G=G^{\rm ess}_V\ltimes G^{\rm lin}_V$. Here the subgroup $G^{\rm ess}_V$ of G_V consists of the transformations of the corresponding form from the theorem with $\Lambda=0$ and with the natural domains, which coincide with the entire space with the coordinates (t,x,ψ,ψ^*). Thus, the subgroup $G^{\rm ess}_V$ is a five-dimensional Lie group consisting of two components. We call the subgroup $G^{\rm ess}_V$ the *essential point symmetry group* of the equation $\mathcal{F}_V$. Its identity component $G^{\rm ess}_{V,\rm id}$ is singled out by the constraint $\varepsilon=1$. The only independent (up to composing with elements of $G^{\rm ess}_{V,\rm id}$) discrete transformation in $G^{\rm ess}_V$ is the composition of the the Wigner time reflection and the space reflection, $\tilde t=-t$, $\tilde x=-x$, $\tilde\psi=\psi^*$ for the potentials $V=i\alpha x+\delta x^2$ with $\delta\in\{-1,0,1\}$ and a more complicated transformation $\tilde t=-t$, $\tilde x=-x$, $\tilde\psi=\exp\big(2it(x-t^2)+\frac23\alpha t^3\big)\psi^*$ for the potential $V(x)=i\alpha x+x$

Theorem 6. *(i) The point symmetry pseudogroup G_0 of the free (1+1)-dimensional linear Schrödinger equation consists of the point transformations*

$$\tilde t=\frac{\lambda_1t+\lambda_2}{\lambda_3t+\lambda_4},\quad \tilde x=\frac{x+\kappa t+\nu}{\lambda_3t+\lambda_4},$$
$$\tilde\psi=\sigma\sqrt{|\lambda_3t+\lambda_4|}\,\exp\left(-i\varepsilon'\frac{\lambda_3x^2-(\kappa\lambda_4-\nu\lambda_3)(2x+\kappa t+\nu)}{4(\lambda_3t+\lambda_4)}\right)(\hat\psi+\hat\Lambda),\tag{2}$$

where λ_1, λ_2, λ_3, λ_4, κ and ν are arbitrary real constants with $\lambda_1\lambda_4-\lambda_2\lambda_3:=\varepsilon'=\pm1$, σ is an arbitrary nonzero complex constant, and $\Lambda=\Lambda(t,x)$ is an arbitrary solution of this equation.

(ii) The point symmetry pseudogroup G_V of the (1+1)-dimensional linear Schrödinger equation of the form (1) with the potential $V(x)=\mu/x^2$ is constituted by the point transformations of the form (2) with $\kappa=\nu=0$ if $\mu\in\mathbb R\setminus\{0\}$ and in addition with $\varepsilon'=1$ if $\mu\in\mathbb C\setminus\mathbb R$, where $\Lambda=\Lambda(t,x)$ is an arbitrary solution of this equation.

Corollary 7. *The point symmetry pseudogroup G_{-x^2} of the (1+1)-dimensional linear Schrödinger equation of the form (1) with the potential $V(x)=-x^2$ consists of the point transformations*

$$
\begin{aligned}
&\tilde t=\frac12\arctan\frac{2\varrho}{\varsigma},\quad \tilde x=\frac{2x+\vartheta}{\varsigma(1+4\varrho^2/\varsigma^2)^{1/2}},\\
&\tilde\psi=\sigma(\varsigma^2+4\varrho^2)^{1/4}\exp\left(\frac{-i\varrho(2x+\vartheta)^2}{\varsigma(\varsigma^2+4\varrho^2)}-i\frac{\varepsilon'}{4}\frac{\varsigma_t}{\varsigma}x^2+i\varepsilon'(\kappa\lambda_4-\nu\lambda_3)\frac{4x+\vartheta}{4\varsigma}\right)(\hat\psi+\hat\Lambda),
\end{aligned}
\tag{3}
$$

where $\varrho:=\lambda_1\sin2t+2\lambda_2\cos2t$, $\varsigma:=\lambda_3\sin2t+2\lambda_4\cos2t$, $\vartheta:=\kappa\sin2t+2\nu\cos2t$, λ_1, λ_2, λ_3, λ_4, κ and ν are arbitrary real constants with $\lambda_1\lambda_4-\lambda_2\lambda_3:=\varepsilon'=\pm1$, σ is an arbitrary nonzero complex constant, and $\Lambda=\Lambda(t,x)$ is an arbitrary solution of this equation.

The point symmetry pseudogroup G_V of the (1+1)-dimensional linear Schrödinger equation of the form (1) with the potential $V(x)=\mu/x^2-x^2$ is constituted by the point transformations of the form (3) with $\kappa=\nu=0$ if $\mu\in\mathbb R\setminus\{0\}$ and in addition with $\varepsilon'=1$ if $\mu\in\mathbb C\setminus\mathbb R$, where $\Lambda=\Lambda(t,x)$ is an arbitrary solution of this equation.

Corollary 8. *The point symmetry pseudogroup G_{x^2} of the (1+1)-dimensional linear Schrödinger equation of the form (1) with the potential $V(x)=x^2$ consists of the point transformations*

$$
\begin{aligned}
&\tilde t=\frac14\ln\left|4\frac{\varrho}{\varsigma}\right|,\quad \tilde x=\frac{4x+\vartheta}{2\varsigma|\varrho/\varsigma|^{1/2}},\\
&\tilde\psi=\sigma|\varrho\varsigma|^{1/4}\exp\left(-i\frac{(4x+\vartheta)^2}{8|\varrho\varsigma|}-i\frac{\varepsilon'}{4}\frac{\varsigma_t}{\varsigma}x^2+i\varepsilon'(\kappa\lambda_4-\nu\lambda_3)\frac{8x+\vartheta}{4\varsigma}\right)(\hat\psi+\hat\Lambda),
\end{aligned}
\tag{4}
$$

where $\varrho:=\lambda_1\mathrm e^{2t}+4\lambda_2\mathrm e^{-2t}$, $\varsigma:=\lambda_3\mathrm e^{2t}+4\lambda_4\mathrm e^{-2t}$, $\vartheta:=\kappa\mathrm e^{2t}+4\nu\mathrm e^{-2t}$, λ_1, λ_2, λ_3, λ_4, κ and ν are arbitrary real constants with $\lambda_1\lambda_4-\lambda_2\lambda_3:=\tilde\varepsilon=\pm1$, $\varepsilon':=\tilde\varepsilon\,\mathrm{sgn}(\varrho\varsigma)$, σ is an arbitrary nonzero complex constant, and $\Lambda=\Lambda(t,x)$ is an arbitrary solution of this equation.

The point symmetry pseudogroup G_V of the (1+1)-dimensional linear Schrödinger equation of the form (1) with the potential $V(x)=\mu/x^2+x^2$ is constituted by the point transformations of the form (4) with $\kappa=\nu=0$ if $\mu\in\mathbb R\setminus\{0\}$ and in addition with $\varepsilon'=1$ if $\mu\in\mathbb C\setminus\mathbb R$, where $\Lambda=\Lambda(t,x)$ is an arbitrary solution of this equation.

Corollary 9. *The point symmetry pseudogroup G_x of the (1+1)-dimensional linear Schrödinger equation of the form (1) with the potential $V(x)=x$ consists of the point transformations*

$$
\begin{aligned}
&\tilde t=\frac{\lambda_1t+\lambda_2}{\lambda_3t+\lambda_4},\quad \tilde x=\frac{x-t^2+\kappa t+\nu}{\lambda_3t+\lambda_4}+\left(\frac{\lambda_1t+\lambda_2}{\lambda_3t+\lambda_4}\right)^2,\\
&\tilde\psi=\sigma\sqrt{|\lambda_3t+\lambda_4|}\,\exp\left(i\frac{\lambda_1t+\lambda_2}{(\lambda_3t+\lambda_4)^2}(x-t^2+\kappa t+\nu)+\frac23 i\left(\frac{\lambda_1t+\lambda_2}{\lambda_3t+\lambda_4}\right)^3\right)\\
&\qquad\times\exp\left(-i\varepsilon'\frac{\lambda_3(x-t^2)^2-(\kappa\lambda_4-\nu\lambda_3)(2x-2t^2+\kappa t+\nu)}{4(\lambda_3t+\lambda_4)}-i\varepsilon'tx+i\varepsilon'\frac{t^3}{3}\right)(\hat\psi+\hat\Lambda),
\end{aligned}
$$

where λ_1, λ_2, λ_3, λ_4, κ and ν are arbitrary real constants with $\lambda_1\lambda_4-\lambda_2\lambda_3:=\varepsilon'=\pm1$, σ is an arbitrary nonzero complex constant, and $\Lambda=\Lambda(t,x)$ is an arbitrary solution of this equation.

In [4], Niederer claimed that the free (1+1)-dimensional linear Schrödinger equation admits the single independent “kinematical” discrete point symmetry transformation

$$\mathcal{K}'\colon\ \tilde t=-\frac1t,\quad \tilde x=\frac xt,\quad \tilde\psi=|t|^{\frac12}\,\mathrm{e}^{-i\frac{x^2}{4t}}\psi.$$

Consider the continuous one-parameter subgroup of G_0 that is singled out by the constraints $\lambda_1=\lambda_4=\cos\epsilon$, $\lambda_2=-\lambda_3=\sin\epsilon$, $\kappa=\nu=0$, $\sigma=1$ and $\Lambda=0$,

$$\mathcal{Q}^+(\epsilon)\colon\ \tilde t=\frac{\sin\epsilon+t\cos\epsilon}{\cos\epsilon-t\sin\epsilon},\ \tilde x=\frac{x}{\cos\epsilon-t\sin\epsilon},\ \tilde\psi=|\cos\epsilon-t\sin\epsilon|^{\frac12}\exp\left(-\frac{ix^2\sin\epsilon}{4(\cos\epsilon-t\sin\epsilon)}\right)\psi,$$

where ϵ is an arbitrary constant parameter, which is defined by the corresponding transformation up to a summand $2\pi k$, $k\in\mathbb{Z}$. The Jacobian of $\mathcal{Q}^+(\epsilon)$ is positive and negative for all values of (t,x,ψ) if $\epsilon=0$ and $\epsilon=\pi$, respectively. For $\epsilon\in(0,\pi)\cup(\pi,2\pi)$, the transformation $\mathcal{Q}^+(\epsilon)$ is not defined if $t=\cot\epsilon$, and for the other values of (t,x,ψ) the sign of its Jacobian coincides with $\operatorname{sgn}(\cos\epsilon-t\sin\epsilon)$. The free (1+1)-dimensional linear Schrödinger equation is invariant with respect to the involution $\mathcal{J}:=\mathcal{Q}^+(\pi)$ only alternating the sign of x and the transformation $\mathcal{K}':=\mathcal{Q}^+(-\frac12\pi)$. The transformations $\mathcal{J}$ and $\mathcal{K}'$ seem to be discrete point symmetry transformations of the free (1+1)-dimensional linear Schrödinger equation. However, this is not the case when considering the natural group multiplication in G_0, which is a modified composition [3] of transformations of the specific form (1). The Jacobian of $\mathcal{J}$ is equal to -1 for all values of (t,x,ψ). Nevertheless, this involutive transformation belongs to the one-parameter subgroup $\{\mathcal{Q}^+(\epsilon)\}$ of G_0, and hence it lies in the identity component of the pseudogroup G_0. A similar situation occurs for the transformation $\mathcal{K}'$, the sign of whose Jacobian is equal to $\operatorname{sgn}t$. Therefore, the above Niederer's claim is incorrect since $\mathcal{K}'$ is not a discrete symmetry: it belongs to the same connected component of the symmetry pseudogroup G_0 as the identity transformation.

Summing up, each of the (1+1)-dimensional linear Schrödinger equations with the potentials $V=\mu/x^2+\delta x^2$, where $\mu\in\mathbb{R}$ and $\delta\in\{-1,0,1\}$, including the free case, or $V(x)=x$ admits a single independent discrete point symmetry, which is the Wigner time reflection $\tilde t=-t$, $\tilde x=x$, $\tilde\psi=\psi^*$. The (1+1)-dimensional linear Schrödinger equations with the potentials $V=\mu/x^2+\delta x^2$, where $\mu\in\mathbb{C}\setminus\mathbb{R}$ and $\delta\in\{-1,0,1\}$, possess no discrete point symmetries.

References

[1] Kurujyibwami C., Basarab-Horwath P. and Popovych R.O., Algebraic method for group classification of (1+1)-dimensional linear Schrödinger equations, *Acta Appl. Math.* **157** (2018), 171–203, arXiv:1607.04118.

[2] Kurujyibwami C., Popovych D.R. and Popovych R.O., Algebraic method of group classification for semi-normalized classes of differential equations, arXiv:2408.16897, 42 pp.

[3] Koval S.D. and Popovych R.O., Point and generalized symmetries of the heat equation revisited, *J. Math. Anal. Appl.* **527** (2023), 127430, arXiv:2208.11073.

[4] Niederer U., The maximal kinematical invariance group of the free Schrödinger equation, *Helv. Phys. Acta.* **45** (1972), 802–810

[5] Nikitin A.G. and Popovych R.O., Group classification of nonlinear Schrödinger equations, *Ukrainian Math. J.* **53** (2001), 1255–1265, arXiv:math-ph/0301009.

[6] Nikitin A.G. and Zasadko T.M., Group classification of Schrödinger equations with position dependent mass, *J. Phys. A* **49** (2016), 365204, arXiv:1603.00890.