When does a conflict dynamical system lack equilibrium states?

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Let us consider in $\mathbb{R}^2$ the conflict dynamical system

$$\mathbf{p}^t \to \mathbf{p}^{t+1}, \quad \mathbf{p}^t = (p_1^t, p_2^t), \quad 0 \le p_1^t, p_2^t \le 1, \quad p_1^t + p_2^t = 1, \quad t = 0, 1, \dots, \tag{1}$$

given by the equations

$$\begin{aligned} p_{\min}^{t+1} &= \frac{(p_{\min}^t + b)p_{\min}^t}{z^t} = \frac{(p_{\min}^t)^2 + bp_{\min}^t}{z^t}, \\ p_{\max}^{t+1} &= \frac{(p_{\max}^t)^2}{z^t}, \quad 0 < b < 1, \end{aligned} \tag{2}$$

where $p_{\min}^t := \min_{i=1,2}\{p_1^t, p_2^t\}$, $p_{\max}^t := 1 - p_{\min}^t$, the denominator $z^t = (p_{\min}^t)^2 + bp_{\min}^t + (p_{\max}^t)^2$, and the parameter b does not change in time. Here we discuss more specific situation than in [2–5]. We restrict our attention to trajectories (orbits) that never map to the point 1/2.

Theorem. *For trajectories of conflict dynamical system* (1) *given by equations* (2) *there exist two regimes of behaviour, which are defined by the conditions on initial coordinates* $p_1 \equiv p_1^{t=0}$, $p_2 \equiv p_2^{t=0}$*:* (1) $0 < b \le |p_1 - p_2| \le 1$, (2) $0 < |p_1 - p_2| < b < 1$. *In regime* (1) *the dynamical system has four fixed points:* $\mathbf{p}_{0,1} = (0, 1)$*,* $\mathbf{p}_{1,0} = (1, 0)$*,* $\mathbf{p}_{\kappa,1-\kappa} = (\kappa, 1 - \kappa)$*, and* $\mathbf{p}_{1-\kappa,\kappa} = (1 - \kappa, \kappa)$*, where* $\kappa = \frac{1-b}{2}$*. In regime* (2) *the dynamical system has no one equilibrium state, its trajectories are quasy-chaotic, chaotic, or eventually periodic.*

We illustrate the behavior of the system in the regime (2) by Figures 1–4.

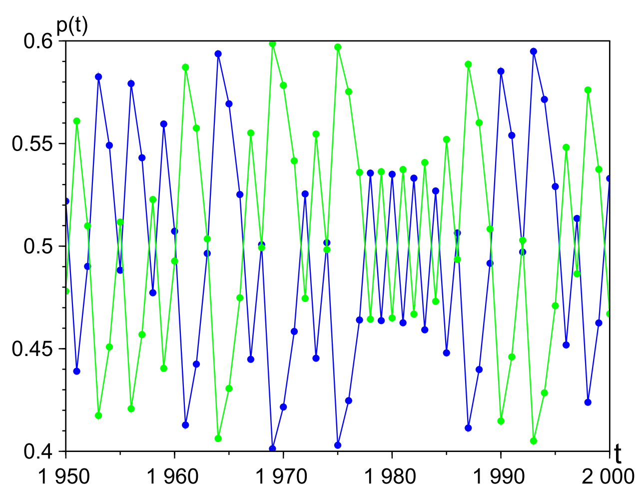

Figure 1. $b = 0.25$, $\mathbf{p}^0 = (0.4, 0.6)$. Chaotic behavior of the coordinates trajectories when $0 < |p_1 - p_2| < b < 1$.

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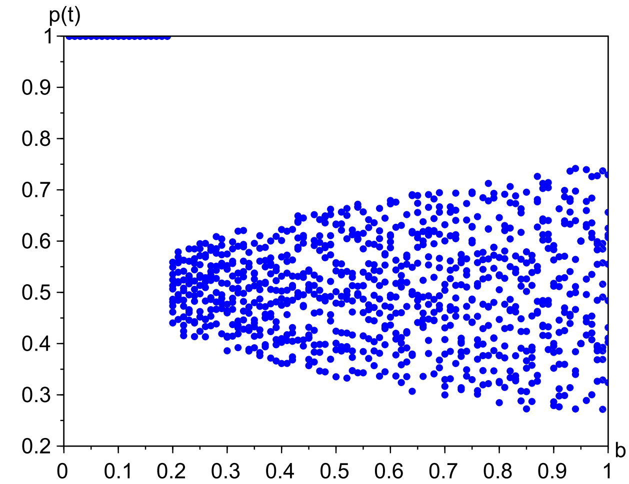

Figure 2. $\mathbf{p} = (0.4, 0.6)$, b changes with step 0.01 from 0 to 1. We plot 10 values of p_2^t at moments t from 1990 till 2000 for each value of b. The points that are plotted approximate either fixed attracting set or randomly fill out the a subinterval $[0, 1]$.

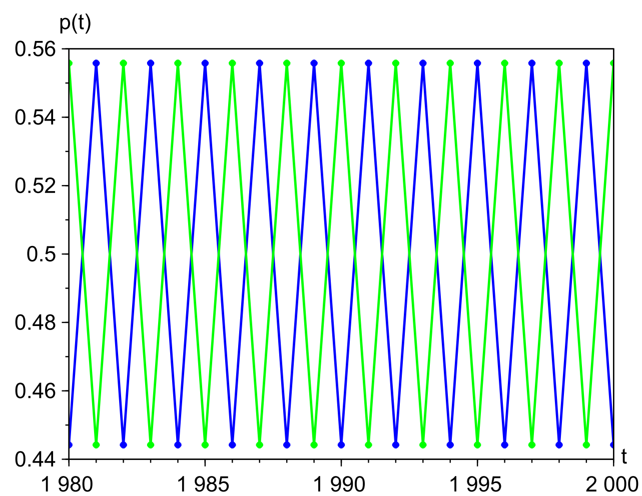

Figure 3. $\mathbf{p}^0 = (0.4442, 0.5558)$. Unstable period 2 cycle when $0 < |p_1 - p_2| < b < 1$, $b = |p_1^3 - p_2^3|/\min^2(p_1, p_2)$.

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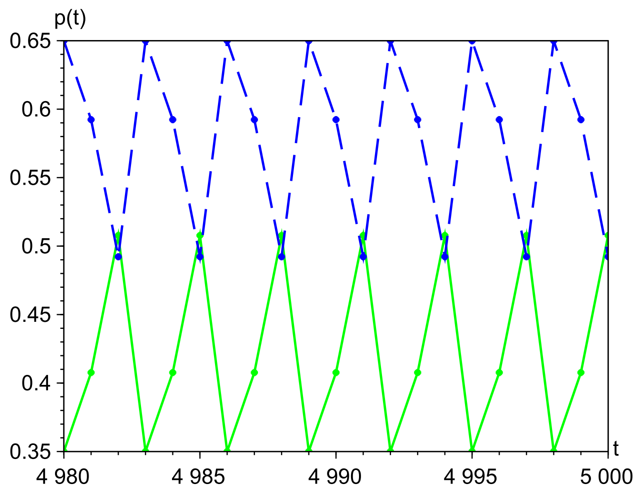

Figure 4. $b = 0.48020751000483968$, $\mathbf{p}^0 = (0.3501, 1 - 0.3501)$. Unstable period 3 cycle when $0 < |p_1 - p_2| < b < 1$.

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