Kontsevich graphs act on Nambu–Poisson brackets. VI. Open problems

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Abstract

Kontsevich's graphs from deformation quantisation allow encoding multi-vectors whose coefficients are differential-polynomial in components of Poisson brackets on finite-dimensional affine manifolds. The calculus of Kontsevich graphs can be made dimension-specific for the class of Nambu–Poisson brackets given by Jacobian determinants. Using the Kontsevich–Nambu micro-graphs in dimensions $d \geqslant 2$, we explore the open problem of (non)triviality for Kontsevich's tetrahedral graph cocycle action on the space of Nambu–Poisson brackets. We detect a conjecturally infinite new set of differential-polynomial identities for Jacobian determinants of arbitrary sizes $d \times d$.

1 Introduction

The Nambu-determinant Poisson bracket $\{f,g\}_d$ on $\mathbb{R}^d \ni \boldsymbol{x} = (x^1,\ldots,x^d)$,

$$\{f,g\}_d(\boldsymbol{x}) = \varrho(\boldsymbol{x}) \cdot \det\Big(\partial\big(f,g,a^1,\ldots,a^{d-2}\big)/\partial\big(x^1,\ldots,x^d\big)\Big), \tag{1}$$

was introduced in [1]; its Casimirs $a^i \in C^1(\mathbb{R}^d)$ satisfy $\{f,a^i\}_d \equiv 0$. Following [2], we study infinitesimal symmetries of the Jacobi identity, i.e. deformations of brackets which preserve their property to be Poisson (see [3–5] and references therein). By using 'good' cocycles in the graph complex, Kontsevich revealed in [2] a class of such symmetries; the tetrahedron γ_3 is the smallest 'good' cocycle. The conjecture in [3] is that the γ_3-flow *restricts* to the Nambu subset of Poisson brackets on $\mathbb{R}^{d\geqslant 3}$. Recall the standard problem in deformation theory: is the γ_3-flow *non*trivial, i.e. does the change of Poisson bivector not amount to a change of coordinates along a vector field $\vec{X}^{\gamma_3}_d$ on $\mathbb{R}^d$? Solution $\vec{X}^{\gamma_3}_{d=2}$ was hinted in [2], the field $\vec{X}^{\gamma_3}_{d=3}$ is known from [3], and $\vec{X}^{\gamma_3}_{d=4}$ was found in [5]. In this note, we sum up the surprising properties of graphs that encode solutions $\vec{X}^{\gamma_3}_{d\geqslant 2}(\varrho,\boldsymbol{a})$ for the γ_3-flow of Nambu–Poisson brackets.[1]

We know the solutions $\vec{X}^{\gamma_3}_{d\leqslant 4}$ separately in each dimension d. There can be no universal formula of $\vec{X}^{\gamma_3}$ – for all Poisson brackets in all dimensions – for the nontrivial graph cocycle γ_3 of [2]; we do not know a universal formula of $\vec{X}^{\gamma_3}_d$ – working in each $d \geqslant 3$ – for Nambu brackets (1) in particular.[2] Studying the step $d \mapsto d+1$, we saw in [6] that solution $\vec{X}^{\gamma_3}_{d=4}$ is found *economically* by knowing (*i*) the combinatorics of formulas in $\vec{X}^{\gamma_3}_{d=2}$ and $\vec{X}^{\gamma_3}_{d=3}$ and (*ii*) the

[1] It remains an open problem to find an example of *nontrivial* graph cocycle action on a Poisson bracket, thus spreading it to a family of inequivalent structures not related by a coordinate change. Beyond the tetrahedron γ_3, there are countably many 'good' nontrivial graph cocycles: γ_5, γ_7, ... (see [3, 4] and references therein); they stem from the Grothendieck–Teichmüller Lie algebra $\mathfrak{grt}$.

[2] To obtain the solution $\vec{X}^{\gamma_3}_{d=4}$ over $\mathbb{R}^4$, in [5] we reduced the size of the problem circa 300 times w.r.t. the worst-case scenario in [4]. This reduction effort continued in [6] for steps $d \mapsto d+1$ in any dimension, given a solution $\vec{X}^{\gamma_3}_d$ and some 'invisible' structures over $\mathbb{R}^d$, which we now tackle.

structures that encode identically vanishing[3] 1-vectors built of whole Nambu brackets over $\mathbb{R}^3$. These vanishing objects are tensor-valued invariants of $GL(d)$-action; we study their collective behaviour under $d \mapsto d+1$ and internal construction of individual invariants. For them, we discover a likely infinite set of new identities, differential-polynomial w.r.t. the Casimirs a^i in the Jacobian determinants and coefficients $\varrho(\boldsymbol{x})$ of d-vectors. In this note we describe the procedure of (micro-)graph embeddings to obtain such identities and illustrate how they hold.

Preliminaries. Kontsevich's directed graphs from [2] encode polydifferential operators – in practice, multivectors – made of copies of Poisson bi-vectors P as building blocks: the subgraph of P is the wedge $\leftarrow\!\bullet\!\rightarrow$. By definition, each digraph Γ (possibly, in an $\mathbb{R}$-linear combination) consists of the ordered set of m sinks (where operator's arguments are placed) and n internal vertices, the wedge tops; for each top, the pair of outgoing edges is ordered Left $\prec$ Right. Label the ordered sinks by $\mathsf{0}, \ldots, \mathsf{m-1}$ and label the wedge tops by $\mathsf{m}, \ldots, \mathsf{m+n-1}$. The encoding of Γ is the ordered list of n ordered pairs[4] (L $\prec$ R) of the arrowhead vertices for the two edges issued from the respective wedge top, i.e. from the arrowtail.[5] Each edge carries its own summation index ranging $[1, \ldots, \dim M^d]$ for the affine[6] Poisson manifold at hand. An edge decorated by index i encodes the derivative $\partial/\partial x^i$ w.r.t. the affine coordinates $x^1, \ldots, x^d$ of a base point $\boldsymbol{x} \in M^d$. The formula[7] of polydifferential operator is the sum (over all indices running $\mathsf{1}, \ldots, \mathsf{d}$) of the product of vertices' content.

By the number $d-2$ of its Casimirs a^k, the Nambu–Poisson bracket in Eq. (1) is specific to dimension d. Take a microscope and magnify each internal vertex of Kontsevich's graph; it telescopes to a brush: the Levi-Civita arrowtail vertex contains $\varrho(\boldsymbol{x}) \cdot \varepsilon^{i_1 \cdots i_d}$, with $\partial_{i_1} \wedge \cdots \wedge \partial_{i_d}$ on the wedge-ordered d-tuple of outgoing edges, and (still within the magnified zone) there are $d-2$ terminal vertices with the Casimirs $a^1, \ldots, a^{d-2}$ of the respective copy of Nambu–Poisson bracket. By definition, a *Nambu micro-graph* over M^d is made of whole copies of Nambu–Poisson bracket (1) as building blocks.[8] A Nambu micro-graph over M^d consists of m sinks $\mathsf{0}, \ldots, \mathsf{m-1}$ and of n copies of bracket (1), which provide the Levi-Civita vertices (with $\varrho(\boldsymbol{x}) \cdot \varepsilon^{\vec{\imath}}$) labelled $\mathsf{m}, \ldots, \mathsf{m+n-1}$ and the Casimir vertices,[9] namely: $\mathsf{m+n}, \ldots, \mathsf{m+2n-1}$ with a^1, then $\mathsf{m+2n}, \ldots, \mathsf{m+3n-1}$ with a^2, etc. Every Nambu micro-graph is now encoded by the ordered n-tuple of ordered lists (of length d) of arrowhead vertices for the d-tuples of edges issued from the Levi-Civita arrowtail vertices.

Example 1. Over $d=3$, we shall refer from Table 1 to the twelve *vanishing* (as formulas) 1-vector Nambu micro-graphs build over $m=1$ sink of $n=3$ tridents: the sink is $\mathsf{0}$, the Levi-Civita trident tops are $\mathsf{1,2,3}$, and the Casimir vertices $\mathsf{4,5,6}$ (with a^1) are terminal; their encodings are given in [5, Lemma 2], typeset in boldface. Each encoding is the ordered list of $n=3$ ordered $(d=3)$-tuples of arrowhead vertices for the triples of edges issued from the arrowtails $\mathsf{1,2,3}$.

[3] Examples of this, not exhausting the full range, are given by formulas, differential-polynomial in the components of Poisson tensor, which equal minus themselves under a relabelling of summation indices. We stress the existence of other mechanisms for objects' vanishing.

[4] Swapping the order of two outgoing edges in a wedge reverses the sign in front of the graph.

[5] The sunflower [2,4] is $X^{\gamma_3} = (\mathsf{0,1;1,3;1,2}) + 2 \cdot (\mathsf{0,2;1,3;1,2})$, here $(m,n) = (1,3)$.

[6] The Poisson manifold is affine to make formulas, differential-polynomial in Poisson bracket's components and encoded by Kontsevich's graphs, independent of coordinate changes $\boldsymbol{x} = A\mathbf{x} + \boldsymbol{b}$.

[7] Unlike its formula, the Kontsevich graph, with a copy of Poisson bi-vector $P = P^{ij}\, \partial_i \otimes \partial_j$ in each internal vertex, does not depend on the dimension d; its topology and the ordering L $\prec$ R of outgoing edges in every wedge encodes the operator's formula for every $d \geqslant 2$.

[8] Every Kontsevich's graph can be expanded to a sum of Nambu micro-graphs over given $d \geqslant 3$ by working out the Leibniz rules for arrows which entered the internal vertices. Yet not every linear combination of Nambu micro-graphs over M^d is Kontsevich's, and not every digraph built over m sinks from suitably many Levi-Civita and Casimir vertices is Nambu.

[9] The label of a Casimir a^k differs by $k \cdot n$ from the label of its 'parent' Levi-Civita vertex.

2 Vanishing micro-graphs and embedding: $d \hookrightarrow d+1$

We now see two ways how, from a Nambu micro-graph over dimension d, we can construct (a linear combination of) Nambu micro-graph(s) over the next dimension $d{+}1$. Namely, equip every Levi-Civita vertex with one extra Casimir a^{d-1} in its new terminal vertex (to which the new, ordered *last* arrow is now sent from its 'parent' Levi-Civita vertex at the top of the $(d{+}1)$-brush).

Descendants. If, in dimension d, an external arrow – issued from another copy of Nambu–Poisson bivector – acted on a Casimir a^k, $1 \leqslant k \leqslant d-2$, within a subgraph of Nambu structure, then let the external arrow run – via Leibniz rule[10]– over its old target and the newly attached Casimir vertex with a^{d-1}. Every expansion of Leibniz rule thus yields a linear combination of $(d+1)$-dimensional Nambu micro-graph *descendants* of the originally taken d-dimensional Nambu micro-graph.

Embedding. After the new Casimirs a^{d-1} are attached, one per each Levi-Civita vertex, none of the old edges is re-directed and no Leibniz rules are worked out.[11] This yields the *embedding* $\Gamma_d \hookrightarrow \widehat{\Gamma}_{d+1}$ of the original (micro) graph Γ from dimension $d \geqslant 2$ to the Nambu micro-graph over dimension $d+1$.

Consider the formula of the object (e.g., 1-vector on $\mathbb{R}^{d+1}$) encoded by the Nambu micro-graph $\widehat{\Gamma}_{d+1}$ after embedding Γ_d as its subgraph. The new edge(s) to new Casimir(s) carry new indice(s). When *each* new index equals $d+1$, encoding $\partial/\partial x^{d+1}(a^{d-1})$ in the new vertex, the old formula $\phi(\Gamma_d)$ reappears:

$$\phi\bigl(\widehat{\Gamma}_{d+1}\bigr) = \phi(\Gamma_d)\cdot\bigl(\partial a^{d-1}/\partial x^{d+1}\bigr)^n + \langle\text{cross-terms}\rangle, \tag{2}$$

n being the number of Levi-Civita vertices. The cross-terms are those where at least one new $\partial/\partial x^{d+1}$ acts on the content of old vertices from the subgraph Γ_d. We discover a curious property of Jacobian determinants in brackets (1): the vanishing $\phi(\Gamma_d) = 0$ is preserved by the embedding $\Gamma_d \hookrightarrow \widehat{\Gamma}_{d+1}$, so $\phi\bigl(\widehat{\Gamma}_{d+1}\bigr) = 0$; all the cross-terms cancel out! Exploring the open problem – is the tetrahedral graph cocycle action on the space of Poisson brackets *non*trivial? – we study the vanishing mechanism(s) and the work of (micro-)graph embeddings.[12]

The tetrahedral flow $\dot{P} = Q^{\gamma_3}\bigl(P^{\otimes^4}\bigr)$ needs 1-vectors $\vec{X}^{\gamma_3}\bigl(P^{\otimes^3}\bigr)$ to be trivialised (if, indeed, Q^{γ_3} is a Poisson coboundary $[\![P, \vec{X}^{\gamma_3}]\!]$); to find $\vec{X}^{\gamma_3}_{d\geqslant 2}$ for the Nambu class, we use micro-graphs on $m = 1$ sink and $n = 3$ copies of bracket (1) over $\mathbb{R}^d$. To make the task for $d+1$ smaller, in [5] and [6] we learned to use the descendants of 'sunflower' graph from 2D (see footnote 5 on p. 2), adjoining the set of $(d{+}1)$-descendants of *vanishing* 1-vector Nambu micro-graphs, which were invisible in lower dimension d. Specifically for $d = 3 \hookrightarrow d+1 = 4$, consider sunflower's twelve vanishing descendants: see Example 1 on p. 2 and Table 1.[13] Of them, Nos. 38 and 41 are *zero*: they have a symmetry $g \in \mathrm{Aut}(\Gamma)$ that acts on the vertices and edges (thus relabelling the summation indices) and makes $\Gamma = -\Gamma$ w.r.t. the wedge ordering $\mathrm{L} \prec \mathrm{M} \prec \mathrm{R}$ of edges in the tridents, so $\phi(\Gamma) = -\phi(\Gamma) = 0$. The other ten descendants have no symmetry with such effect; being nonzero, they vanish in another way.

[10] These Leibniz rules work independently one from another for each incoming edge.

[11] See [6, Example 3] for an illustration: $\Gamma_{d=3} \hookrightarrow \widehat{\Gamma}_{d=4}$; similar is Definition 5 and Propositions 4,5 in the paper [I.] (arXiv:2409.18875 [q-alg]) or Definition 3 and Example 2 in [III.], which is arXiv:2409.15932 [q-alg].

[12] **Example.** For $d = 4$ and $(m, n) = (0, 2)$, the only vanishing (as formula) Hamiltonian is $H^{(9)}_{d=4} = [1,2,3,5; 3,4,5,6]$ from [III., Lemma 16]: here 1,2 are Levi-Civita vertices, 3,4 are Casimirs a^1, and 5,6 are a^2; its embedding still vanishes, $\phi\bigl(\widehat{H}^{(9)}_{d=5}\bigr) = 0$.

[13] **Legend** (to Table 1)**.** Row 1 (R1) contains the indices (in a list of 48) of vanishing 3D micro-graph descendants $\Gamma \in \mathrm{Van}_{d=3}(\text{sunflower})$ from 2D. • Micro-graphs with a nontrivial automorphism group are labelled by **aut**. • Micro-graphs which equal minus themselves under some automorphism are labelled by **zero**. • Row 2 (R2) contains the number of 4D-descendants of each micro-graph $\Gamma \in \mathrm{Van}_{d=3}(\text{sunflower})$. • Row 3 (R3) contains the number of vanishing 4D-descendants of each micro-graph $\Gamma \in \mathrm{Van}_{d=3}(\text{sunflower})$. • Row 4 (R4) states the nature of vanishing 4D-descendants for each micro-graph $\Gamma \in \mathrm{Van}_{d=3}(\text{sunflower})$; the embedding map is denoted by **e**, the contra-embedding **c** then amounts to $a^1 \rightleftarrows a^2$.

Table 1. Vanishing 4D-descendants of 3D micro-graphs $\mathrm{Van}_{d=3}$(sunflower).

R1:	10 aut	13	20	21	24 aut	25	29	32 aut	33 aut	37	38 zero	41 zero	Total
R2:	8	2	4	4	8	8	4	8	8	8	16	32	118
R3:	2	2	4	4	8	2	4	2	2	8	2	12	54
R4:	e,c	e,c	e,c + 2	e,c + 2	e,c + 6	e,c	e,c + 2	e,c	e,c	e,c + 6	e,c	e,c + 10	e,c **vanish**

Case $\mathrm{Aut}(\Gamma) \neq \{\mathbf{1}\}$**.** Four nonzero micro-graphs in Table 1 still have nontrivial symmetry groups (Nos. 10, 24, 32, 33). We detect[14] that for each of them, its formula splits into mutually cancelling disjoint pairs of terms; these pairs are marked by exactly those Casimirs which are effectively moved by at least one element $g \neq \mathbf{1}$ from $\mathrm{Aut}(\Gamma)$. In brief, automorphisms highlight the key factors.

Case $\mathrm{Aut}(\Gamma) = \{\mathbf{1}\}$**.** For the six remaining nonzero vanishing descendants of the sunflower, the identity $\phi(\Gamma) = 0$ is still due to the cancellation of disjoint pairs of terms, without accumulation of longer linear combinations; yet no vertices are marked by the effective action of a symmetry. To reveal this mechanism in full is a standing problem; the impact of topology (in Γ) on arithmetic (in $\phi(\Gamma)$) will make possible the study of $d \gtrsim 5$, so far costly.

We argue that these two mechanisms of $\phi(\Gamma_d) = 0$ for (non)zero (micro-)graphs persist under $\Gamma_d \hookrightarrow \widehat{\Gamma}_{d+1}$. For zero $\Gamma \cong g(\Gamma) = -\Gamma$ the extension of g after new Casimirs are adjoined is verbatim. For nonzero Γ_d, disjoint pairs of terms cancelled out as the summation indices ran, independently one from another, up to d. Yet the adjoining of new Casimirs and new summation index in every Levi-Civita symbol $\varepsilon^{i_1 \cdots i_{d+1}}$ does not alter the (grouping of) previously existing factors and terms, only lifting the summation limit to $d+1$. The cancellations work as before, now on a larger set of indices and a longer range of each index.[15]

Let us compare the set $\mathrm{Van}_{d=4}$ of vanishing 4D-descendants[16] of the 'sunflower' in 2D with, on the other hand, the set of vanishing 4D-descendants of the twelve (from Table 1) *vanishing* 3D-descendants $\Gamma \in \mathrm{Van}_{d=3}$ of the 'sunflower'.

Proposition 1. $\mathrm{Van}_{d=4} =$ *4D-descendants* $\bigl(\mathrm{Van}_{d=3}\bigr)$.

This means that in $d+1 = 4$, we do not seek for the vanishing 1-vectors anywhere *else*; there appear no new (starting to work) mechanisms of $\phi(\Gamma_{d+1}) = 0$ w.r.t. $\phi(\Gamma_d) = 0$. This is important (see [6]): we reduce the intractable problem in dimension $d+1$ before trying to solve it; e.g., $d = 5$ is beyond the power of Hábrók computing cluster at hand.

Open problems, specific to Kontsevich's graph cocycle flows on the (sub)spaces of Nambu–Poisson brackets (1) on $\mathbb{R}^{d \geqslant 3}$, are in particular these:

- If $\dot{P} = Q^{\gamma_3}_{d+1}\bigl(P^{\otimes^4}\bigr)$ is trivial, $Q^{\gamma_3}_{d+1} = [\![P, \vec{X}^{\gamma_3}_{d+1}\bigl(P^{\otimes^3}\bigr)]\!]$, then is $\vec{X}^{\gamma_3}_{d+1}$ found over linear combinations of the 'sunflower' descendants from 2D ?
- If yes, is $\vec{X}^{\gamma_3}_{d+1}$ found over the union of $(d+1)$-descendants for the underlying solution $\vec{X}^{\gamma_3}_{d}$ and vanishing 1-vector micro-graphs over $\mathbb{R}^d$?
- Do these indispensable (cf. [6]) vanishing micro-graphs stem *only* from the underlying vanishing set in $\mathbb{R}^{d-1}$ through their descendants ?
- What is the group-theoretic and topological mechanism of vanishing for Nambu micro-graphs and their linear combinations ?

[14] This claim is verified by brute force calculation, see [6, Example 5] for graph No. 10 in Table 1.

[15] **Example.** Taking micro-graph No. 10 from Table 1 in $d = 3$, one easily upgrades $\phi(\Gamma_{d=3}) = 0$ to $\phi(\widehat{\Gamma}_{d=4}) = 0$ by three new factors.

[16] There are 324 4D-descendants of the 'sunflower'; 54 of them vanish (see [5]).

A Resilience of the graph calculus in the dimensional shift $d \mapsto d+1$. Examples

In a series of examples we examine the mechanism behind vanishing graphs – graphs whose formulas obtained via the graph calculus are equal to zero. We detect a partial answer behind the vanishing mechanism for a certain class of graphs; and we observe a consistent pattern in how the components of the graph formulas cancel out.

Example 2 (Embedding of 3D graph into 4D)**.** We take the graph built of 3D Nambu–Poisson structures given by the encoding $e = (0,2,4;1,3,5;1,2,6)$ and embed it into 4D (that is, we apply the embedding map to e): embedding$(e) = (0,2,4,\mathbf{7};1,3,5,\mathbf{8};1,2,6,\mathbf{9})$, where the new Casimirs $a^2 \in \{7,8,9\}$ which appear in the 4D Nambu–Poisson structure are in bold font. Recall that each tuple (separated by a semi-colon) in the encoding e corresponds to the outgoing arrows of each Nambu–Poisson structure. In embedding(e) the arrows from the graph built of 3D Nambu–Poisson structures (the first three vertex numbers in each tuple of the encoding e) remain as they were, with the only difference being that each structure has an outgoing edge to the new Casimir acquired in the dimensional step 3D↦4D (the last vertex number in each tuple). ■

It is *a priori* not obvious that the embeddings of vanishing (micro-)graphs will again vanish due to the vanishing of their sub-structures. Indeed, the assembly of formulas using the graph calculus implies the creation of a new family of cross-terms.

Example 3. Let us take the formula of the 3D vanishing sunflower micro-graph g with index 10 in Table 1 given by the encoding e_g, and embed it into 4D: $e_g = (0,1,4;1,6,5;4,5,6)$, embedding$(e_g) = (0,1,4,\mathbf{7};1,6,5,\mathbf{8};4,5,6,\mathbf{9})$, with the index of the new Casimir a^2 in bold. We write the inert sum of the formula of g in 3D:

$$\phi(g) = \sum\nolimits_{\vec{\imath},\vec{\jmath},\vec{k}}^{d=3} \varepsilon^{i_1 i_2 i_3} \varepsilon^{j_1 j_2 j_3} \varepsilon^{k_1 k_2 k_3}\, \varrho^2 \varrho_{i_2 j_1} a_{i_3 k_1} a_{j_3 k_2} a_{j_2 k_3} \partial_{i_1}(),$$

and the inert sum of the embedding of g into 4D:

$$\phi\bigl(\text{embedding}(g)\bigr) = \sum_{\vec{\imath},\vec{\jmath},\vec{k}}^{\boldsymbol{d=4}} \varepsilon^{i_1 i_2 i_3 \boldsymbol{i_4}} \varepsilon^{j_1 j_2 j_3 \boldsymbol{j_4}} \varepsilon^{k_1 k_2 k_3 \boldsymbol{k_4}}\, \varrho^2 \varrho_{i_2 j_1} a^1_{i_3 k_1} a^1_{j_3 k_2} a^1_{j_2 k_3} \boldsymbol{a^2_{i_4} a^2_{j_4} a^2_{k_4}} \partial_{i_1}(),$$

where the terms concerning the new Casimir a^2 and dimension 4D are in bold. That is, each Nambu–Poisson structure which composes the graph embedding(g) in 4D has four outgoing edges (instead of three, as in 3D). Therefore, the indices in the inert sum which correspond to the outgoing edges of the Nambu–Poisson structures will run over $\{1,2,3,4\}$, which will create cross-terms in such a way that we lose track of the formula of the 3D vanishing micro-graph g. That is, the 3D formula is reproduced and multiplied by the terms in bold with $\boldsymbol{i_4, j_4, k_4} = 4$. But there appear many other terms, when the indices are permuted over $\{1,2,3,4\}$. ■

The approach we take – to investigate why the embeddings of 3D vanishing sunflower micro-graphs vanish – is to understand why they themselves vanish in 3D.

Example 4. Let us look at the above example of the 3D vanishing sunflower micro-graph g with index 10 in Table 1 with a non-trivial automorphism group, where we show in bold the Casimirs on which there acts the non-trivial automorphism group:

$$\phi(g) = \sum\nolimits^{d=3}_{\substack{i_1,i_2,i_3\\ j_1,j_2,j_3,\\ k_1,k_2,k_3=1}} \varepsilon^{i_1 i_2 i_3} \varepsilon^{j_1 j_2 j_3} \varepsilon^{k_1 k_2 k_3}\, \varrho^2 \varrho_{i_2 j_1} a_{i_3 k_1} \boldsymbol{a_{j_3 k_2} a_{j_2 k_3}} \partial_{i_1}().$$

We plug in a certain permutation of the $\vec{\imath}$ terms into the inert sum, $\vec{\imath} = (1,2,3)$, that is $i_1 = 1, i_2 = 2, i_3 = 3$:

- $\vec{\imath} = (1,2,3)$: $\sum_{j,k} \varepsilon^{123} \varepsilon^{j_1 j_2 j_3} \varepsilon^{k_1 k_2 k_3} \varrho^2 \varrho_{2j_1} a_{3k_1} a_{j_3 k_2} a_{j_2 k_3} \partial_1()$. We now plug in two consecutive permutations of the $\vec{\jmath}$ terms:
- $(\vec{\imath}, \vec{\jmath}) = \big(\vec{\imath} = (1,2,3), \vec{\jmath} = (1,2,3)\big)$:

$$\sum_k \varepsilon^{123} \varepsilon^{123} \varepsilon^{k_1 k_2 k_3} \varrho^2 \varrho_{21} a_{3k_1} \boldsymbol{a_{3k_2}} \boldsymbol{a_{2k_3}} \partial_1() = \sum_k \varepsilon^{k_1 k_2 k_3} \varrho^2 \varrho_{21} a_{3k_1} \boldsymbol{a_{3k_2}} \boldsymbol{a_{2k_3}} \partial_1(),$$

- $(\vec{\imath}, \vec{\jmath}) = \big(\vec{\imath} = (1,2,3), \vec{\jmath} = (1,3,2)\big)$:

$$\sum_k \varepsilon^{123} \varepsilon^{132} \varepsilon^{k_1 k_2 k_3} \varrho^2 \varrho_{21} a_{3k_1} \boldsymbol{a_{2k_2}} \boldsymbol{a_{3k_3}} \partial_1() = \sum_k -\varepsilon^{k_1 k_2 k_3} \varrho^2 \varrho_{21} a_{3k_1} \boldsymbol{a_{2k_2}} \boldsymbol{a_{3k_3}} \partial_1().$$

Here, we can see without expanding the sum any further, that the inert sums of $\big(\vec{\imath} = (1,2,3), \vec{\jmath} = (1,2,3)\big)$ and $\big(\vec{\imath} = (1,2,3), \vec{\jmath} = (1,3,2)\big)$ will cancel out due to the Casimirs in bold on which the non-trivial automorphism group acts. Indeed, for a given permutation $\vec{\imath}$ the set of permutations $\vec{\jmath}$ is partitioned into three odd-even pairs, which differ by one transposition, hence by the +/- sign. These pairs of terms cancel out for every particular value of the permutation $\vec{k}$.

We find that the other $\big(\vec{\imath} = (1,2,3), \vec{\jmath}\big)$ terms cancel in the same way for any $\vec{\jmath}$, meaning that all terms with $\vec{\imath} = (1,2,3)$ vanish on their own. There is no cross-cancellation with other $\vec{\imath}$ terms, that is:

$$\begin{aligned} \phi(g) = &\sum_{\vec{\imath}=(1,2,3),\vec{\jmath},\vec{k}}^{d=3} + \sum_{\vec{\imath}=(1,3,2),\vec{\jmath},\vec{k}}^{d=3} + \sum_{\vec{\imath}=(2,1,3),\vec{\jmath},\vec{k}}^{d=3} + \sum_{\vec{\imath}=(2,3,1),\vec{\jmath},\vec{k}}^{d=3} + \sum_{\vec{\imath}=(3,2,1),\vec{\jmath},\vec{k}}^{d=3} + \sum_{\vec{\imath}=(3,1,2),\vec{\jmath},\vec{k}}^{d=3} \\ &= 0+0+0+0+0+0 = 0. \end{aligned}$$

In this example, we showed how the non-trivial automorphism group of the micro-graph acting on the Casimirs induced the formula of the micro-graph to vanish. ■

Let us again look at the same graph as in Example 4:

$$\phi(g) = \sum\nolimits_{\vec{\imath},\vec{\jmath},\vec{k}}^{d=3} \varepsilon^{i_1 i_2 i_3} \varepsilon^{j_1 j_2 j_3} \varepsilon^{k_1 k_2 k_3} \varrho^2 \varrho_{i_2 j_1} a_{i_3 k_1} a_{j_3 k_2} a_{j_2 k_3} \partial_{i_1}(),$$

$$\phi\big(\text{embedding}(g)\big) = \sum_{\vec{\imath},\vec{\jmath},\vec{k}}^{\boldsymbol{d=4}} \varepsilon^{i_1 i_2 i_3 \boldsymbol{i_4}} \varepsilon^{j_1 j_2 j_3 \boldsymbol{j_4}} \varepsilon^{k_1 k_2 k_3 \boldsymbol{k_4}} \varrho^2 \varrho_{i_2 j_1} a^1_{i_3 k_1} a^1_{j_3 k_2} a^1_{j_2 k_3} \boldsymbol{a^2_{i_4}} \boldsymbol{a^2_{j_4}} \boldsymbol{a^2_{k_4}} \partial_{i_1}().$$

It is clear that the cancellation structure is preserved under the embedding.

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