Symmetry analysis and exact solutions of a diffusive Lotka–Volterra type system with convection

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We investigate a nonlinear reaction–diffusion–convection model arising in the description of viscous fingering induced by a chemical reaction of the form $A + B \to C$. The starting point is a multicomponent mathematical model proposed in [1], which couples incompressible flow in a porous medium with diffusion and reaction processes. Following [1], the governing equations read

$$\begin{aligned} &\nabla \cdot U = 0, \\ &\kappa \nabla p + \mu(w) U = 0, \\ &u_t + U \cdot \nabla u = d_1 \Delta u - kuv, \\ &v_t + U \cdot \nabla v = d_2 \Delta v - kuv, \\ &w_t + U \cdot \nabla w = d_3 \Delta w + kuv, \end{aligned} \tag{1}$$

where the operator ∇ and the Laplacian $\triangle$ are taken in $\mathbb{R}^2$. The functions and coefficients in (1) have the following physical meanings: the functions $u(t,x,y)$, $v(t,x,y)$ and $w(t,x,y)$ denote two reactants A and B and their product C, respectively; k is a kinetic constant; $p(t,x,y)$ is pressure; $U = (U_1, U_2)$ is two-dimensional velocity field; d_1, d_2 and d_3 are diffusion coefficients; κ is permeability; $\mu(w)$ is viscosity of the fluid.

Introducing a stream function Ψ according to the well-known formulae:

$$U_1 = \frac{\partial \Psi}{\partial y}, \quad U_2 = -\frac{\partial \Psi}{\partial x},$$

the above five-component system can be reduced to a three-component diffusive Lotka–Volterra type system with convection

$$\begin{aligned} &u_t + \Psi_y u_x - \Psi_x u_y = d_1 (u_{xx} + u_{yy}) - kuv, \\ &v_t + \Psi_y v_x - \Psi_x v_y = d_2 (v_{xx} + v_{yy}) - kuv, \\ &w_t + \Psi_y w_x - \Psi_x w_y = d_3 (w_{xx} + w_{yy}) + kuv, \end{aligned} \tag{2}$$

while the pressure equation becomes semiautonomous.

System (2) is examined using the classical Lie method. A complete Lie symmetry classification is derived via a rigorous algorithm. The most interesting cases, from the symmetry and applicability points of view, are studied in order to derive exact solutions. A wide range of exact solutions is constructed for radially symmetric stream functions. It is shown that some of the exact solutions can be used to demonstrate the spatiotemporal evolution of the concentrations of two reactants and their product.

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References

[1] Gerard T. and De Wit A., Miscible viscous fingering induced by a simple $A + B \to C$ chemical reaction, *Phys. Rev. E* **79** (2009), 016308.