Point symmetries with linear fractional components

Serhii D. Koval† and Roman O. Popovych‡

†*Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's (NL) A1C 5S7, Canada*
Department of Mathematics, Kyiv Academic University, 36 Vernads'koho Blvd, 03142 Kyiv, Ukraine

‡ *Mathematical Institute, Silesian University in Opava, Na Rybníčku 1, 746 01 Opava, Czech Republic*
Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivska Str., 01024 Kyiv, Ukraine

E-mails: skoval@mun.ca, rop@imath.kiev.ua

To properly study the structures of the complete and the essential point symmetry pseudogroups G and $G^{\rm ess}$ of the remarkable Fokker–Planck equation $u_t + xu_y = u_{xx}$, in [1] we redefine the group operation in G via extending the domains of compositions of transformations from G. By this redefining, the pseudogroup $G^{\rm ess}$ was turned into an eight-dimensional Lie group, which is isomorphic to the group $({\rm SL}(2,\mathbb{R}) \ltimes {\rm H}(2,\mathbb{R})) \times \mathbb{Z}_2$. Here ${\rm SL}(2,\mathbb{R})$ and ${\rm H}(2,\mathbb{R})$ denote the real degree-two special linear group and the real rank-two Heisenberg group, respectively. This approach is based on the fact that the t-component of any transformation from G is linear fractional in t.

The application of the above approach to the linear (1+1)-dimensional heat equation $u_t = u_{xx}$ allowed us to correctly classify subalgebras of its essential Lie invariance algebra and to accurately describe the discrete elements of its complete point symmetry pseudogroup in [2]. In particular, an unexpected result is that the alternating the sign of x is in fact not a discrete symmetry transformation of the heat equation, in contrast to that stated earlier in the literature.

In this talk, we extend the approach to an arbitrary pseudogroup of point transformations whose components with respect to a subset of variables are linear fractional functions with respect to these variables. Then we apply the developed formalism to many both linear and nonlinear systems of differential equations with point symmetry pseudogroups of such kind. Among the considered systems are the Burgers equation, the nonlinear diffusion equation with power nonlinearity of power $-4/3$, the Harry Dym equation, the two-dimensional Burgers system, the free (1+1)-dimensional Schrödinger equation, the (1+1)-dimensional Schrödinger and linear heat equations with inverse square potentials, the homogeneous Monge–Ampere equation, and the Chazy equations.

It is obvious that analogous point equivalence pseudogroups of classes of differential equations can be studied within the same framework.

References

[1] Koval S.D., Bihlo A. and Popovych R.O., Extended symmetry analysis of remarkable (1+2)-dimensional Fokker-Planck equation, 2022, arXiv:2205.13526.

[2] Koval S.D. and Popovych R.O., Point and generalized symmetries of the heat equation revisited, 2022, arXiv:2208.11073.