Equivalence transformations and group classification of (1+2)-dimensional degenerate Fokker–Planck equations

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We find the equivalence group $G^{\sim}_{\bar{\mathcal F}}$ of the class $\bar{\mathcal F}$ of the (1+2)-dimensional degenerate Fokker–Planck equations of the general form

$$u_t = B(t,x,y)u_y + A^2(t,x,y)u_{xx} + A^1(t,x,y)u_x + A^0(t,x,y)u + C(t,x,y),$$

where A^0, A^1, A^2, B and C are arbitrary smooth functions of (t,x,y) with $A^2B_x \neq 0$, and prove that this class is normalized. We use the normalization of the class $\bar{\mathcal F}$ and its equivalence group $G^{\sim}_{\bar{\mathcal F}}$ in the course of computing the equivalence group of the class $\mathcal F$ that consists of the equations

$$u_t + xu_y = |x-\alpha|^\beta u_{xx},$$

parameterized by the arbitrary constants α and β. These equations have a wide range of applications in theoretical biology, especially for describing the evolution of cell populations.

Gauging α to zero by a wide subset of the action groupoid of the group $G^{\sim}_{\mathcal F}$, we obtain the class $\mathcal F'$ of equations of the form

$$u_t + xu_y = |x|^\beta u_{xx},$$

where β remains to be the only arbitrary element. The class $\mathcal F'$ is convenient for solving the group classification problem, and the group classification of the $\mathcal F$ reduces to that of the class $\mathcal F'$. By embedding $\mathcal F'$ in $\bar{\mathcal F}$, we compute the equivalence group $G^{\sim}_{\mathcal F'}$ and carry out the group classification $\mathcal F'$ up to the $G^{\sim}_{\mathcal F'}$-equivalence.

The equation $\mathcal F'_0$: $u_t + xu_y = u_{xx}$ is singled out among the above equations by its remarkable symmetry properties. Its essential Lie invariance algebra $\mathfrak g_0$ is nonsolvable and is of specific structure that has never been studied within the framework of subalgebra analysis. The optimal systems of one- and two-dimensional subalgebras of $\mathfrak g_0$ are constructed for purpose of carrying out Lie reductions of the equation $\mathcal F'_0$.