ARITHMETIC-HARMONIC MEAN INEQUALITY FOR SYMMETRIZATIONS OF CONVEX SETS

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The arithmetic-geometric-harmonic mean inequality states in the two-argument case

$$\min\{a,b\} \le \left(\frac{a^{-1}+b^{-1}}{2}\right)^{-1} \le \sqrt{ab} \le \frac{a+b}{2} \le \max\{a,b\} \tag{1}$$

for any $a, b > 0$, with equality in any of the inequalities if and only if $a = b$ (see [HLP, Sch]).

For any $X \subset \mathbb{R}^n$ let $\operatorname{conv}(X)$ denote the *convex hull*, i.e., the smallest convex set containing X. A *segment* is the convex hull of $\{x, y\} \subset \mathbb{R}^n$, which we abbreviate by $[x, y]$. For any $X, Y \subset \mathbb{R}^n$, $\rho \in \mathbb{R}$ let $X + Y = \{x + y : x \in X, y \in Y\}$ be the *Minkowski sum* of X and Y, and $\rho X = \{\rho x : x \in X\}$ the *ρ-dilatation* of X. We abbreviate $(-1)X$ by $-X$. The family of all *convex bodies* (full-dimensional compact convex sets) is denoted by $\mathcal{K}^n$ and for any $C \in \mathcal{K}^n$ we write $C^\circ = \{a \in \mathbb{R}^n : a^T x \le 1,\ x \in C\}$ for the *polar* of C.

One may identify means of numbers by means of segments via associating $a, b > 0$ with $[-a, a]$ and $[-b, b]$. Thus, e.g., the arithmetic mean of a and b is identified with $[-\frac{1}{2}(a+b), \frac{1}{2}(a+b)] = \frac{1}{2}([-a, a] + [-b, b])$. In general, the *arithmetic mean* of $K, C \in \mathcal{K}^n$ is defined by $\frac{1}{2}(K + C)$, the *minimum* by $K \cap C$, and the *maximum* by $\operatorname{conv}(K \cup C)$. Since polarity can be regarded as the higher-dimensional counterpart of the inversion operation $x \to 1/x$ (cf. [MR]), the *harmonic mean* of K and C is defined by $\left(\frac{1}{2}(K^\circ + C^\circ)\right)^\circ$.

Firey's extension of the harmonic-arithmetic mean inequality ([Fi]) states

Proposition 0.1. *Let $C, K \in \mathcal{K}^n$ with 0 in their interior. Then*

$$K \cap C \subset \left(\frac{K^\circ + C^\circ}{2}\right)^\circ \subset \frac{K + C}{2} \subset \operatorname{conv}(K \cup C), \tag{2}$$

with equality between any of the means if and only if $K = C$.

We analyze sharpness of the set-containment inequalities w.r.t. optimal containment: For $C, K \in \mathcal{K}^n$ we say K is *optimally contained* in C ($K \subset^{opt} C$), if $K \subset C$ and $K \not\subset \rho C + t$ for any $\rho \in [0, 1)$ and $t \in \mathbb{R}^n$.

Theorem 0.2. *Let $C, K \in \mathcal{K}^n$ with $0 \in \operatorname{int}(K \cap C)$. Then*

$$K \cap C \subset^{opt} \operatorname{conv}(K \cup C) \iff \left(\frac{1}{2}(K^\circ + C^\circ)\right)^\circ \subset^{opt} \frac{1}{2}(K + C).$$

If $C = -C + t$ for some $t \in \mathbb{R}^n$, we say C is *symmetric*, and if $C = -C$, we say C is *0-symmetric*. The family of 0-symmetric convex bodies is denoted by $\mathcal{K}^n_0$.

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We focus on optimal containments of means of C and $-C$ for a convex body C, which are all symmetrizations of C. Symmetrizations are frequently used in convex geometry, e.g., as extreme cases of a variety of geometric inequalities. Consider, e.g., the Bohnenblust inequality [Bo], which bounds the ratio of the circumradius and the diameter of convex bodies in arbitrary normed spaces. The equality case in this inequality is reached in normed spaces with $S \cap (-S)$ or $\frac{1}{2}(S-S)$ as their unit balls [BK], where S denotes an n-simplex with center of gravity in 0. These means also appear in characterizations of spaces for which C is complete or reduced [BGJM, Prop. 3.5 – 3.10].

The most common choice of an asymmetry measure and a corresponding center are the *Minkowski asymmetry* of $C \in \mathcal{K}^n$, which is defined by

$$s(C) := \inf\{\rho > 0 : C - c \subset \rho(C - c), c \in \mathbb{R}^n\},$$

and the (not necessarily unique) *Minkowski center* of C, which is any $c \in \mathbb{R}^n$ fulfilling $C - c \subset s(C)(c - C)$ [Gr, BG]. If the Minkowski center is 0, we say C is *Minkowski centered*. Note that $s(C) \in [1, n]$, with $s(C) = 1$ if and only if C is symmetric, and $s(C) = n$ if and only if C is an n-dimensional simplex [Gr]. Moreover, the Minkowski asymmetry $s : \mathcal{K}^n \to [1, n]$ is continuous w.r.t. the Hausdorff metric (see [Gr], [Sch] for some basic properties) and invariant under non-singular affine transformations. We believe that the Minkowski asymmetry is most suitable for studying optimal containments and consequently focus on Minkowski centered convex sets.

The classical norm relations $\|x\|_\infty \le \|x\|_2 \le \|x\|_1$ with $x \in \mathbb{R}^n$ can be naturally reversed by the inequalities $\|x\|_1 \le \sqrt{n}\|x\|_2 \le n\|x\|_\infty$, which both transfer to left-to-right optimal containments between the corresponding unit ball of these ℓ_p-spaces. Similarly, we consider the norms induced by the means of K and C. Doing so, (2) can be read as follows:

$$\|x\|_{\mathrm{conv}(K \cup C)} \le \|x\|_{\frac{K+C}{2}} \le \|x\|_{\left(\frac{K^\circ + C^\circ}{2}\right)^\circ} \le \|x\|_{K \cap C}. \tag{3}$$

In order to reverse this chain of inequalities, we need to provide a chain of (optimal) inclusions, which is reverse to (2). While this is not possible for general convex bodies, since the scaling factors of the reverse inclusions cannot be bounded, but assuming the Minkowski centeredness of the considered body, this problem can be fixed for the symmetrizations.

Theorem 0.3. *Let $C \in \mathcal{K}^n$ be Minkowski centered. Then*

(i) $\mathrm{conv}(C \cup (-C)) \subset^{opt} s(C)(C \cap (-C))$,
(ii) $\mathrm{conv}(C \cup (-C)) \subset^{opt} \frac{2s(C)}{s(C)+1} \frac{C-C}{2}$,
(iii) $\left(\frac{C^\circ - C^\circ}{2}\right)^\circ \subset^{opt} \frac{2s(C)}{s(C)+1}(C \cap (-C))$,
(iv) $\frac{C-C}{2} \subset^{opt} \frac{s(C)+1}{2}(C \cap (-C))$, *and*
(v) $\mathrm{conv}(C \cup (-C)) \subset^{opt} \frac{s(C)+1}{2} \left(\frac{C^\circ - C^\circ}{2}\right)^\circ$.
(vi) $\frac{C-C}{2} \subset \frac{s(C)+1}{2} \left(\frac{C^\circ - C^\circ}{2}\right)^\circ$, *and for all $s \in [n]$ there exists a Minkowski centered $C \in \mathcal{K}^n$ with $s(C) = s$, such that the containment is optimal.*

We proceed with a stability result. First we introduce several parameters.

$$\psi := \psi(n,s) := \frac{(n-s+1)(s+1)}{1-n(n-s)(n+s(n+1))} - n,$$
$$\mu := \mu(n,s) = \frac{n+1}{s+1}\left(1 - \frac{s(n+1)(n-s)}{1-n(n-s)}\right),$$
$$\gamma_1 := \gamma_1(n) := \frac{1}{2}(n-1+\sqrt{(n-2)n+5}),$$
$$\gamma_2 := \gamma_2(n) := \frac{n^4+n^3+2n^2+\sqrt{n^8+6n^7+17n^6+28n^5+28n^4+12n^3-4n^2-12n-4}}{2(n^3+2n^2+3n+1)},$$
$$\gamma_3 := \gamma_3(n) := \frac{n^4+3n^3+2n^2+1+\sqrt{n^8+6n^7+13n^6+8n^5-14n^4-22n^3+8n+1}}{2(n^3+2n^2+2n)}.$$

One can check that $n-\frac{1}{n} < \gamma_2 < \gamma_3 < n$ and that both ψ and μ become one in case $n = s$. Moreover, we will see that $\psi\frac{n}{n+1} > 1$ for all $s > \gamma_2$, while $\mu\psi\frac{n(n+2)}{(n+1)^2} < 1$ for all $s > \gamma_3$.

Theorem 0.4. *Let n be even and $C \in \mathcal{K}^n$ be Minkowski centered with $s(C) = s$. Then*

(i) $C \cap (-C) \subset \psi\,\frac{n}{n+1}\operatorname{conv}(C \cup (-C))$, *if* $s \geq \gamma_2(n)$, *and*

(ii) $\left(\frac{C^\circ + (-C)^\circ}{2}\right)^\circ \subset \mu\psi\,\frac{n(n+2)}{(n+1)^2}\,\frac{C-C}{2}$, *if* $s \geq \gamma_3(n)$.

We determine the smallest number $\gamma(n) \in [n-1, n]$ such that for every Minkowski centered $C \in \mathcal{K}^n$ with $s(C) \geq \gamma(n)$ the harmonic mean of C and $-C$ is not optimally contained in their arithmetic mean and call it the *asymmetry threshold of means*

Theorem 0.5. *Let n be even. Then*

$$n-1 < \gamma_1 \leq \gamma(n) \leq \gamma_2 < n.$$

The asymmetry threshold provides us with a lower bound for the values of s such that (2) cannot be left-to-right optimal. In the following we want to go one step further and determine the possible values for the contraction factors $\alpha(s)$, $\beta(s)$ for which the minimum is optimally contained in the according contraction of the maximum and for which the harmonic mean is optimally contained in the contraction of the arithmetic mean, respectively.

Theorem 0.6. *Let $C \in \mathcal{K}^n$ be Minkowski centered with $s(C) = s$.*

a) Let $\alpha(s) \in \mathbb{R}$ such that $C \cap (-C) \subset^{opt} \alpha(s)\operatorname{conv}(C \cup (-C))$ and $\alpha_1(s)$, $\alpha_2(s)$ be the optimal lower and upper bounds on $\alpha(s)$, respectively. Then
   - *(i) $\alpha_1(s) \geq \frac{2}{s+1}$ with equality at least for $s \leq 2$.*
   - *(ii) $\alpha_2(s) = 1$ for $s \leq \gamma_1$, $\alpha_2(s) \leq \psi\frac{n}{n+1}$, for $s > \gamma_2$ and $\alpha_2(s) \geq \frac{s}{s^2-1}$ for $s \leq 2$.*

b) Let $\beta(s) \in \mathbb{R}$ such that $\left(\frac{1}{2}(C^\circ - C^\circ))\right)^\circ \subset^{opt} \beta(s)\,\frac{1}{2}(C-C)$ and $\beta_1(s), \beta_2(s)$ be the optimal lower and upper bounds on $\beta(s)$, respectively. Then
   - *(i) $\beta_1(s) \geq \frac{4s}{(s+1)^2}$ with equality at least for $s \leq 2$.*
   - *(ii) $\beta_2(s) = 1$ for $s \leq \gamma_1$, $\beta_2(s) \leq \mu\psi\frac{n(n+2)}{(n+1)^2}$ for $s > \gamma_3$ and $\beta_2(s) \geq \max\left\{\frac{s}{s^2-1}, \frac{4s}{(s+1)^2}\right\}$ for $s \leq 2$.*

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