Symmetries and exact solutions of isothermal no-slip drift flux model

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We carry out extended group analysis of the isothermal no-slip drift flux model given by the system of differential equations

$$\rho^1_t + u\rho^1_x + u_x\rho^1 = 0, \quad \rho^2_t + u\rho^2_x + u_x\rho^2 = 0, \quad \big(\rho^1 + \rho^2\big)(u_t + uu_x) + a^2\big(\rho^1_x + \rho^2_x\big) = 0.$$

First of all, it was noted that this system is inconvenient for group analysis. Introducing the new dependent variables $v = \ln\big(\rho^1 + \rho^2\big)$ and $w = \rho^1/\rho^2$, we obtain the equivalent system $\mathcal{S}$:

$$u_t + uu_x + v_x = 0, \quad v_t + uv_x + u_x = 0, \quad w_t + uw_x = 0.$$

Its maximal Lie invariance algebra $\mathfrak{g}$ turns out to be infinite-dimensional,

$$\mathfrak{g} = \langle t\partial_t + x\partial_x,\ t\partial_x + \partial_u,\ \partial_t,\ \partial_x,\ \partial_v,\ \kappa(w)\partial_w\rangle,$$

where κ runs through the set of smooth functions of w.

Using the combined algebraic method we find the complete point symmetry group G of the system $\mathcal{S}$ consisting of the transformations of the form

$$\tilde t = T^1t + T^0, \quad \tilde x = T^1x + T^1U^0t + X^0, \quad \tilde u = u + U^0, \quad \tilde v = v + V^0, \quad \tilde w = W(w),$$

where T^0, T^1, X^0, U^0 and V^0 are arbitrary constants and $W(w)$ runs through the set of smooth functions of w with $T^1W_w \neq 0$.

Following the standard procedure of Lie reduction, we obtain optimal lists of one- and two-dimensional subalgebras of the algebra $\mathfrak{g}$, using which we construct ansatzes for (u, v, w). Substituting them into the system $\mathcal{S}$ we find families of its invariant and partially invariant solutions.

Since the system $\mathcal{S}$ is semi-coupled, it is possible to consider separately the subsystem $\mathcal{S}_0$ of the first two equations. Such a choice is justified by the fact that the system $\mathcal{S}_0$ has wider symmetry than its counterpart $\mathcal{S}$, namely its maximal Lie invariance algebra $\mathfrak{g}_0$ is spanned by vector fields

$$\mathcal{D} = t\partial_t + x\partial_x, \quad \mathcal{G} = t\partial_x + \partial_u, \quad \mathcal{P}\big(\tau^0, \xi^0\big) = \tau^0(u,v)\partial_t + \xi^0(u,v)\partial_x,$$
$$\mathcal{P}^v = \partial_v, \quad \mathcal{J} = (2tu - x)\partial_t + \big(tu^2 - 2tv - t\big)\partial_x - 2v\partial_u - 2u\partial_v,$$

where τ^0, ξ^0 run through the solution set of the system $u\tau^0_u - \tau^0_v = \xi^0_u$ and $u\tau^0_v - \tau^0_u = \xi^0_v$. Besides, the structure of $\mathfrak{g}_0$ makes an allusion that the system $\mathcal{S}_0$ is linearized via the two-dimensional hodograph transformation, with $(p, q) = (t, x)$, $(y, z) = (u, v)$ being the new dependent and independent variables, respectively. This leads to the potential system

$$q_z - yp_z + p_y = 0, \quad -q_y - p_z + yp_y = 0$$

of the telegraph equation $p_{yy} = p_{zz} + p_z$. After the change $\tilde p = e^{-z/2}p$ we obtain the famous Klein–Gordon equation $\tilde p_{yy} = \tilde p_{zz} - \tilde p/4$. As the latter equation is well studied, we can use a lot of its known solutions and recover q afterwards. Rewriting the third equation of the system $\mathcal{S}$ in the new variables, we find solutions of the initial system $\mathcal{S}$ in parameterized form. Two examples are provided to underscore advantages of the above approach.

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