Solitons, compactons and other invariant wave patterns in the generalized convection–reaction–diffusion equation

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Plan of the talk:

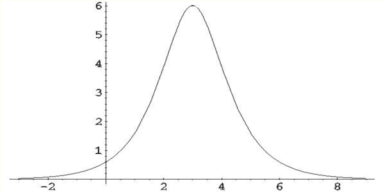
- Solitons and compactons from the geometric point of view
- Solitons, compactons and other patterns within covection-reaction-diffusion equation:
- results of
  - qualitative analysis
  - and numerical simulation.

Korteveg-de Vries (KdV) hierarchy

$$K(m) = u_t + \beta\, u^m\, u_x + u_{xxx} = 0. \tag{1}$$

Solitary wave solution for m=1 [9]:

$$u = \frac{3\,V}{\beta}\, sech^2\left[\sqrt{\frac{V}{4}}(x - V\,t)\right]$$



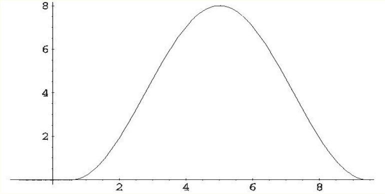
Rosenau-Hyman generaization of KdV hierarchy

$K(m, n)$ hierarchy(Rosenau, Hyman, 1993) [10]:

$$K(m, n) = u_t + \alpha\,(u^m)_x + \beta\,(u^n)_{xxx} = 0, \qquad m \geq 2, \quad n \geq 2. \tag{2}$$

Solitary wave solution, corresponding to $\alpha = \beta = 1$ and $m = n = 2$ [10]:

$$u = \begin{cases} \frac{4V}{3}\,\cos^2\frac{\xi}{4} \text{ when } |\xi| \leq 2\pi, \\ 0 \text{ when } |\xi| > 2\pi, \end{cases} \qquad \xi = x - V\,t. \tag{3}$$



Solitons and compactons from geometric point of view [11].

Reduction of KdV equation

In order to describe solitons, we use the TW reduction

$$u(t, x) = U(\xi), \;\text{ with }\; \xi = x - V\, t.$$

Inserting $U(\xi)$ into the KdV equation

$$u_t + \beta\, u\, u_x + u_{xxx} = 0$$

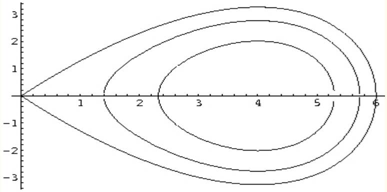
we get, after one integration, Hamiltonian system:

$$\dot{U}(\xi) = -W(\xi) = -H_W, \tag{4}$$

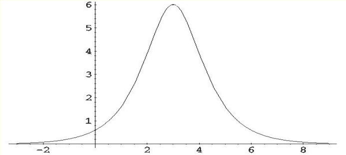
$$\dot{W}(\xi) = \frac{\beta}{2}\, U(\xi) \left(U(\xi) - \frac{2\, v}{\beta} \right) = H_U.$$

$$H = \frac{1}{2} \left(W^2 + \frac{\beta}{3}\, U^3 - V\, U^2 \right). \tag{5}$$

Level curves of the Hamiltonian $H = \frac{1}{2}\,\left(W^2 + \frac{\beta}{3}\,U^3 - V\,U^2\right) = K = \text{const}$



Solution to KdV, corresponds to the homoclinic trajectory (HCL). Being bi-asymptotic to a saddle $(0,0)$, HCL is penetrated in infinite "time"!



Reduction of $K(m, n)$ equation

Inserting ansatz $u(t, x) = U(\xi) \equiv U(x - V\,t)$ into

$$K(m, n) \equiv u_t + \alpha\,(u^m)_x + \beta\,(u^n)_{xxx} = 0,$$

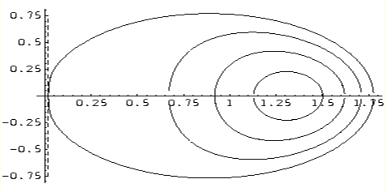
we obtain, after one integration and employing the integrating multiplier $\varphi[U] = U^{n-1}$, the Hamiltonian system:

$$\begin{cases} n\,\beta\,U^{2(n-1)}\,\frac{dU}{d\xi} = -n\,\beta\,U^{2(n-1)}\,W = -H_W, \\ n\,\beta\,U^{2(n-1)}\,\frac{dW}{d\xi} = U^{n-1}\left[-v\,U + \alpha U^m + n(n-1)\,\beta\,U^{n-2}W^2\right] = H_U \end{cases}$$

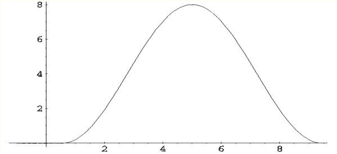
Every trajectory of the above system can be identified with some level curve $H = const$ of the Hamiltonian

$$H = \frac{\alpha}{m+n}\,U^{m+n} - \frac{v}{n+1}\,U^{n+1} + \frac{\beta\,n}{2}\,U^{2(n-1)}\,W^2.$$

Level curves of the Hamiltonian
$H = \frac{\alpha}{m+2} U^{m+2} - \frac{v}{3} U^3 + \beta U^2 W^2 = L = \text{const}$, corresponding to the reduced $K(m, 2)$ equation



Solution to $K(2, 2)$ equation corresponding to HCL. Since HCL is bi-asymptotic to a saddle lying on the singular line $n \beta U^{2(n-1)} = 0$, the "time" needed to penetrate HCL is finite !



Conclusions:

Soliton-like TW solution is represented in the phase space of the factorized system by the trajectory bi-asymptotic to a saddle.

Compacton-like TW solution is represented by the trajectory bi-asymptotic to a saddle, **lying on a singular manifold of dynamical system**

Modeling system [1–8] and its factorization

$$u_t + u\, u_x - \kappa\, (u^n\, u_x)_x = (u - U_1)\, \varphi(u), \qquad U_1 > 0. \tag{6}$$

We are going to analyze the set of TW solutions to (6), having the following form:

$$u(t,\, x) = U(\xi) \equiv U\,(x - V\,t)\,. \tag{7}$$

Inserting ansatz (7) into the GBE one can obtain, after some manipulation, the following dynamical system:

$$\Delta(U)\,\dot{U} = \Delta(U)\,W, \tag{8}$$

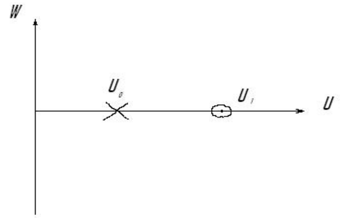
$$\Delta(U)\,\dot{W} = -\left[(U - U_1)\,\varphi(U) + \kappa\, n\, U^{n-1}\, W^2 + (V - U)\; W\right],$$

where $\Delta(U) = \kappa\, U^n$.

Further assumptions: concerning function $\varphi(U) = 0$

- We assume that $\exists\, U_0 \in [0, U_1)\, : \varphi(U_0) = 0$.
- We assume that function $\varphi(U)$ does not change its sign within the open interval $(U_0,\ U_1)$.

Under these assumption our system has two stationary points $(U_0, 0)$ and $(U_1, 0)$ lying on the horizontal axis of the phase space $(U,\ W)$, and no any other stationary point inside the segment $(U_0,\ U_1 + L)$ for some $L > 0$.



Our further strategy:

- to state the condition for which the stable limit cycle appearance in proximity of $(U_1, 0)$
- to state further conditions, assuring that the other point $(U_0, 0)$ is a saddle.
- to check numerically the possibility of homoclinic bifurcation appearing as the bifurcation parameter V changes.

In case when the singular manifold $\Delta(U) = 0$ contains the saddle $(U_0, 0)$ (i.e. $\Delta(U_0) = 0$) the homoclinic loop is the image of either compacton, or soliton.

Local asymptotic analysis [16] of solution in proximity of the stationary point $(U_0, 0)$ enables to distinguish the compactly-supported solution.

Creation of a stable limit cycle [14,15]

Analysis of normal form [14,15] built in proximity of the critical point $(U_1,\ 0)$ enables to formulate the following statement concerning the limit cycle appearance:

Theorem 1. *If $\Delta(U_1)$ and $\varphi(U_1)$ are both positive and inequality*

$$U_1\,\dot{\varphi}(U_1) + n\varphi(U_1) > 0. \tag{9}$$

is fulfilled then in proximity of the critical value of the wave pack velocity $V_{cr} = U_1$ a stable limit cycle appears.

Result of local asymptotic analysis near the saddle point (0, 0)

Further analysis is performed to in case when $\varphi(u) = u^m$:

$$u_t + u\,u_x - \kappa\,(u^n\,u_x)_x = (u - U_1)\,u^m, \qquad U_1 > 0. \qquad (10)$$

The other stationary point is placed at the origin!

Proposition 1. *The homoclinic loop bi-asymptotic to the saddle point* (0, 0) *corresponds to compacton -like solution of equation (10) for any natural* n *if* $m < 1$.

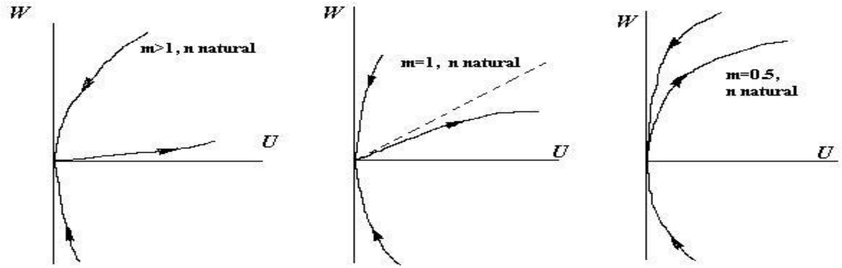


Figure: Vicinity of the origin for various combinations of the parameters $m,\ n$

What sort of TW corresponds to the homoclinic loop in case when $m \geq 1$?

Asymptotic arguments [16] enable to state that the "tail" of TW in this case spreads up to $-\infty$ whereas the front sharply ends, forming some sort of a semi-compacton or in other words, a shock wave with infinitely long relaxing "tail"

Numerical investigation of factorized system

Case 1. $\kappa = 1, \ m = \frac{1}{2}, \ n = 1, \ U_1 = 3.$

Numerical simulations show the appearance of stable limit cycle when V is slightly less than $V_{cr} = U_1 = 3.$ The radius of the limit cycle grows as V decreases

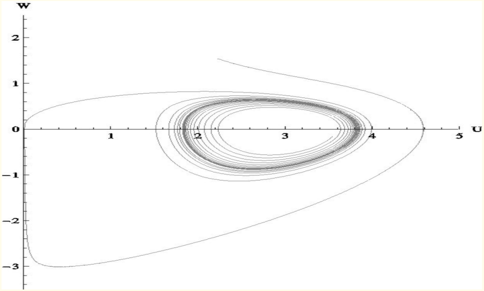


Figure: Phase portrait corresponding to $V = 2.82$

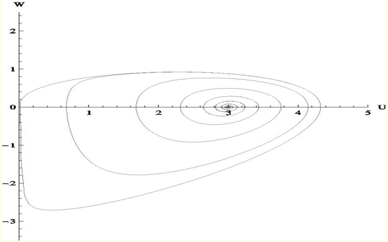


Figure: Homoclinic bifurcation occurred at $V = 2.77786585$

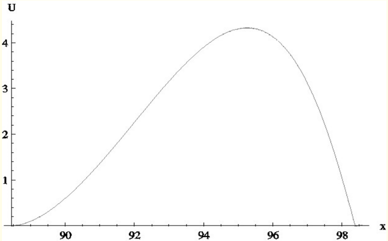
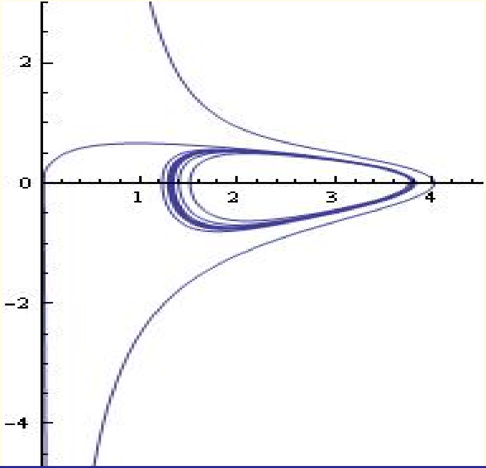
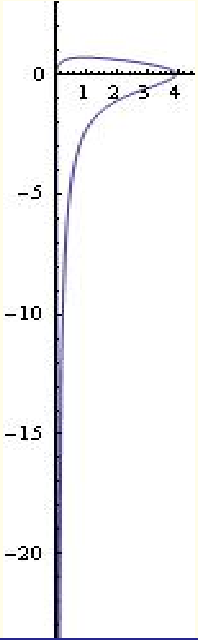


Figure: Compactly-supported solution to the initial PDE

Case 2. $\kappa = 1$, $m = \frac{1}{2}$, $n = 2$, $U_1 = 3$. Scenario is the same. Peculiarity: a huge asymmetry between the ingoing and outgoing separatrices is caused by the growth of parameter n



Case 2. $\kappa = 1$, $m = \frac{1}{2}$, $n = 2$, $U_1 = 3$: the homoclinic loop



Peculiarity: the face of the compacton has become sharp.

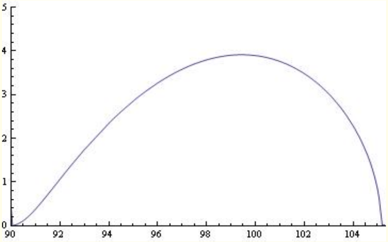


Figure: Compacton corresponding to the above homoclinic loop

Case 3. $\kappa = 1,\ m = 1,\ n = 1,\ U_1 = 3.$

Scenario is the same: limit cycle $\mapsto$ homoclinic bifurcation.

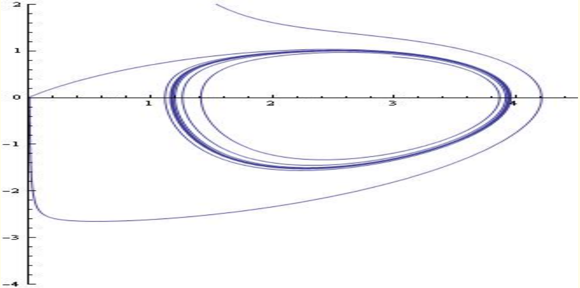


Figure: Phase portrait corresponding to $V = 2.8$

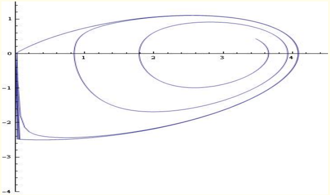


Figure: Homoclinic bifurcation occuring at $V \simeq 2.453875$

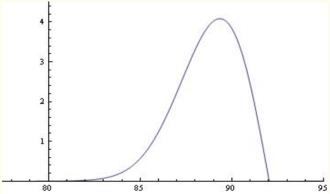


Figure: Solitary wave corresponding to the homoclinic loop

Case 4. $\kappa = 1$, $m = 2$, $n = 1$, $U_1 = 3$.

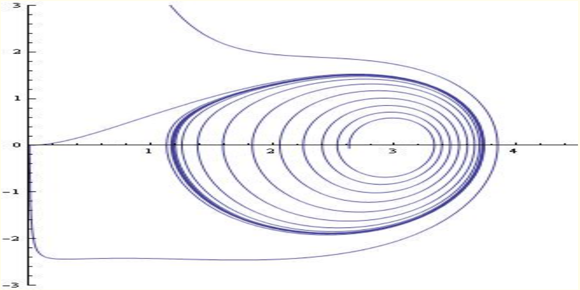


Figure: Phase portrait corresponding to $V = 2.8$

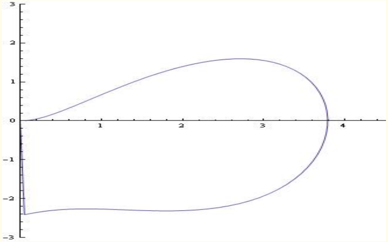


Figure: Homoclinic bifurcation occurred at $V = 2.453875$

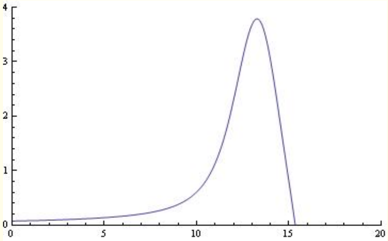
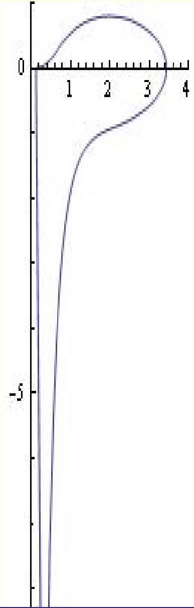


Figure: Solitary wave corresponding to the above homoclinic loop

Case 5. Further growth of nonlinearity ($m = 3,\ n = 1$) leads to the huge asymmetry of the homoclinic loop



and appearance of wave pattern reminding shock or detonation wave

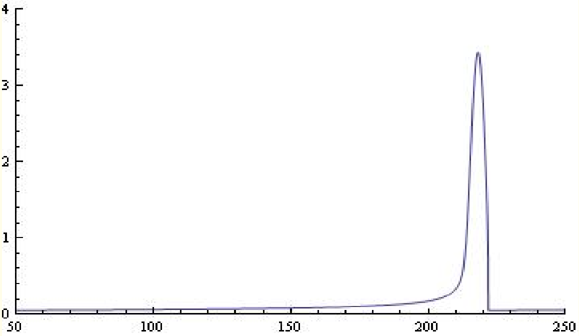


Figure: Solitary wave corresponding to $m = 3$, $n = 1$, $V = 2.53$

Final remarks

1. Existence of compacton-like and soliton-like solutions within the convection-reaction-diffusion model is possible for wide range of the parameters' values.
2. Existence of compacton-like solutions is possible if the diffusion coefficient is a function of dependent variable u.
3. The shape of solitary wave strongly depends on degree of nonlinearity
4. Presented results are not completely rigorous.
5. To obtain above solutions **symmetry-related methods [17,18,19] can be applied**.
6. Yet it is little chances to obtain them for all possible values of the parameters for the source equation is not completely integrable
7. It would be desired to apply the computer-assisted proofs for these problems.
8. The problem of prime importance is the investigation of the stability and asymptotic features [22,23,11] of compactons and solitons appearing in convection-reaction-diffusion equation.

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