Analysis of the spectrum of some Hamiltonians based on a generalization of the Dolan-Grady condition

Tobias Verhulst Jan Naudts Ben Anthonis

Universiteit Antwerpen (Belgium)

22/6/2009

Outline

Introduction
- Dolan-Grady condition
- Our condition

The theory
- Ladder operators
- The hierarchy
- Construction theorem

Examples
- 1D Hubbard model
- The Jaynes-Cummings model

Ongoing research...

The Dolan-Grady condition

The condition

- In 1982 Dolan and Grady introduced a method for constructing conserved charges for a Hamiltonian H.

The Dolan-Grady condition

The condition

- In 1982 Dolan and Grady introduced a method for constructing conserved charges for a Hamiltonian H.
- If $H = KB + \Gamma\tilde{B}$, and B, $\tilde{B}$ satisfy the condition:

$$[B,[B,[B,\tilde{B}]]] = 16[B,\tilde{B}]$$

The Dolan-Grady condition

The condition

- In 1982 Dolan and Grady introduced a method for constructing conserved charges for a Hamiltonian H.
- If $H = KB + \Gamma\tilde{B}$, and B, $\tilde{B}$ satisfy the condition:

$$[B,[B,[B,\tilde{B}]]] = 16[B,\tilde{B}]$$

What they did with it

then there exists an infinite set of conserved charges Q_{2n}:

$$Q_{2n} = K(W_{2n} - \tilde{W}_{2n-2}) + \Gamma(\tilde{W}_{2n} - W_{2n-2}) \qquad Q_0 = H$$

$$W_{2n} = -\frac{1}{8}[B,[\tilde{B},W_{2n-2}]] - \tilde{W}_{2n-2} \qquad W_0 = 0$$

Our condition

The condition

Given a Hamiltonian H, suppose there exists an Hermitian operator M satisfying

$$[[[H, M], M], M] = \gamma^2 [H, M]$$

for some $\gamma \neq 0$.

Our condition

The condition

Given a Hamiltonian H, suppose there exists an Hermitian operator M satisfying

$$[[[H, M], M], M] = \gamma^2 [H, M]$$

for some $\gamma \neq 0$.

Some remarks

- M does not have to be some part of H, (altough it might be).

Our condition

The condition

Given a Hamiltonian H, suppose there exists an Hermitian operator M satisfying

$$[[[H, M], M], M] = \gamma^2 [H, M]$$

for some $\gamma \neq 0$.

Some remarks

- M does not have to be some part of H, (altough it might be).
- Given H, M is not unique.

Our condition

The condition

Given a Hamiltonian H, suppose there exists an Hermitian operator M satisfying

$$[[[H, M], M], M] = \gamma^2 [H, M]$$

for some $\gamma \neq 0$.

Some remarks

- M does not have to be some part of H, (altough it might be).
- Given H, M is not unique.
- γ can be chosen equal to one.

Ladder operators

Given H and M satisfying $[[[H, M], M], M] = [H, M]$, define:

$$R = \frac{1}{2}[[H, M], M] + \frac{1}{2}[H, M]$$

Ladder operators

Given H and M satisfying $[[[H, M], M], M] = [H, M]$, define:

$$R = \frac{1}{2}[[H, M], M] + \frac{1}{2}[H, M]$$

then

$$[R, M] = R$$
$$[R^\dagger, M] = -R^\dagger$$

Ladder operators

Given H and M satisfying $[[[H, M], M], M] = [H, M]$, define:

$$R = \frac{1}{2}[[H, M], M] + \frac{1}{2}[H, M]$$

then

$$[R, M] = R$$
$$[R^\dagger, M] = -R^\dagger$$

- R and $R^\dagger$ act like annihilation and creation operators, M acts as a counting operator.

Ladder operators

Given H and M satisfying $[[[H,M],M],M]=[H,M]$, define:

$$R=\frac{1}{2}[[H,M],M]+\frac{1}{2}[H,M]$$

then

$$[R,M]=R$$
$$[R^\dagger,M]=-R^\dagger$$

- R and $R^\dagger$ act like annihilation and creation operators, M acts as a counting operator.
- We call the algebra generated by R, $R^\dagger$ and M the ladderalgebra $\mathcal{L}$.

The hierarchy

- The Hamiltonian can be written as

$$H = H_{\mathrm{ref}} + R + R^{\dagger}$$

The hierarchy

- The Hamiltonian can be written as

$$H = H_{\mathrm{ref}} + R + R^{\dagger}$$

- The eigenvalue equation $H\psi = E\psi$ can be projected on the eigenspace of M. This gives a hierarchy of equations:

$$(H_{\mathrm{ref}} - E)(\mathcal{P}_{\mu}\psi) + R(\mathcal{P}_{\mu+1}\psi) + R^{\dagger}(\mathcal{P}_{\mu-1}\psi) = 0$$

Construction theorem

- Given a set $\{\phi_i\}$ of eigenstates of M

Construction theorem

- Given a set $\{\phi_i\}$ of eigenstates of M with different eigenvalues $\{\lambda_i\}$

Construction theorem

- Given a set $\{\phi_i\}$ of eigenstates of M with different eigenvalues $\{\lambda_i\}$ and with $R\phi_i = \xi_{i-1}\phi_{i-1}$ and $R^\dagger\phi_i = \xi_i\phi_{i+1}$.

Construction theorem

- Given a set $\{\phi_i\}$ of eigenstates of M with different eigenvalues $\{\lambda_i\}$ and with $R\phi_i = \xi_{i-1}\phi_{i-1}$ and $R^\dagger\phi_i = \xi_i\phi_{i+1}$.
- Then there exists h_i^k such that $\psi^k = \sum_i h_i^k\phi_i$ are eigenstates of H

Construction theorem

- Given a set $\{\phi_i\}$ of eigenstates of M with different eigenvalues $\{\lambda_i\}$ and with $R\phi_i = \xi_{i-1}\phi_{i-1}$ and $R^\dagger\phi_i = \xi_i\phi_{i+1}$.
- Then there exists h_i^k such that $\psi^k = \sum_i h_i^k\phi_i$ are eigenstates of H with eigenvalues $E^k = \lambda_0 + \xi_0\frac{h_1^k}{h_0^k}$.

Construction theorem

- Given a set $\{\phi_i\}$ of eigenstates of M with different eigenvalues $\{\lambda_i\}$ and with $R\phi_i = \xi_{i-1}\phi_{i-1}$ and $R^\dagger\phi_i = \xi_i\phi_{i+1}$.
- Then there exists h_i^k such that $\psi^k = \sum_i h_i^k\phi_i$ are eigenstates of H with eigenvalues $E^k = \lambda_0 + \xi_0\frac{h_1^k}{h_0^k}$.
- The coefficients h_i^k can be calculated from the hierarchy

$$\forall j : h_j^k(\lambda_j - E^k) + h_{j+1}^k\xi_j + h_{j-1}^k\xi_{j-1} = 0$$

Construction theorem

- Given a set $\{\phi_i\}$ of eigenstates of M with different eigenvalues $\{\lambda_i\}$ and with $R\phi_i = \xi_{i-1}\phi_{i-1}$ and $R^\dagger\phi_i = \xi_i\phi_{i+1}$.
- Then there exists h_i^k such that $\psi^k = \sum_i h_i^k \phi_i$ are eigenstates of H with eigenvalues $E^k = \lambda_0 + \xi_0 \frac{h_1^k}{h_0^k}$.
- The coefficients h_i^k can be calculated from the hierarchy

$$\forall j : h_j^k(\lambda_j - E^k) + h_{j+1}^k \xi_j + h_{j-1}^k \xi_{j-1} = 0$$

- We call the states $\{\psi^k\}$ a multiplet.

Construction theorem

- The $\{\phi_i\}$ span a vectorspace on which there is a N-dimensional simple representation of $\mathcal{L}$ (of course, $\{\psi^k\}$ span the same space).

Construction theorem

- The $\{\phi_i\}$ span a vectorspace on which there is a N-dimensional simple representation of $\mathcal{L}$ (of course, $\{\psi^k\}$ span the same space).
- If all simple representations of $\mathcal{L}$ are of this form, we call M ideal.

Construction theorem

- The $\{\phi_i\}$ span a vectorspace on which there is a N-dimensional simple representation of $\mathcal{L}$ (of course, $\{\psi^k\}$ span the same space).
- If all simple representations of $\mathcal{L}$ are of this form, we call M ideal.
- If M is ideal, all eigenvectors of H can be constructed by the above procedure.

Construction theorem

- The $\{\phi_i\}$ span a vectorspace on which there is a N-dimensional simple representation of $\mathcal{L}$ (of course, $\{\psi^k\}$ span the same space).
- If all simple representations of $\mathcal{L}$ are of this form, we call M ideal.
- If M is ideal, all eigenvectors of H can be constructed by the above procedure.

Thus, if M is ideal, the eigenstates (and eigenvalues) can be constructed and classified into multiplets corresponding to one of the simple representations of $\mathcal{L}$.

The Hubbard model

Consider the Hubbard Hamiltonian

$$H(\alpha) = -\sum_{i,j=1}^{N} t_{ij} \sum_{\sigma=\uparrow,\downarrow} b_{i,\sigma}^{\dagger} b_{j,\sigma} + \alpha \sum_{k=1}^{N} \hat{n}_{k,\uparrow} \hat{n}_{k,\downarrow}$$

The Hubbard model

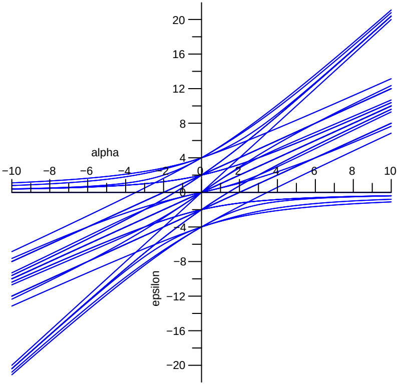
Consider the Hubbard Hamiltonian

$$H(\alpha) = -\sum_{i,j=1}^{N} t_{ij} \sum_{\sigma=\uparrow,\downarrow} b_{i,\sigma}^{\dagger} b_{j,\sigma} + \alpha \sum_{k=1}^{N} \hat{n}_{k,\uparrow} \hat{n}_{k,\downarrow}$$

Assume:

1. one dimensional lattice
2. periodic boundary conditions
3. nearest neighbour hopping

The eigenvectors and eigenvalues for small latices are known[1,2]. The spectrum for $N = 4$ at half filling and with $S = 0$ is:



[1] R. Schumann, Ann. Phys. (Leipzig) **11**, 49 (2002), cond-mat/0101476v1

[2] C. Noce, M. Cuoco, Phys. Rev. B **54**, 13047 (1996)

Counting operator

For this example there exists an ideal counting operator:

$$M = \sum_i n_{i,\uparrow} n_{i,\downarrow} \left(1 + \sum_{j=-1,1} \sum_{\sigma=\uparrow,\downarrow} n_{i+j,\sigma} n_{i-j,\bar{\sigma}} \mathcal{F} \right)$$

where $\mathcal{F}$ substitutes empty $\leftrightarrow$ doubly occupied places.

Counting operator

For this example there exists an ideal counting operator:

$$M = \sum_i n_{i,\uparrow} n_{i,\downarrow} \left(1 + \sum_{j=-1,1} \sum_{\sigma=\uparrow,\downarrow} n_{i+j,\sigma} n_{i-j,\bar{\sigma}} \mathcal{F} \right)$$

where $\mathcal{F}$ substitutes empty $\leftrightarrow$ doubly occupied places.

What does it count?

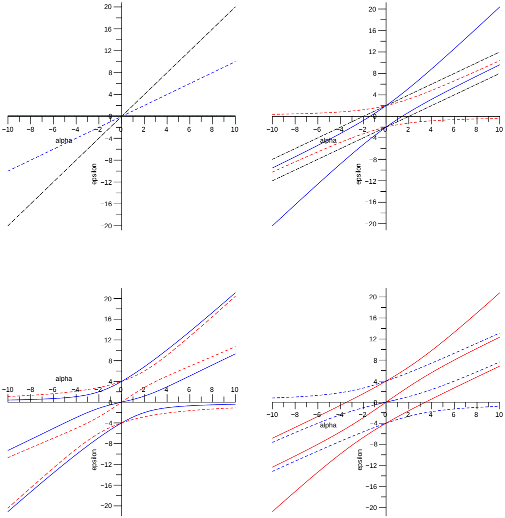
M counts the number of pairs plus or minus the number of pairs with no empty space nex to it, depending on the symmetry of the state under $\mathcal{F}$.

The multiplets

The eigenvalues can be classified into the following multiplets:

1. three singlets (one with six-fold degeneracy)
2. three doublets (two with two-fold and one with four-fold degeneracy)
3. four triplets

The multiplets



The Jaynes-Cummings model

- Consider the Hamiltonian

$$H = \frac{1}{2}\hbar\omega\{b^\dagger, b\} + \frac{1}{2}\hbar\omega_0\sigma_z + \hbar\kappa(b^\dagger\sigma_- + b\sigma_+)$$

The Jaynes-Cummings model

- Consider the Hamiltonian

$$H = \frac{1}{2}\hbar\omega\{b^\dagger, b\} + \frac{1}{2}\hbar\omega_0\sigma_z + \hbar\kappa(b^\dagger\sigma_- + b\sigma_+)$$

- As an ideal M we one can use the non-interacting part of H:

$$M = \frac{1}{2}\hbar\omega\{b^\dagger, b\} + \frac{1}{2}\hbar\omega_0\sigma_z$$

The Jaynes-Cummings model

- Consider the Hamiltonian

$$H = \frac{1}{2}\hbar\omega\{b^\dagger, b\} + \frac{1}{2}\hbar\omega_0\sigma_z + \hbar\kappa(b^\dagger\sigma_- + b\sigma_+)$$

- As an ideal M we one can use the non-interacting part of H:

$$M = \frac{1}{2}\hbar\omega\{b^\dagger, b\} + \frac{1}{2}\hbar\omega_0\sigma_z$$

- The spectrum then consits of one singlet and an infinite number of doublets.

Ongoing research...

Mathematical questions

- What is the set of possible counting operators M, given H?
- Given H, under what conditions is there at least one ideal M? And how can it be found?

Ongoing research...

Mathematical questions

- What is the set of possible counting operators M, given H?
- Given H, under what conditions is there at least one ideal M? And how can it be found?

Practical use

- Use this theory to construct eigenstates, for example in the 2D-Hubbard model.
- What are the properties of these multiplets?

References

1. J. Naudts, T. Verhulst and B. Anthonis: *Counting operator analysis of the discrete spectrum of some model Hamiltonians*, arXiv:0811.3073.
2. T. Verhulst, B. Anthonis and J. Naudts: *Analysis of the* $N = 4$ *Hubbard ring using counting operators*, Phys. Lett. A **373**, 2109–2113, (2009) arXiv:0811.3077.

References

1. J. Naudts, T. Verhulst and B. Anthonis: *Counting operator analysis of the discrete spectrum of some model Hamiltonians*, arXiv:0811.3073.
2. T. Verhulst, B. Anthonis and J. Naudts: *Analysis of the $N = 4$ Hubbard ring using counting operators*, Phys. Lett. A **373**, 2109–2113, (2009) arXiv:0811.3077.

That's all, thank you!