On Universality of Bulk Local Regime of the Deformed Laguerre Ensemble

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Examples:

Gaussian unitary ensemble (GUE):

$$M = n^{-1/2} W, \tag{1}$$

where W is a Hermitian $n \times n$ matrix whose entries $\Re W_{jk}$ and $\Im W_{jk}$ are independent identically distributed Gaussian random variables with expectation 0 and dispersion 1/2.

Hermitian matrix model:

$$P(d\,M) = \frac{1}{Z_n} \exp\{-n \mathrm{Tr}\, V(M)\} d\,M, \tag{2}$$

where V is some function and Z_n is a normalizing constant. If we take $V(x) = x^2/2$ we obtain GUE.

Deformed Laguerre ensemble:

$$H_n = \frac{1}{n} A^*_{m,n} A_{m,n} + H_n^{(0)}, \tag{3}$$

where $H_n^{(0)}$ is a Hermitian $n \times n$ matrix (random or not random) with eigenvalues $\{h_j^{(n)}\}_{j=1}^n$ and $A_{m,n}$ is a $m \times n$ matrix, whose entries $\Re a_{\alpha j}$ and $\Im a_{\alpha j}$ are independent Gaussian random variables such that

$$\mathbf{E}\{a_{\alpha j}\} = \mathbf{E}\{a_{\alpha j}^2\} = 0, \; \mathbf{E}\{|a_{\alpha j}|^2\} = 1, \; \alpha = \overline{1,m}, \; j = \overline{1,n}, \tag{4}$$

moreover $m/n \to c > 1$ (as $m, n \to \infty$).

Denote by $\lambda_1^{(n)}, \ldots, \lambda_n^{(n)}$ the eigenvalues of the random matrix. Define the normalized eigenvalue counting measure of the matrix as

$$N_n(\triangle) = \sharp\{\lambda_j^{(n)} \in \triangle, j = \overline{1,n}\}/n, \quad N_n(\mathbb{R}) = 1, \tag{5}$$

where $\triangle$ is an arbitrary interval of the real axis.

For many known random matrices the expectation $\overline{N}_n = \mathbf{E}\{N_n\}$ is absolutely continuous and its density ρ_n is called the density of states.

Let

$$N_n^{(0)}(\triangle) = \frac{1}{n}\sharp\{h_j^{(n)} \in \triangle, j = \overline{1,n}\},$$

be the Normalized Counting Measure of eigenvalues of $H_n^{(0)}$.

The global regime for the ensemble (3) − (4): It was shown in the paper of Marchenko, Pastur [3] that if $N_n^{(0)}$ converges weakly with probability 1 to a non-random measure $N^{(0)}$ as $n \to \infty$, then N_n also converges weakly with probability 1 to a measure N. The measure N is normalized to unity and is absolutely continuous and its density ρ is called the limiting density of states of the ensemble.

It follows from the definition of N_n and the above result that any n-independent interval Δ such that $N(\Delta) > 0$ contains $O(n)$

eigenvalues. Thus, to deal with a finite number of eigenvalues one has to consider spectral intervals, whose length tends to zero as $n \to \infty$. This is the local regime of the random matrix theory. In particular, in the local bulk regime we are about intervals of the length $O(n^{-1})$.

Define also **the k-point correlation function $\mathbf{R_k^{(n)}}$** by the equality:

$$\mathbf{E}\left\{\sum_{j_1\neq\ldots\neq j_k}\varphi_k(\lambda_{j_1},\ldots,\lambda_{j_k})\right\}$$
$$=\int_{\mathbb{R}}\varphi_k(\lambda_1,\ldots,\lambda_m)R_k^{(n)}(\lambda_1,\ldots,\lambda_k)d\lambda_1,\ldots,d\lambda_k, \quad (6)$$

where $\varphi_k:\mathbb{R}^k\to\mathbb{C}$ is bounded, continuous and symmetric in its arguments and the summation is over all k-tuples of distinct integers $j_1,\ldots,j_k=\overline{1,n}$. We will call the spectrum the support of N and define the bulk of the spectrum as

$$\text{bulk}\, N = \{\lambda | \exists (a,b) \subset \text{supp}\, N : \lambda \in (a,b), \inf_{\mu \in (a,b)} \rho(\mu) > 0\}. \tag{7}$$

The bulk local regime for the ensemble (3) – (4):

The universality hypothesis on the bulk of the spectrum says that for $\lambda_0 \in \text{bulk}\, N$ we have:

(i) for any fixed k uniformly in $x_1, x_2, \ldots, x_k$ varying in any compact set in $\mathbb{R}$

$$\lim_{n\to\infty} \frac{1}{(n\rho_n(\lambda_0))^k} R_k^{(n)} \left(\lambda_0 + \frac{x_1}{\rho_n(\lambda_0)\, n}, \ldots, \lambda_0 + \frac{x_k}{\rho_n(\lambda_0)\, n} \right) = \det\{S(x_i - x_j)\}_{i,j=1}^k, \tag{8}$$

where

$$S(x_i - x_j) = \frac{\sin \pi (x_i - x_j)}{\pi (x_i - x_j)}; \tag{9}$$

(ii) if

$$E_n(\triangle) = \mathbf{P}\{\lambda_i^{(n)} \notin \triangle,\ i = \overline{1, n}\}, \tag{10}$$

is the gap probability, then

$$\lim_{n\to\infty} E_n\left(\left[\lambda_0 + \frac{a}{\rho_n(\lambda_0)\, n}, \lambda_0 + \frac{b}{\rho_n(\lambda_0)\, n}\right]\right) = \det\{1 - S_{a,b}\}, \tag{11}$$

where the operator $S_{a,b}$ is defined on $L_2[a, b]$ by the formula

$$S_{a,b} f(x) = \int_a^b S(x - y) f(y) d\, y,$$

and S is defined in (9).

The main result of the paper is following theorem

Theorem 1 *Let* $c > 1$ *and the eigenvalues* $\{h_j^{(n)}\}_{j=1}^n$ *of* $H_n^{(0)}$ *in (3) be a collection of random variables independent of* A_n. *Assume that there exists a non-random measure* $N^{(0)}$ *of a bounded support such that such that* $N_n^{(0)}$ *converges weakly with probability 1 to* $N^{(0)}$. *Then for any* $\lambda_0 \in \text{bulk}\, N$ *the universality properties (8) and (11) hold.*

Harish-Chandra/Itzykson-Zuber formula:

$$\int \exp\{\mathrm{Tr} AU^* BU\} d\,\mu(U) = \frac{\det[\exp\{a_i b_j\}]_{i,j=1}^n}{\triangle(A)\triangle(B)}, \qquad (12)$$

where a_i, b_i are eigenvalues of matrices A and B correspondingly and $\triangle(A)$ is a Van der Monde determinant of eigenvalues of matrix A.

Proposition 1 *Let H_n be the random matrix defined in (3) and $R_k^{(n)}$ be the correlation function (6). Then we have*

$$R_k^{(n)}(\lambda_1, \dots, \lambda_k) = \mathbf{E}^{(h)}\{\det\{K_n(\lambda_i, \lambda_j)\}_{i,j=1}^k\}, \tag{13}$$

with

$$K_n(\lambda, \mu) =$$
$$\frac{m}{4\pi^2} \oint_L \oint_\omega \frac{\exp\{n(u-t)\}\,(t+\lambda)^{m-1}}{(u-t)(u+\mu)^{m+1}} \prod_{j=1}^{n} \left(\frac{u+h_j^{(n)}}{t+h_j^{(n)}} \right) d\,t\,d\,u, \tag{14}$$

where the contour L is a closed contour, encircling $\{-h_j^{(n)} : h_j^{(n)} < \lambda\}$ and ω is any closed contour encircling $-\mu$ and not intersect L.

This proposition reduces (8) to the limiting transition in (14). The limiting transition is done using the steepest descent method.

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