On Universality of Bulk Local Regime of the Deformed Laguerre Ensemble

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Examples:

Gaussian unitary ensemble (GUE):

\[ M = n^{-1/2}W, \]  \hspace{1cm} (1)

where \( W \) is a Hermitian \( n \times n \) matrix whose entries \( \Re W_{jk} \) and \( \Im W_{jk} \) are independent identically distributed Gaussian random variables with expectation 0 and dispersion 1/2.

Hermitian matrix model:

\[ P(dM) = \frac{1}{Z_n} \exp\{-n\text{Tr} V(M)\}dM, \]  \hspace{1cm} (2)

where \( V \) is some function and \( Z_n \) is a normalizing constant. If we take \( V(x) = x^2/2 \) we obtain GUE.
Deformed Laguerre ensemble:

\[ H_n = \frac{1}{n} A_{m,n}^* A_{m,n} + H_n^{(0)}, \]  

(3)

where \( H_n^{(0)} \) is a Hermitian \( n \times n \) matrix (random or not random) with eigenvalues \( \{h^{(n)}_j\}_{j=1}^n \) and \( A_{m,n} \) is a \( m \times n \) matrix, whose entries \( \Re a_{\alpha j} \) and \( \Im a_{\alpha j} \) are independent Gaussian random variables such that

\[ \mathbb{E}\{a_{\alpha j}\} = \mathbb{E}\{a_{\alpha j}^2\} = 0, \quad \mathbb{E}\{|a_{\alpha j}|^2\} = 1, \quad \alpha = \overline{1,m}, \quad j = \overline{1,n}, \]  

(4)

moreover \( m/n \to c > 1 \) (as \( m, n \to \infty \)).

Denote by \( \lambda_1^{(n)}, \ldots, \lambda_n^{(n)} \) the eigenvalues of the random matrix. Define the normalized eigenvalue counting measure of the matrix as

\[ N_n(\triangle) = \frac{\#\{\lambda_j^{(n)} \in \triangle, j = \overline{1,n}\}}{n}, \quad N_n(\mathbb{R}) = 1, \]  

(5)

where \( \triangle \) is an arbitrary interval of the real axis.
For many known random matrices the expectation $\overline{N}_n = \mathbf{E}\{N_n\}$ is absolutely continuous and its density $\rho_n$ is called the density of states.

Let

$$N_n^{(0)}(\Delta) = \frac{1}{n} \# \{ h_j^{(n)} \in \Delta, j = 1, n \},$$

be the Normalized Counting Measure of eigenvalues of $H_n^{(0)}$.

**The global regime for the ensemble (3) – (4):** It was shown in the paper of Marchenko, Pastur [3] that if $N_n^{(0)}$ converges weakly with probability 1 to a non-random measure $N^{(0)}$ as $n \to \infty$, then $N_n$ also converges weakly with probability 1 to a measure $N$. The measure $N$ is normalized to unity and is absolutely continuous and its density $\rho$ is called the limiting density of states of the ensemble.

It follows from the definition of $N_n$ and the above result that any $n$-independent interval $\Delta$ such that $N(\Delta) > 0$ contains $O(n)$
eigenvalues. Thus, to deal with a finite number of eigenvalues one has to consider spectral intervals, whose length tends to zero as \( n \to \infty \).

This is the local regime of the random matrix theory. In particular, in the local bulk regime we are about intervals of the length \( O(n^{-1}) \).

Define also the **k-point correlation function** \( R_k^{(n)} \) by the equality:

\[
E \left\{ \sum_{j_1 \neq \ldots \neq j_k} \varphi_k(\lambda_{j_1}, \ldots, \lambda_{j_k}) \right\} = \int_{\mathbb{R}} \varphi_k(\lambda_1, \ldots, \lambda_m) R_k^{(n)}(\lambda_1, \ldots, \lambda_k) d\lambda_1, \ldots, d\lambda_k, \quad (6)
\]

where \( \varphi_k : \mathbb{R}^k \to \mathbb{C} \) is bounded, continuous and symmetric in its arguments and the summation is over all \( k \)-tuples of distinct integers \( j_1, \ldots, j_k = \overline{1, n} \). We will call the spectrum the support of \( N \) and
define the bulk of the spectrum as

\[
\text{bulk } N = \{ \lambda \mid \exists (a, b) \subset \text{supp } N : \lambda \in (a, b), \inf_{\mu \in (a,b)} \rho(\mu) > 0 \}. \quad (7)
\]

The bulk local regime for the ensemble (3) – (4):

The universality hypothesis on the bulk of the spectrum says that for \( \lambda_0 \in \text{bulk } N \) we have:

(i) for any fixed \( k \) uniformly in \( x_1, x_2, \ldots, x_k \) varying in any compact set in \( \mathbb{R} \)

\[
\lim_{n \to \infty} \frac{1}{(n\rho_n(\lambda_0))^k} R_k^{(n)} \left( \lambda_0 + \frac{x_1}{\rho_n(\lambda_0)n}, \ldots, \lambda_0 + \frac{x_k}{\rho_n(\lambda_0)n} \right) = \det \{ S(x_i - x_j) \}_{i,j=1}^k, \quad (8)
\]
where
\[ S(x_i - x_j) = \frac{\sin \pi(x_i - x_j)}{\pi(x_i - x_j)}; \quad (9) \]

(ii) if
\[ E_n(\triangle) = \mathbf{P}\{\lambda_i^{(n)} \notin \triangle, i = 1, n\}, \quad (10) \]
is the gap probability, then
\[
\lim_{n \to \infty} E_n \left( \left[ \lambda_0 + \frac{a}{\rho_n(\lambda_0) n}, \lambda_0 + \frac{b}{\rho_n(\lambda_0) n} \right] \right) = \det\{1 - S_{a,b}\}, \quad (11)
\]
where the operator \( S_{a,b} \) is defined on \( L_2[a, b] \) by the formula
\[
S_{a,b}f(x) = \int_a^b S(x - y)f(y)dy,
\]
and \( S \) is defined in (9).
The main result of the paper is following theorem

**Theorem 1** Let $c > 1$ and the eigenvalues $\{h_j^{(n)}\}_{j=1}^n$ of $H_n^{(0)}$ in (3) be a collection of random variables independent of $A_n$. Assume that there exists a non-random measure $N^{(0)}$ of a bounded support such that such that $N_n^{(0)}$ converges weakly with probability 1 to $N^{(0)}$. Then for any $\lambda_0 \in \text{bulk } N$ the universality properties (8) and (11) hold.

**Harish-Chandra/Itzykson-Zuber formula:**

$$
\int \exp\{\text{Tr}AU^*BU\}d\mu(U) = \frac{\det[\exp\{a_ib_j\}]_{i,j=1}^n}{\triangle(A)\triangle(B)}, \quad (12)
$$

where $a_i, b_i$ are eigenvalues of matrices $A$ and $B$ correspondingly and $\triangle(A)$ is a Van der Monde determinant of eigenvalues of matrix $A$. 
**Proposition 1** Let $H_n$ be the random matrix defined in (3) and $R^{(n)}_k$ be the correlation function (6). Then we have

$$R^{(n)}_k(\lambda_1, \ldots, \lambda_k) = \mathbb{E}^{(h)} \{ \det \{ K_n(\lambda_i, \lambda_j) \}_{i,j=1}^k \}, \quad (13)$$

with

$$K_n(\lambda, \mu) =$$

$$\frac{m}{4\pi^2} \oint_L \oint_\omega \frac{\exp \{ n(u - t) \} (t + \lambda)^{m-1}}{(u - t)(u + \mu)^{m+1}} \prod_{j=1}^n \left( \frac{u + h^{(n)}_j}{t + h^{(n)}_j} \right) dt \, du, \quad (14)$$

where the contour $L$ is a closed contour, encircling $\{ -h^{(n)}_j : h^{(n)}_j < \lambda \}$ and $\omega$ is any closed contour encircling $-\mu$ and not intersect $L$.

This proposition reduces (8) to the limiting transition in (14). The limiting transition is done using the steepest descent method.

