Polynomials orthogonal on the unit circle with n-varying exponential weights

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The Random Matrix Theory (RMT) is an active field of mathematics and theoretic physics. The big interest to this subject is conditioned by a numerous applications of RMT related to quantum mechanics and quantum chaos theory, probabilistic theory, combinatorics, integrable systems, orthogonal polynomials theory, mathematic biology and other branches.

We study a class of random matrix ensembles, known as unitary matrix models, that are defined by the probability law

$$p_n(U)\,d\mu_n(U) = Z_{n,2}^{-1}\exp\left\{-n\mathrm{Tr}V\left(\frac{U+U^*}{2}\right)\right\}d\mu_n(U), \qquad (1)$$

where $U = \{U_{jk}\}_{j,k=1}^{n}$ is a $n\times n$ unitary matrix, $\mu_n(U)$ is the Haar measure on the group $U(n)$, $Z_{n,2}$ is the normalization constant and $V:[-1,1]\to\mathbb{R}^+$ is a continuous function, called the potential of the model.

Let $e^{i\lambda_j}$ be an eigenvalues of the unitary matrix U. The joint probability density of λ_j, corresponding to (1), is given by (see [1])

$$p_n(\lambda_1,\ldots,\lambda_n) = \frac{1}{Z_n}\prod_{1\le j<k\le n}\left|e^{i\lambda_j}-e^{i\lambda_k}\right|^2\exp\left\{-n\sum_{j=1}^{n}V(\cos\lambda_j)\right\}. \qquad (2)$$

below we will write $V(x)$ instead of $V(\cos x)$. Normalized Counting Measure of eigenvalues (NCM) is given by

$$N_n(\Delta) = n^{-1}\sharp\left\{\lambda_l^{(n)}\in\Delta,\, l=1,\ldots,n\right\}, \quad \Delta\subset[-\pi,\pi].$$

Eigenvalue distribution is one of the main question in RMT. The random matrix theory deals with several asymptotic regimes of the eigenvalue distribution. The global regime is centered around weak convergence of Normalized Counting Measure of eigenvalues. Global regime for unitary matrix models was studied in [2].

Theorem 1 *Assume that the potential V of the model* (1) *is a $C^2(-\pi,\pi)$ function. Then:*

- *there exists a measure* $N \in \mathcal{M}_1([-\pi,\pi])$ *with a compact support* σ, *such that NCM* N_n *converges in probability to* N;
- N *has a bounded density* ρ *(DOS), that satisfies the equation*

$$V'(\lambda) = v.p. \int_{-\pi}^{\pi} \cot\frac{\lambda-\mu}{2}\rho(\mu)\,d\mu, \quad for\,\lambda\in\sigma; \tag{3}$$

- *denote* $\rho_n := p_1^{(n)}$ *the first marginal density, then for any* $\phi \in H^1(-\pi,\pi)$

$$\left|\int\phi(\lambda)\,\rho_n(\lambda)\,d\lambda - \int\phi(\lambda)\,\rho(\lambda)\,d\lambda\right| \le C\,\|\phi\|_2^{1/2}\,\|\phi'\|_2^{1/2}\,n^{-1/2}\ln^{1/2} n, \tag{4}$$

where $\|\cdot\|_2$ *denotes* L_2 *norm on* $[-\pi,\pi]$

Local regime is responsible for eigenvalue statistics on a small enough intervals. Length of these intervals is corresponding to the average distance between adjacent eigenvalues. One of the main topics of local regime is universality of local eigenvalue statistics. Let

$$p_l^{(n)}(\lambda_1,\dots,\lambda_l) = \int p_n(\lambda_1,\dots,\lambda_l,\lambda_{l+1},\dots,\lambda_n)\,d\lambda_{l+1}\dots d\lambda_n \tag{5}$$

be the l -th marginal density of p_n. We suppose that $\rho(\lambda)$ behaves asymptotically as $const\cdot|\lambda\mp\theta|^{1/2}$ in a neighborhood of edges $\pm\theta$ of the density $\rho(\lambda)$ defined in (3).

Theorem 2 (Universality conjecture) *Let* V *be a real analytic function, then*

$$\lim_{n\to\infty}\left[\gamma n^{2/3}\right]^{-l} p_l^{(n)}\left(\theta+\frac{x_1}{\gamma n^{2/3}},\dots,\theta+\frac{x_l}{\gamma n^{2/3}}\right) = \det\left\{S(x_j,x_k)\right\}_{j,k=1}^{l}, \tag{6}$$

where γ *some constant and*

$$S(x,y) = \frac{Ai(x)\,Ai'(y) - Ai'(x)\,Ai(y)}{x-y}. \tag{7}$$

This conjecture for all matrix ensembles first appeared in [3], but first rigorous proofs of this universality property were given in [4] and [5] for the hermitian matrix models. All these proofs lean on the orthogonal polynomial techniques and determinant formulas [1].

Consider the system of functions $\{e^{ik\lambda}\}_{k=0}^{\infty}$ and use for them the Gram-Shmidt procedure in $L_2\left([-\pi,\pi], e^{-nV(\lambda)}\right)$. For any n we get the system of functions $\left\{P_k^{(n)}(\lambda)\right\}_{k=0}^{\infty}$ which are orthogonal and normalized in $L_2\left([-\pi,\pi], e^{-nV(\lambda)}\right)$. Denote

$$\psi_k^{(n)}(\lambda) = P_k^{(n)}(\lambda)\, e^{-nV(\lambda)/2}. \tag{8}$$

The reproducing kernel of the system (8) is given by

$$K_n(\lambda,\mu) = \sum_{j=0}^{n-1} \psi_l^{(n)}(\lambda)\, \overline{\psi_l^{(n)}(\mu)}. \tag{9}$$

Now we can represent marginal densities using the reproducing kernel

$$p_l^{(n)}(\lambda_1,\ldots,\lambda_l) = \frac{(n-l)!}{n!} \det \left\| K_n(\lambda_j,\lambda_k) \right\|_{j,k=1}^{l}. \tag{10}$$

Thereby universality conjecture can be reduce to the limiting relation for $K(\lambda,\mu)$. Set $\mathcal{K}_n(x,y) = \gamma^{-1} n^{-2/3} K_n\left(\theta + x\gamma^{-1}n^{-2/3}, \theta + y\gamma^{-1}n^{-2/3}\right)$ and prove that

$$\mathcal{K}_n(x,y) \to S(x,y). \tag{11}$$

It was shown in [7] and [5] that relation (11) for Hermitian ensembles can be proved from the asymptotic relations for elements of the corresponding Jacobi matrix. Polynomials $P_k^{(n)}$ satisfy the analogously recurrence relations but corresponding matrix is Heisenberg and not Jacobi. In [8] authors proposed to modify this polynomials to obtain a five-diagonal recurrence relation.

Denote by $c_{k,l}^{(n)}$ coefficients of $e^{il\lambda}$ in $P_k^{(n)}(\lambda)$. Since V is even, it is easy to see that all coefficients $c_{k,l}^{(n)}$ are real. According to [8] we consider the reverse polynomials $Q_k^{(n)}$, defined by

$$Q_k^{(n)}(\lambda) = \sum_{l=0}^{k} c_{k,l}^{(n)} e^{i(k-l)\lambda}. \tag{12}$$

Consider $\left\{\chi_k^{(n)}(\lambda)\right\}_{k=0}^{\infty}$ the sequence of right orthonormal L-polynomials

$$\begin{aligned} \chi_{2k}^{(n)}(\lambda) &= e^{-ik\lambda} Q_{2k}^{(n)}(\lambda) \\ \chi_{2k+1}^{(n)}(\lambda) &= e^{-ik\lambda} P_{2k+1}^{(n)}(\lambda). \end{aligned} \tag{13}$$

Lemma 3 *Let $\chi_k^{(n)}$ be the system of orthogonal polynomials, defined above. Then*

$$e^{i\lambda}\chi_{2k-1}^{(n)}(\lambda) = -\alpha_{2k}^{(n)}\rho_{2k-1}^{(n)}\chi_{2k-2}^{(n)}(\lambda) - \alpha_{2k}^{(n)}\alpha_{2k-1}^{(n)}\chi_{2k-1}^{(n)}(\lambda) - \alpha_{2k+1}^{(n)}\rho_{2k}^{(n)}\chi_{2k}^{(n)}(\lambda) + \rho_{2k}^{(n)}\rho_{2k+1}^{(n)}\chi_{2k+1}^{(n)}(\lambda), \quad (14)$$

$$e^{i\lambda}\chi_{2k}^{(n)}(\lambda) = \rho_{2k}^{(n)}\rho_{2k-1}^{(n)}\chi_{2k-2}^{(n)}(\lambda) + \alpha_{2k-1}^{(n)}\rho_{2k}^{(n)}\chi_{2k-1}^{(n)}(\lambda) - \alpha_{2k+1}^{(n)}\alpha_{2k}^{(n)}\chi_{2k}^{(n)}(\lambda) + \alpha_{2k}^{(n)}\rho_{2k+1}^{(n)}\chi_{2k+1}^{(n)}(\lambda), \quad (15)$$

where $\alpha_k^{(n)} = \frac{c_{k,0}^{(n)}}{c_{k,k}^{(n)}}$ *and* $\rho_k^{(n)} = \frac{c_{k-1,k-1}^{(n)}}{c_{k,k}^{(n)}}$ *the Schur parameters of the system* $\left\{P_k^{(n)}(\lambda)\right\}_{k=0}^{\infty}$ *and*

$$\left(\rho_k^{(n)}\right)^2 + \left(\alpha_k^{(n)}\right)^2 = 1. \quad (16)$$

The last lemma allows us to construct a five diagonal CMV matrix ([8]). Consider the space $\mathrm{L} = L_2\left([-\pi,\pi], e^{-nV(\lambda)}\right)$ and a multiplication operator $C^{(n)} : \mathrm{L} \to \mathrm{L}$, defined by

$$C^{(n)}[f](\lambda) = e^{i\lambda} f(\lambda). \quad (17)$$

Using the orthogonal polynomials $\left\{\chi_k^{(n)}(\lambda)\right\}_{k=0}^{\infty}$ as a basis, we obtain a matrix representation of $C^{(n)}$.

$$C^{(n)} = \begin{pmatrix} -\alpha_1^{(n)} & \rho_1^{(n)} & 0 & 0 & 0 & 0 & \cdots \\ -\rho_1^{(n)}\alpha_2^{(n)} & -\alpha_1^{(n)}\alpha_2^{(n)} & -\rho_2^{(n)}\alpha_3^{(n)} & \rho_2^{(n)}\rho_3^{(n)} & 0 & 0 & \cdots \\ \rho_1^{(n)}\rho_2^{(n)} & \alpha_1^{(n)}\rho_2^{(n)} & -\alpha_2^{(n)}\alpha_3^{(n)} & \alpha_2^{(n)}\rho_3^{(n)} & 0 & 0 & \cdots \\ 0 & 0 & -\rho_3^{(n)}\alpha_4^{(n)} & -\alpha_3^{(n)}\alpha_4^{(n)} & -\rho_4^{(n)}\alpha_5^{(n)} & \rho_4^{(n)}\rho_5^{(n)} & \cdots \\ 0 & 0 & \rho_3^{(n)}\rho_4^{(n)} & \alpha_3^{(n)}\rho_4^{(n)} & -\alpha_4^{(n)}\alpha_5^{(n)} & \alpha_4^{(n)}\rho_5^{(n)} & \cdots \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \quad (18)$$

The main result of our work is the next theorem.

Theorem 4 *If the support of the DOS is the interval $[-\theta,\theta]$ then*

$$\alpha_{n+k,n+k}^{(n)} = (-1)^{n+k} s\left(\cos\frac{\theta}{2} + \frac{k}{n}c*\right) + O\left(n^{-4/3} + \frac{k^2}{n^2}\right), \quad (19)$$

for $|k| \le \varepsilon n$ with some small enough n independent ε, where $s = \pm 1$ and $c*$ some constant.

First we derive from (3) the expression for the DOS.

$$\rho(\mu) = \frac{1}{4\pi^2}\chi(\mu)\, P(\mu), \tag{20}$$

where

$$\chi(\mu) = \sqrt{\cos\mu - \cos\theta}, \quad P(\mu) = \int_{-\theta}^{\theta} \frac{V'(\lambda) - V'(\mu)}{\sin\dfrac{\lambda-\mu}{2}} \frac{d\lambda}{\chi(\lambda)}. \tag{21}$$

From determinant formulas [1] we obtain the equation valid for any twice differentiable on the unit circle function ϕ.

$$\begin{aligned} v.p. \int d\lambda \int_{-\theta}^{\theta} d\mu \phi(\mu) \cot\frac{\lambda-\mu}{2} \rho(\mu) \left|\psi_{k+n}^{(n)}(\lambda)\right|^2 + \\ + \frac{1}{2\pi}\int_{\sigma^c} \operatorname{sign}\lambda \phi(\lambda)\, P(\lambda) \sqrt{\cos\theta - \cos\lambda} \left|\psi_{k+n}^{(n)}(\lambda)\right|^2 d\lambda = \\ = \underline{O}\left(\left(n^{-1/2}\log^{1/2} n + \frac{|k|+1}{n}\right)\left(\|\phi'\|_\infty + \|\phi''\|_\infty\right)\right). \end{aligned} \tag{22}$$

Taking in (22) $\phi(\mu) = P^{-1}(\mu)\cos\dfrac{\mu}{2}\cot\dfrac{z-\mu}{2}$, we obtain a diagonal elements of the Herglotz transformation for matrix $C^{(n)}$. From this relation we get a system of equations on Shur parameters and solving it obtain a first order asymptotics

$$\alpha_{n+k}^{(n)} = (-1)^{n+k} s \cos\frac{\theta}{2} + O\left(n^{-1/4}\log^{1/2} n + \left(\frac{|k|}{n}\right)^{1/2}\right). \tag{23}$$

Further we use a well-known string equation

$$\begin{aligned} \int_0^{2\pi} \sin\lambda V'(\cos\lambda)\, \chi_{n+k}^{(n)}\left(e^{i\lambda}\right) \overline{\chi_{n+k-1}^{(n)}(e^{i\lambda})} d\lambda \\ = i(-1)^{n+k-1}\frac{n+k}{n}\frac{\alpha_{n+k}^{(n)}}{\rho_{n+k}^{(n)}}. \end{aligned} \tag{24}$$

We consider these equations for different k as a system of nonlinear equations with respect to the coefficients $\alpha_{n+k}^{(n)}$. Using the perturbation theory we obtain from this system the second order asymptotics (19).

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