Realizations of certain five-dimensional Lie algebras

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Realizations of certain five-dimensional Lie algebras by vector fields on spaces of arbitrary finite numbers of variables are classified by means of using the direct method.

1 Megaideals and realizations of Lie algebras

Now we define the notion of megaideal that is useful for constructing realizations and proving their inequivalence in a simpler way. Let A be an m-dimensional (real or complex) Lie algebra ($m \in \mathbb{N}$) and let $\mathrm{Aut}(A)$ and $\mathrm{Int}(A)$ denote the groups of all the automorphisms of A and of its inner automorphisms respectively. The Lie algebra of the group $\mathrm{Aut}(A)$ coincides with the Lie algebra $\mathrm{Der}(A)$ of all the derivations of the algebra A. (A derivation D of A is called a linear mapping from A into itself which satisfy the condition $D[u,v] = [Du,v] + [u,Dv]$ for all $u, v \in A$.) $\mathrm{Der}(A)$ contains as an ideal the algebra $\mathrm{Ad}(A)$ of inner derivations of A, which is the Lie algebra of $\mathrm{Int}(A)$. (The inner derivation corresponding to $u \in A$ is the mapping $\mathrm{ad}\, u \colon v \to [v,u]$.) Fixing a basis $\{e_\mu, \mu = \overline{1,m}\}$ in A, we associate an arbitrary linear mapping $l : A \to A$ (e.g., an automorphism or a derivation of A) with a matrix $\alpha = (\alpha_{\nu\mu})_{\mu,\nu=1}^{m}$ by means of the expanding $l(e_\mu) = \alpha_{\nu\mu} e_\nu$. Then each group of automorphisms of A is associated with a subgroup of the general linear group $GL(m)$ of all the non-degenerated $m \times m$ matrices (over $\mathbb{R}$ or $\mathbb{C}$) as well as each algebra of derivations of A is associated with a subalgebra of the general linear algebra $gl(m)$ of all the $m \times m$ matrices.

Definition. We call a vector subspace of A, which is invariant under any transformation from $\mathrm{Aut}(A)$, a *megaideal* of A.

Since $\mathrm{Int}(A)$ is a normal subgroup of $\mathrm{Aut}(A)$, it is clear that any megaideal of A is a subalgebra and, moreover, an ideal in A. But when $\mathrm{Int}(A) \neq \mathrm{Aut}(A)$ (e.g., for nilpotent algebras) there exist ideals in A, which are not megaideals. Moreover, any megaideal I of A is invariant with respect to all the derivations of A: $\mathrm{Der}(A)I \subset I$, i.e. it is a characteristic subalgebra. Characteristic subalgebras which are not megaideals can exist only if $\mathrm{Aut}(A)$ is a disconnected Lie group.

Both improper subsets of A (the empty set and A itself) are always megaideals in A. The following lemmas are obvious.

Lemma 1. If I_1 and I_2 are megaideals of A then so are $I_1 + I_2$, $I_1 \cap I_2$ and $[I_1, I_2]$, i.e. sums, intersections and Lie products of megaideals are also megaideals.

Similar statements are true for both ideals and characteristic ideals.

Lemma $1'$**.** If I' is a megaideal of I and I is a megaideal of A then I' is a megaideal of A, i.e. megaideals of megaideals are also megaideals.

In contrast to Lemma 1, a similar statement is true for characteristic ideals but not for usual ideals.

Corollary 1. All the members of the commutator (derived) and the lower central series of A, i.e. all the derivatives $A^{(n)}$ and all the Lie powers A^n ($A^{(n)} = [A^{(n-1)}, A^{(n-1)}]$, $A^n = [A, A^{n-1}]$, $A^{(0)} = A^0 = A$) are megaideals in A.

This corollary follows from Lemma 1 by induction since A is a megaideal in A.

Corollary 2. The center $A_{(1)}$ and all the other members of the upper central series $\{A_{(n)},\ n \geq 0\}$ of A are megaideals in A.

Let us remind that $A_{(0)} = \{0\}$ and $A_{(n+1)}/A_{(n)}$ is the center of $A/A_{(n)}$.

Lemma 2. The radical (i.e. the maximal solvable ideal) and the nil-radical (i.e. the maximal nilpotent ideal) of A are its megaideals.

The above lemmas give a number of invariant subspaces of all the automorphisms in A and, therefore, simplify calculating $\mathrm{Aut}(A)$.

Let M denote a n-dimensional smooth manifold and $\mathrm{Vect}(M)$ denote the Lie algebra of smooth vector fields (i.e. first-order linear differential operators) on M with the Lie bracket of vector fields as a commutator. Here and below smoothness means analyticity.

Definition. *A realization of a Lie algebra A in vector fields on M* is called a homomorphism R: $A \to \mathrm{Vect}(M)$. The realization is said *faithful* if $\ker R = \{0\}$ and *unfaithful* otherwise. Let G be a subgroup of $\mathrm{Aut}(A)$. The realizations R_1: $A \to \mathrm{Vect}(M_1)$ and R_2: $A \to \mathrm{Vect}(M_2)$ are called *G-equivalent* if there exist $\varphi \in G$ and a diffeomorphism f from M_1 to M_2 such that $R_2(v) = f_* R_1(\varphi(v))$ for all $v \in A$. Here f_* is the isomorphism from $\mathrm{Vect}(M_1)$ to $\mathrm{Vect}(M_2)$ induced by f. If G contains only the identical transformation, the realizations are called *strongly equivalent.* The realizations are *weakly equivalent* if $G = \mathrm{Aut}(A)$. A restriction of the realization R on a subalgebra A_0 of the algebra A is called *a realization induced by R* and is denoted as $R\big|_{A_0}$.

Within the framework of local approach that we use M can be considered as an open subset of $\mathbb{R}^n$ and all the diffeomorphisms are local.

Usually realizations of a Lie algebra have been classified with respect to the weak equivalence. This it is reasonable although the equivalence used in the representation theory is similar to the strong one. The strong equivalence suits better for construction of realizations of algebras using realizations of their subalgebras and is verified in a simpler way than the weak equivalence. It is not specified in some papers what equivalence has been used, and this results in classification mistakes.

To classify realizations of a m-dimensional Lie algebra A in the most direct way, we have to take m linearly independent vector fields of the general form $e_i = \xi^{ia}(x)\partial_a$, where $\partial_a = \partial/\partial x_a$, $x = (x_1, x_2, \dots, x_n) \in M$, and require them to satisfy the commutation relations of A. As a result, we obtain a system of first-order PDEs for the coefficients ξ^{ia} and integrate it, considering all the possible cases. For each case we transform the solution into the simplest form, using either local diffeomorphisms of the space of x and automorphisms of A if the weak equivalence is meant or only local diffeomorphisms of the space of x for the strong equivalence. A drawback of this method is the necessity to solve a complicated nonlinear system of PDEs. Another way is to classify sequentially realizations of a series of nested subalgebras of A, starting with a one-dimensional subalgebra and ending up with A.

Let V be a subset of $\mathrm{Vect}(M)$ and $r(x) = \dim\langle V(x)\rangle$, $x \in M$. $0 \le r(x) \le n$. The general value of $r(x)$ on M is called the *rank* of V and is denoted as $\operatorname{rank} V$.

Lemma 3. Let B be a subset and R_1 and R_2 be realizations of the algebra A. If $R_1(B)$ and $R_2(B)$ are inequivalent with respect to endomorphisms of $\mathrm{Vect}(M)$ generated by diffeomorphisms on M. Then R_1 and R_2 are strongly inequivalent.

Corollary 2. If there exists a subset B of A such that $\operatorname{rank} R_1(B) \ne \operatorname{rank} R_2(B)$ then the realizations R_1 and R_2 are strongly inequivalent.

Lemma 4. Let I be a megaideal and R_1 and R_2 be realizations of the algebra A. If $R_1\big|_I$ and $R_2\big|_I$ are $\mathrm{Aut}(A)|_I$-inequivalent then R_1 and R_2 are weakly inequivalent.

Corollary 3. If $R_1\big|_I$ and $R_2\big|_I$ are weakly inequivalent then R_1 and R_2 also are weakly inequivalent.

Corollary 4. If there exists a megaideal I of A such that $\operatorname{rank} R_1(I) \ne \operatorname{rank} R_2(I)$ then the realizations R_1 and R_2 are weakly inequivalent.

2 The technique of classification

- For certain five-dimensional algebra A from Mubarakzyanov's classification we find the automorphism group $\mathrm{Aut}(A)$ and the set of megaideals of A. (Our notions of low-dimensional algebras, choice of their basis elements, and, consequently, the form of commutative relations coincide with Mubarakzyanov's ones.) Calculations of this step is quite simple due to low dimensions and simplicity of the canonical commutation relations. Lemmas 1 and 2, Corollary 1 and other similar statements are useful for such calculations. See also the remarks below.

- We choose a maximal proper subalgebra B of A. As rule, dimension of B is equal to $m-1$. So, if A is solvable, it necessarily contains a $(m-1)$-dimensional ideal. The simple algebra $sl(2,\mathbb{R})$ has a two-dimensional subalgebra. The Levi factors of unsolvable four-dimensional algebras ($sl(2,\mathbb{R}) \oplus A_1$ and $so(3) \oplus A_1$)

are three-dimensional ideals of these algebras. Only $so(3)$ does not contain a subalgebra of dimension $m-1=2$ that is a reason of difficulties in constructing realizations for this algebra. Moreover, the algebras $sl(2,\mathbb{R})$, $so(3)$, mA_1, $A_{3.1}$, $A_{3.1}\oplus A_1$ and $2A_{2.1}$ exhaust the list of algebras under consideration that do not contain $(m-1)$–dimensional megaideals.

- Let us suppose that a complete list of strongly inequivalent realizations of B has been already constructed. (If B is a megaideal of A and realizations of A are classified only with respect to the weak equivalence, it is sufficient to use only $\mathrm{Aut}(A)|_B$-inequivalent realizations of B.) For each realization $R(B)$ from this list we make the following procedure. We find the set $\mathrm{Diff}^{R(B)}$ of local diffeomorphisms of the space of x, which preserve $R(B)$. Then, we realize the basis vector e_i (or the basis vectors in the case of $so(3)$) from $A\backslash B$ in the most general form $e_i=\xi^{ia}(x)\partial_a$, where $\partial_a=\partial/\partial x_a$, and require that it satisfied the commutation relations of A with the basis vectors from $R(B)$. As a result, we obtain a system of first-order PDEs for the coefficients ξ^{ia} and integrate it, considering all possible cases. For each case we reduce the found solution to the simplest form, using either diffeomorphisms from $\mathrm{Diff}^{R(B)}$ and automorphisms of A if the weak equivalence is meant or only diffeomorphisms from $\mathrm{Diff}^{R(B)}$ for the strong equivalence.

- The last step is to test inequivalence of the constructed realizations. We associate the N-th one of them with the ordered collection of integers (r_{Nj}), where r_{Nj} is equal to the rank of the elements of S_j in the realization $R(A,N)$. Here S_j is either the j-th subset of basis of A with $|S_j|>1$ in the case of strong equivalence or the basis of the j-th megaideals I_j of A with $\dim I_j>1$ in the case of weak equivalence. Inequivalence of realizations with different associated collection of integers immediately follows from Corollary 2 or Corollary 4 respectively. Inequivalence of realizations in the pairs with identical collections of ranks is proved using another method, e.g. Casimir operators (for simple algebras), Lemmas 2 and 3, Corollary 3 and the rule of constraries (see the following section).

3 Five-dimensional Lie algebras

We present realizations of five-dimensional Lie algebras which contain the algebra $A_{3.1}\oplus A_1$ as an ideal.

	N	
$A_{5.19}$	1	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ \partial_2,\ (1+\alpha)x_1\partial_1+\beta x_2\partial_2+x_3\partial_3+\alpha x_4\partial_4+\partial_5$
$[e_1,e_5]=(1{+}\alpha)e_1$	2	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ \partial_2,\ (1+\alpha)x_1\partial_1+\beta x_2\partial_2+x_3\partial_3+\alpha x_4\partial_4$
$[e_2,e_5]=e_2$	3	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_2+x_5\partial_5,\ \partial_2,\ (1+\alpha)x_1\partial_1+\beta x_2\partial_2+x_3\partial_3+(\beta-\alpha)x_4\partial_4+(1-\alpha)x_5\partial_5$

$[e_3,e_5]=\alpha e_3$	4	$\partial_1,\partial_3,\ x_3\partial_1+Cx_4^{\frac{\beta-\alpha}{1-\alpha}}\partial_2+x_4\partial_3,\ \partial_2,\ (1+\alpha)x_1\partial_1+\beta x_2\partial_2+x_3\partial_3+(1-\alpha)x_4\partial_4$
$[e_4,e_5]=\beta e_4$	5	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_2,\ \partial_2,\ (1+\alpha)x_1\partial_1+\beta x_2\partial_2+x_3\partial_3+(\beta-\alpha)x_4\partial_4$
$[e_2,e_3]=e_1$	6	$\partial_1,\ \partial_3,\ x_3\partial_1,\ \partial_2,\ (1+\alpha)x_1\partial_1+\beta x_2\partial_2+x_3\partial_3+\partial_4$
$\beta\neq 0$	7	$\partial_1,\ \partial_3,\ x_3\partial_1,\ \partial_2,\ (1+\alpha)x_1\partial_1+\beta x_2\partial_2+x_3\partial_3$
	8	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ x_2\partial_1,\ (1+\alpha)x_1\partial_1+(1+\alpha-\beta)x_2\partial_2+x_3\partial_3+\alpha x_4\partial_4$
	9	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_3,\ x_2\partial_1,\ (1+\alpha)x_1\partial_1+(1+\alpha-\beta)x_2\partial_2+x_3\partial_3+(1-\alpha)x_4\partial_4$
	10	$\partial_1,\ \partial_3,\ x_3\partial_1+Cx_2^{\frac{1-\alpha}{1+\alpha-\beta}}\partial_3,\ x_2\partial_1,\ (1+\alpha)x_1\partial_1+(1+\alpha-\beta)x_2\partial_2+x_3\partial_3$
$A_{5.20}$	1	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ \partial_2,\ ((1+\alpha)x_1+x_2)\partial_1+(1+\alpha)x_2\partial_2+x_3\partial_3+\alpha x_4\partial_4+\partial_5$
$[e_1,e_5]=(1+\alpha)e_1$	2	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ \partial_2,\ ((1+\alpha)x_1+x_2)\partial_1+(1+\alpha)x_2\partial_2+x_3\partial_3+\alpha x_4\partial_4$
$[e_2,e_5]=e_2$	3	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_2+x_5\partial_5,\ \partial_2,\ (1+\alpha)x_1\partial_1+(1+\alpha)x_2\partial_2+(x_3+x_4)\partial_3+x_4\partial_4+(1-\alpha)x_5\partial_5$
$[e_3,e_5]=\alpha e_3$	4	$\partial_1,\partial_3,\ x_3\partial_1+Cx_4^{\frac{1}{1-\alpha}}\partial_2+x_4\partial_3,\ \partial_2,\ ((1+\alpha)x_1+x_2)\partial_1+(1+\alpha)x_2\partial_2+(x_3+Cx_4^{\frac{1}{1-\alpha}})\partial_3+(1-\alpha)x_4\partial_4$
$[e_4,e_5]=e_1+$	5	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_2,\ \partial_2,\ (1+\alpha)x_1\partial_1+(1+\alpha)x_2\partial_2+(x_3+x_4)\partial_3+x_4\partial_4$
$+(1+\alpha)e_4$	6	$\partial_1,\ \partial_3,\ x_3\partial_1,\ \partial_2,\ ((1+\alpha)x_1+x_2)\partial_1+(1+\alpha)x_2\partial_2+x_3\partial_3+\partial_4$
$[e_2,e_3]=e_1$	7	$\partial_1,\ \partial_3,\ x_3\partial_1,\ \partial_2,\ (1+\alpha)x_1+x_2)\partial_1+(1+\alpha)x_2\partial_2+x_3\partial_3$
	8	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ x_2\partial_1,\ (1+\alpha)x_1\partial_1-\partial_2+x_3\partial_3+\alpha x_4\partial_4$
	9	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_3,\ x_2\partial_1,\ (1+\alpha)x_1\partial_1-\partial_2+x_3\partial_3+(1-\alpha)x_4\partial_4$
	10	$\partial_1,\ \partial_3,\ x_3\partial_1+e^{C(\alpha-1)x_2}\partial_3,\ x_2\partial_1,\ (1+\alpha)x_1\partial_1-\partial_2+x_3\partial_3$
$A_{5.21}$	1	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ \partial_2,\ (2x_1+\frac{x_3^2}{2})\partial_1+(x_2+x_4)\partial_2+x_3\partial_3+(x_3+x_4)\partial_4+\partial_5$
$[e_1,e_5]=2e_1$	2	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ \partial_2,\ (2x_1+\frac{x_3^2}{2}\partial_1+(x_2+x_4)\partial_2+x_3\partial_3+(x_3+x_4)\partial_4$
$[e_2,e_5]=e_2+e_3$	3	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_2+x_5\partial_3,\ \partial_2,\ (2x_1+\frac{x_3^2}{2})\partial_1+(x_2+x_3x_4)\partial_2+(x_3+x_3x_5)\partial_3+(x_4x_5-1)\partial_4+x_5^2\partial_5$
$[e_3,e_5]=e_3+e_4$	4	$\partial_1,\partial_3,\ x_3\partial_1+(\frac{1}{2x_4}+Cx_4)\partial_2+x_4\partial_3,\ \partial_2,\ (2x_1+\frac{x_3^2}{2}\partial_1+(x_2+\frac{x_3}{2x_4}+Cx_3x_4)\partial_2+(x_3x_4+x_3)\partial_3+x_4^2\partial_4$
$[e_4,e_5]=e_4$	5	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_2,\ \partial_2,\ (2x_1+\frac{x_3^2}{2})\partial_1+(x_2+x_3x_4)\partial_2+x_3\partial_3-\partial_4$
$[e_2,e_3]=e_1$	6	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ x_2\partial_1,\ (2x_1+\frac{x_3^2}{2}+x_2x_4)\partial_1+x_2\partial_2+x_3\partial_3+(x_3+x_4)\partial_4$
	7	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_3,\ x_2\partial_1,\ (2x_1+\frac{x_3^2}{2})\partial_1+x_2\partial_2+(x_3x_4+x_3-x_2)\partial_3+x_4^2\partial_4$
	8	$\partial_1,\ \partial_3,\ x_3\partial_1-\frac{1}{\ln Cx_2}\partial_3,\ x_2\partial_1,\ (2x_1+\frac{x_3^2}{2})\partial_1+x_2\partial_2+(x_3-\frac{x_3}{\ln Cx_2}-x_2)\partial_3$
$A_{5.22}$	1	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ \partial_2,\ \frac{x_3^2}{2}\partial_1+x_2\partial_2+x_3\partial_4+\partial_5$
$[e_2,e_5]=e_3$	2	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ \partial_2,\ \frac{x_3^2}{2}\partial_1+x_2\partial_2+x_3\partial_4$
$[e_4,e_5]=e_4$	3	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_2+x_5\partial_5,\ \partial_2,\ \frac{x_3^2}{2}\partial_1+(x_2+x_3x_4)\partial_2+x_3x_5\partial_3+(x_4+x_4x_5)\partial_4+x_5^2\partial_5$
$[e_2,e_3]=e_1$	4	$\partial_1,\partial_3,\ x_3\partial_1+Cx_4e^{-\frac{1}{x_4}}\partial_2+x_4\partial_3,\ \partial_2,\ \frac{x_3^2}{2}\partial_1+(x_2+Cx_3x_4e^{-\frac{1}{x_4}})\partial_2+x_3x_4\partial_3+x_4^2\partial_4$
	5	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_2,\ \partial_2,\ \frac{x_3^2}{2}\partial_1+(x_2+x_3x_4)\partial_2+x_4\partial_4$
	6	$\partial_1,\ \partial_3,\ x_3\partial_1,\ \partial_2,\ \frac{x_3^2}{2}\partial_1+x_2\partial_2+\partial_4$
	7	$\partial_1,\ \partial_3,\ x_3\partial_1,\ \partial_2,\ \frac{x_3^2}{2}\partial_1+x_2\partial_2$
	8	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ x_2\partial_1,\ \frac{x_3^2}{2}\partial_1-x_2\partial_2+x_3\partial_4$
	9	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_3,\ x_2\partial_1,\ \frac{x_3^2}{2}\partial_1-x_2\partial_2+x_3x_4\partial_3+x_4^2\partial_4$
	10	$\partial_1,\ \partial_3,\ x_3\partial_1+\frac{1}{\ln Cx_2}\partial_3,\ x_2\partial_1,\ \frac{x_3^2}{2}\partial_1-x_2\partial_2+\frac{x_3}{\ln Cx_2}\partial_3$

$A_{5.23}$	1	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ \partial_2,\ (2_1+\frac{x_3^2}{2})\partial_1+bx_2\partial_2+x_3\partial_3+(x_3+x_4)\partial_4+\partial_5$
$[e_1,e_5]=2e_1$	2	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ \partial_2,\ (2_1+\frac{x_3^2}{2})\partial_1+bx_2\partial_2+x_3\partial_3+(x_3+x_4)\partial_4$
$[e_2,e_5]=e_2+e_3$	3	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_2+x_5\partial_5,\ \partial_2,\ (2x_1+\frac{x_3^2}{2})\partial_1+(bx_2+x_3x_4)\partial_2+(x_3x_5+x_3)\partial_3+x_4(x_5+b-1)\partial_4+x_5^2\partial_5$
$[e_3,e_5]=e_3$	4	$\partial_1,\partial_3,\ x_3\partial_1+Cx_4e^{-\frac{b-1}{x_4}}\partial_2+x_4\partial_3,\ \partial_2,\ (2x_1+\frac{x_3^2}{2})\partial_1+(bx_2+Cx_3x_4e^{-\frac{b-1}{x_4}}\partial_2+(x_3+x_3x_4)\partial_3+x_4^2\partial_4$
$[e_4,e_5]=be_4$	5	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_2,\ \partial_2,\ (2x_1+\frac{x_3^2}{2})\partial_1+(bx_2+x_3x_4)\partial_2+x_3\partial_3+(b-1)x_4\partial_4$
$[e_2,e_3]=e_1$	6	$\partial_1,\ \partial_3,\ x_3\partial_1,\ \partial_2,\ (2x_1+\frac{x_3^2}{2})\partial_1+bx_2\partial_2+x_3\partial_3+\partial_4$
$b\neq 0$	7	$\partial_1,\ \partial_3,\ x_3\partial_1,\ \partial_2,\ (2x_1+\frac{x_3^2}{2})\partial_1+bx_2\partial_2+x_3\partial_3$
	8	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ x_2\partial_1,\ (2x_1+\frac{x_3^2}{2})\partial_1+(2-b)x_2\partial_2+x_3\partial_3+(x_3+x_4)\partial_4$
	9	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_3,\ x_2\partial_1,\ (2x_1+\frac{x_3^2}{2})\partial_1+(2-b)x_2\partial_2+(x_4+1)\partial_3+x_4^2\partial_4$
	10	$\partial_1,\ \partial_3,\ x_3\partial_1+\frac{b-2}{\ln Cx_2}\partial_3,\ x_2\partial_1,\ (2x_1+\frac{x_3^2}{2})\partial_1+(2-b)x_2\partial_2+(\frac{(b-2)x_3}{\ln Cx_2}+x_3)\partial_3$
$A_{5.24}$	1	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ \partial_2,\ (2x_1+\varepsilon x_2+\frac{x_3^2}{2})\partial_1+2x_2\partial_2+x_3\partial_3+(x_3+x_4)\partial_4+\partial_5$
$[e_1,e_5]=2e_1$	2	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ \partial_2,\ (2x_1+\varepsilon x_2+\frac{x_3^2}{2})\partial_1+2x_2\partial_2+x_3\partial_3+(x_3+x_4)\partial_4$
$[e_2,e_5]=e_2+e_3$	3	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_2+x_5\partial_3,\ \partial_2,\ (2x_1+\varepsilon x_2+\frac{x_3^2}{2})\partial_1+(2x_2+x_3x_4)\partial_2+(x_3x_5+x_3+\varepsilon x_4)\partial_3+(x_4+x_4x_5)\partial_4+x_5^2\partial_5$
$[e_3,e_5]=e_3$	4	$\partial_1,\partial_3,\ x_3\partial_1+Cx_4e^{-\frac{1}{x_4}}\partial_2+x_4\partial_3,\ \partial_2,\ (2x_1+\varepsilon x_2+\frac{x_3^2}{2})\partial_1+(2x_2+Cx_3x_4e^{-\frac{1}{x_4}})\partial_2+(x_3x_4+x_3+\varepsilon Cx_4e^{-\frac{1}{x_4}})\partial_3+x_4^2\partial_4$
$[e_4,e_5]=\varepsilon e_1+2e_4$	5	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_2,\ \partial_2,\ (2x_1+\varepsilon x_2+\frac{x_3^2}{2})\partial_1+(2x_2+x_3x_4)\partial_2+(x_3+\varepsilon x_4)\partial_3+x_4\partial_4$
$[e_2,e_3]=e_1$	6	$\partial_1,\ \partial_3,\ x_3\partial_1,\ \partial_2,\ (2x_1+\varepsilon x_2+\frac{x_3^2}{2})\partial_1+2x_2\partial_2+x_3\partial_3+\partial_4$
$\varepsilon=\pm 1$	7	$\partial_1,\ \partial_3,\ x_3\partial_1,\ \partial_2,\ (2x_1+\varepsilon x_2+\frac{x_3^2}{2})\partial_1+2x_2\partial_2+x_3\partial_3$
	8	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ x_2\partial_1,\ (2x_1+\frac{x_3^2}{2})\partial_1-\varepsilon\partial_2+x_3\partial_3+(x_3+x_4)\partial_4$
	9	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_3,\ x_2\partial_1,\ (2x_1+\frac{x_3^2}{2})\partial_1-\varepsilon\partial_2+(x_3+x_3x_4)\partial_3+x_4^2\partial_4$
	10	$\partial_1,\ \partial_3,\ x_3\partial_1+Ce^{-\varepsilon x_2}\partial_3,\ x_2\partial_1,\ (2x_1+\frac{x_3^2}{2})\partial_1-\varepsilon\partial_2+(x_3+Cx_3e^{-\varepsilon x_2})\partial_3$
$A_{5.25}$	1	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ \partial_2,\ (2px_1+\frac{x_3^2}{2}+\frac{x_4^2}{2})\partial_1+\beta x_2\partial_2+(px_3-x_4)\partial_3+(x_3+px_4)\partial_4+\partial_5$
$[e_1,e_5]=2pe_1$	2	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ \partial_2,\ (2px_1+\frac{x_3^2}{2}+\frac{x_4^2}{2})\partial_1+\beta x_2\partial_2+(px_3-x_4)\partial_3+(x_3+px_4)\partial_4$
$[e_2,e_5]=pe_2+e_3$	3	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_2+x_5\partial_3,\ \partial_2,\ (2px_1+\frac{x_3^2}{2})\partial_1+(\beta x_2+x_3x_4)\partial_2+(x_3x_5+px_3)\partial_3+x_4(\beta-p+x_5)\partial_4+(x_5^2+1)\partial_5$
$[e_3,e_5]=pe_3-e_2$	4	$\partial_1,\partial_3,\ x_3\partial_1+C\sqrt{x_4^2+1}e^{(\beta-p)\arctan x_4}\partial_2+x_4\partial_3,\ \partial_2,\ (2px_1+\frac{x_3^2}{2})\partial_1+(\beta x_2+Cx_3\sqrt{x_4^2+1}e^{(\beta-p)\arctan x_4})\partial_2+(px_3+x_3x_4)\partial_3+(x_4^2+1)\partial_4$
$[e_4,e_5]=\beta e_4$	5	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ x_2\partial_1,\ (2px_1+\frac{x_3^2}{2}-\frac{x_4^2}{2})\partial_1+(2p-\beta)x_2\partial_2+(px_3-x_4)\partial_3+(x_3+px_4)\partial_4$
$[e_2,e_3]=e_1$	6	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_3,\ x_2\partial_1,\ (2px_1+\frac{x_3^2}{2})\partial_1+(2p-\beta)x_2\partial_2+(px_3+x_3x_4)\partial_3+(x_4^2+1)\partial_4$
$\beta\neq 0$	7	$\partial_1,\ \partial_3,\ x_3\partial_1+\varphi(x_2)\partial_3,\ x_2\partial_1,\ (2px_1+\frac{x_3^2}{2})\partial_1-(2p-\beta)x_2\partial_2+(px_3+x_3\varphi(x_2))\partial_3$
$A_{5.26}$	1	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ \partial_2,\ (2px_1+\varepsilon x_2+\frac{x_3^2}{2}-\frac{x_4^2}{2})\partial_1+p(2x_2+x_3)\partial_2+(px_3-x_4)\partial_3+(x_3+px_4)\partial_4+\partial_5$
$[e_1,e_5]=2pe_1$	2	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ \partial_2,\ (2px_1+\varepsilon x_2+\frac{x_3^2}{2}-\frac{x_4^2}{2})\partial_1+p(2x_2+x_3)\partial_2+(px_3-x_4)\partial_3+(x_3+px_4)\partial_4$
$[e_2,e_5]=pe_2+e_3$	3	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_2+x_5\partial_3,\ \partial_2,\ (2px_1+\varepsilon x_2+\frac{x_3^2}{2})\partial_1+(2px_2+x_3x_4)\partial_2+(x_3x_5+px_3+\varepsilon x_4)\partial_3+x_4(p+x_5)\partial_4+(x_3x_5+1)\partial_5$

$[e_3,e_5]=pe_3-e_2$	4	$\partial_1,\partial_3,\ x_3\partial_1+C\sqrt{x_4^2+1}e^{p\arctan x_4}\partial_2+x_4\partial_3,\ \partial_2,\ (2px_1+\varepsilon x_2\frac{x_3^2}{2})\partial_1+(2px_2+Cx_3\sqrt{x_4^2+1}e^{(p\arctan x_4})\partial_2+(px_3+x_3x_4+\varepsilon C\sqrt{x_4^2+1}e^{p\arctan x_4})\partial_3+(x_4^2+1)\partial_4$
$[e_4,e_5]=\varepsilon e_1+2pe_4$	5	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ x_2\partial_1,\ (2px_1+\frac{x_3^2}{2}-\frac{x_4^2}{2})\partial_1-\varepsilon\partial_2+(px_3-x_4)\partial_3+(x_3+px_4)\partial_4$
$[e_2,e_3]=e_1$	6	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_3,\ x_2\partial_1,\ (2px_1+\frac{x_3^2}{2})\partial_1-\varepsilon x_2\partial_2+(px_3+x_3x_4)\partial_3+(x_4^2+1)\partial_4$
$\varepsilon=\pm1$	7	$\partial_1,\ \partial_3,\ x_3\partial_1+\tan(-\varepsilon x_2+C)\partial_3,\ x_2\partial_1,\ (2px_1+\frac{x_3^2}{2})\partial_1-\varepsilon\partial_2+(px_3+x_3\tan(-\varepsilon x_2+C))\partial_3$
$A_{5.27}$	1	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ \partial_2,\ (x_1+x_2)\partial_1+(x_2+x_4)\partial_2+x_4\partial_4+\partial_5$
$[e_1,e_5]=e_1$	2	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ \partial_2,\ (x_1+x_2)\partial_1+(x_2+x_4)\partial_2+x_4\partial_4$
$[e_3,e_5]=e_3+e_4$	3	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_2+x_5\partial_3,\ \partial_2,\ (x_1+x_2)\partial_1+x_2\partial_2+x_4\partial_3-\partial_4-x_5\partial_5$
$[e_4,e_5]=e_1+e_4$	4	$\partial_1,\partial_3,\ x_3\partial_1+\ln Cx_4\partial_2+x_4\partial_3,\ \partial_2,\ (x_1+x_2)\partial_1+x_2\partial_2+\ln Cx_4\partial_3-x_4\partial_4$
$[e_2,e_3]=e_1$	5	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_2,\ \partial_2,\ (x_1+x_2)\partial_1+x_2\partial_2+x_4)\partial_3-\partial_4$
	6	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ x_2\partial_1,\ (x_1+x_2x_4)\partial_1-\partial_2+x_4\partial_4$
	7	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_3,\ x_2\partial_1,\ x_1\partial_1-\partial_2-x_2\partial_3-x_4\partial_4$
	8	$\partial_1,\ \partial_3,\ x_3\partial_1+Ce^{x_2}\partial_3,\ x_2\partial_1,\ x_1\partial_1-\partial_2-x_2\partial_3$
$A_{5.28}$	1	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ \partial_2,\ ((\alpha+1)x_1+\frac{\alpha x_3^2}{2}-\alpha x_3x_4)\partial_1+(x_2+x_4)\partial_2+(x_3+x_4)\partial_4+\partial_5$
$[e_1,e_5]=(1+\alpha)e_1$	2	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ \partial_2,\ ((\alpha+1)x_1+\frac{\alpha x_3^2}{2}-\alpha x_3x_4)\partial_1+(x_2+x_4)\partial_2+(x_3+x_4)\partial_4$
$[e_2,e_5]=\alpha e_3$	3	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_2+x_5\partial_5,\ \partial_2,\ ((1+\alpha)x_1+\frac{\alpha x_3^2}{2})\partial_1+(x_2+\alpha x_3x_4)\partial_2+\alpha x_3(x_5+1)\partial_3+(\alpha x_4x_5-1)\partial_4+(\alpha x_5^2-x_5)\partial_5$
$[e_3,e_5]=e_3+e_4$	4	$\partial_1,\partial_3,\ x_3\partial_1+\varphi(x_4)\partial_2+x_4\partial_3,\ \partial_2,\ ((1+\alpha)x_1+\frac{x_3^2}{2})\partial_1+(x_2+\alpha x_3\varphi(x_4))\partial_2+\alpha x_3(x_4+1)\partial_3+(\alpha x_4^2-x_4)\partial_4$
$[e_4,e_5]=e_4$	5	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_2,\ \partial_2,\ ((1+\alpha)x_1+\frac{\alpha x_3^2}{2})\partial_1+(x_2+\alpha x_3x_4)\partial_2+\alpha x_3\partial_3-\partial_4$
$[e_2,e_3]=e_1$	6	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ x_2\partial_1,\ ((1+\alpha)x_1+\frac{\alpha x_3^2}{2}+x_2x_4-\alpha x_3x_4)\partial_1+\alpha x_2\partial_2+(\alpha x_3+x_4)\partial_4$
	7	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_3,\ x_2\partial_1,\ ((1+\alpha)x_1+\frac{\alpha x_3^2}{2})\partial_1+\alpha x_2\partial_2+(\alpha x_3x_4+\alpha x_3-x_4)\partial_3+(\alpha x_4^2-x_4)\partial_4$
	8	$\partial_1,\ \partial_3,\ x_3\partial_1+\frac{\alpha}{1-Cx_2^\alpha}\partial_3,\ x_2\partial_1,\ ((1+\alpha)x_1+\frac{\alpha x_3^2}{2}\partial_1+\alpha x_2\partial_2+(\frac{\alpha^2x_3}{1-Cx_2^\alpha}+\alpha x_3-x_2\partial_3$
$A_{5.29}$	1	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ \partial_2,\ x_1\partial_1+x_4\partial_2+x_3\partial_3+\partial_5$
$[e_1,e_5]=e_1$	2	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ \partial_2,\ x_1\partial_1+x_4\partial_2+x_3\partial_3+x_5\partial_4$
$[e_2,e_5]=e_2$	3	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ \partial_2,\ x_1\partial_1+x_4\partial_2+x_3\partial_3+C\partial_4$
$[e_3,e_5]=e_4$	4	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_2+x_5\partial_3,\ \partial_2,\ x_1\partial_1+x_3\partial_3-\partial_4+x_5\partial_5$
$[e_2,e_3]=e_1$	5	$\partial_1,\ \partial_3,\ x_3\partial_1+\ln Cx_4\partial_2+x_4\partial_3,\ \partial_2,\ x_1\partial_1+x_3\partial_3+x_4\partial_4$
	6	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_2,\ \partial_2,\ x_1\partial_1+x_3\partial_3-\partial_4$
	7	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ x_2\partial_1,\ x_1\partial_1+x_2\partial_2+x_3\partial_3$
	8	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_3,\ x_2\partial_1,\ x_1\partial_1+x_2\partial_2+(x_3-x_2)\partial_3+x_4\partial_4$
	9	$\partial_1,\ \partial_3,\ x_3\partial_1+Cx_2\partial_3,\ x_2\partial_1,\ x_1\partial_1+x_2\partial_2+(x_3-x_2)\partial_3$