BRANCHING RULES FOR COXETER GROUP ORBITS

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Objective

Reduction or ‘branching rules’ of Coxeter group orbits to the sum of equidimensional orbits of lower Coxeter groups.

The method

- Finding an $n \times m$ matrix P that projects the points of an orbit of a Coxeter group G into the points of orbits of a lower Coxeter group G';
- here n and m are the ranks of G and G' respectively;
- subsequent regrouping of the resulting points into orbits of G'.

Advantages

- For representations, there is no limit for the number of weights, but the orbits are at most of the size of the order of the Weyl group;
- extension of the method to the non-crystallographic Coxeter groups;
- the points of the Coxeter group orbits do not need to be on the weight lattice, but can be anywhere in the Euclidean space R^n.

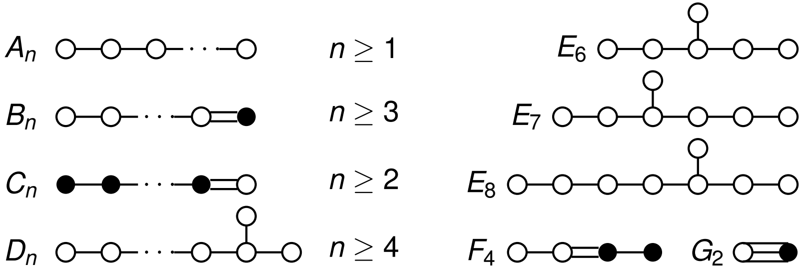
Finite Coxeter Groups

- Finite groups G are generated by reflections in a real Euclidean space R^n;
- the crystallographic ones are the Weyl groups of compact semisimple Lie groups;
- G-orbits are related to weight systems of finite dimensional irreducible representations of semisimple Lie algebras;
- G-orbits constitute the most efficient tool for large scale computations with semisimple Lie groups.

Finite Coxeter Groups

- G is generated by reflections $r_1, \ldots, r_n$, where $r_i = r_{\alpha_i}$, in mirrors that have origin as their common point;
- G is specified by a set of vectors $\{\alpha_1, \ldots, \alpha_n\}$ - the simple roots;
- G is described by its Dynkin diagram or its Cartan matrix.

Dynkin Diagrams



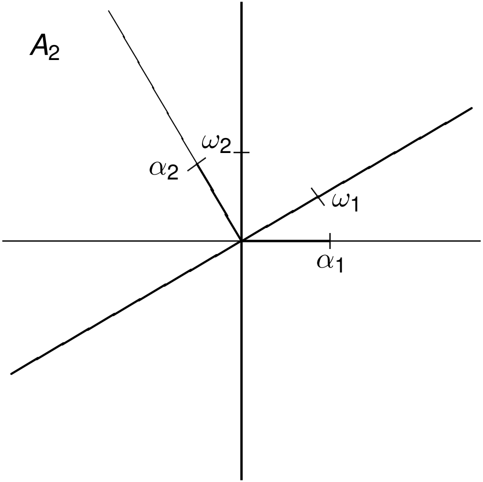
ω-basis

- $\{\omega_1, \omega_2, \ldots, \omega_n\}$ - the weights;
- $\alpha_i = C\omega_i\,, \quad \omega_i = C^{-1}\alpha_i$;
- each orbit contains precisely one point with non-negative coordinates in ω-basis : the dominant point;
- the root lattice Q and the weight lattice P of G are formed respectively by all integer linear combinations of simple roots and of fundamental weights of G;
- $Q \subseteq P$.

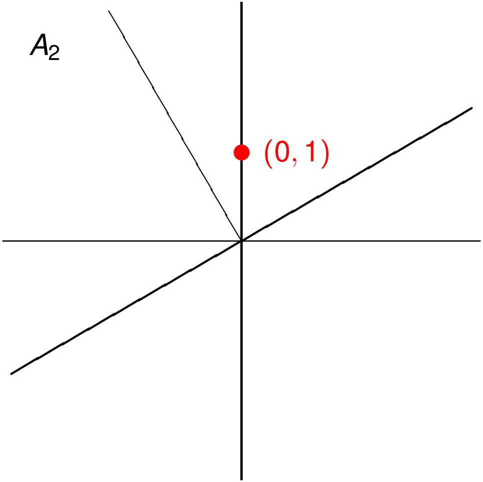
Orbits

- Generated from a single seed point by repeated application of n independent reflections;
- have a finite number of points equidistant from the origin, a generic orbit having the number of points equal to the order of the corresponding Coxeter group.

Example - Coxeter group A_2



Example - Coxeter group A_2



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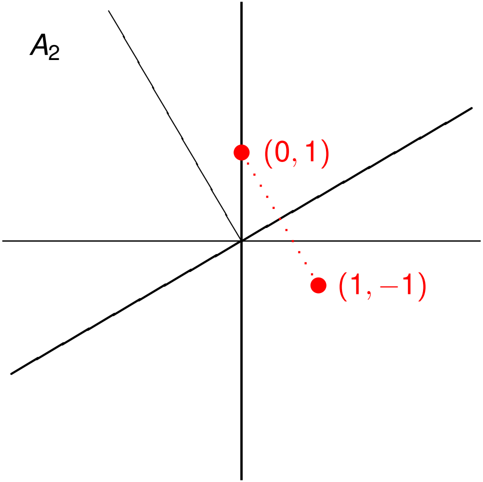

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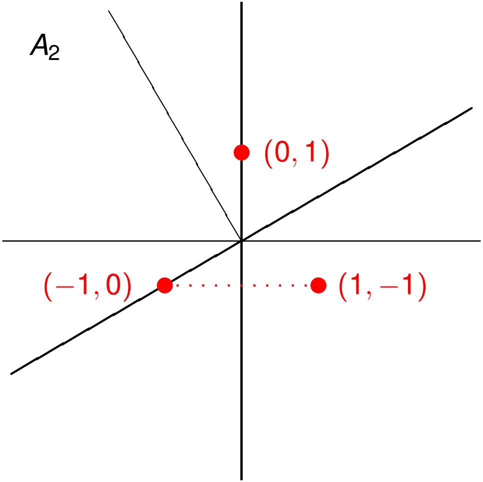

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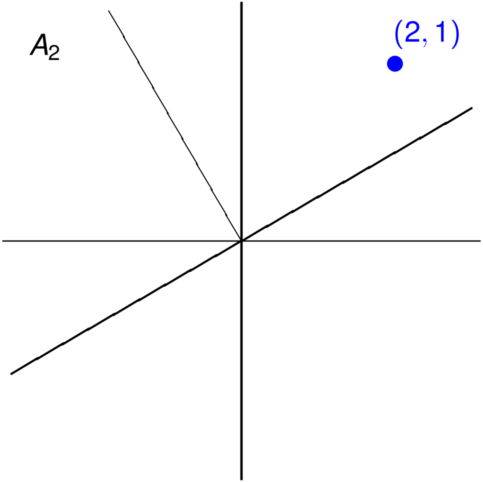

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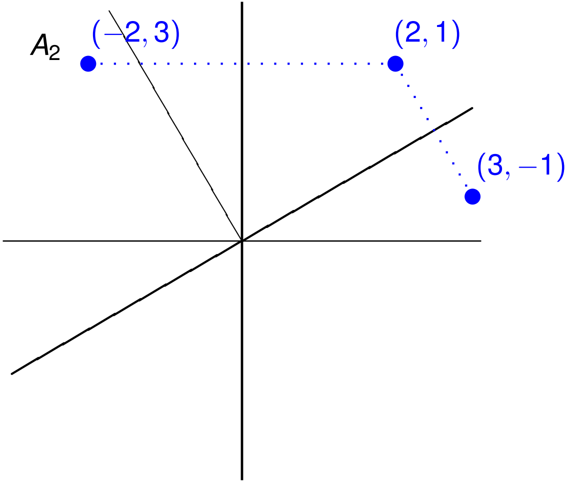

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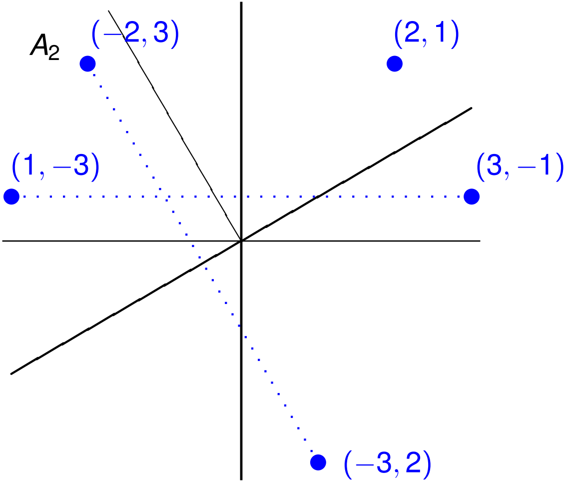

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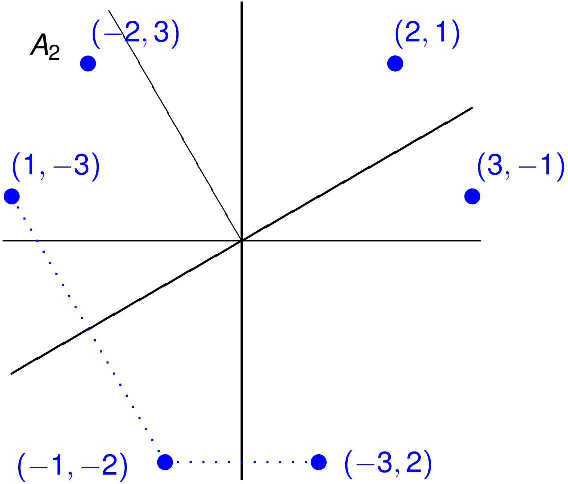
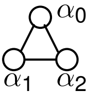
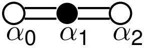


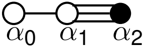
Orbit reduction

- Points of any orbit of G are projected by the $n \times m$ matrix P into the points of the corresponding orbits of G';
- the reduction is to equidimensional orbits, so the points do not move but are given in the new basis of the lower Coxeter group;
- after the projection, the points are sorted out in orbits of G'.

Reduction of Orbits of Coxeter Groups of Rank 2

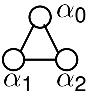
A_2 

C_2 

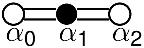
G_2 

Reduction of Orbits of Coxeter Groups of Rank 2

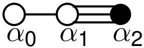
A_2



C_2



G_2



5 cases to consider:

- $A_2 \to A_1 \times U(1)$
- $C_2 \to A_1 \times A_1$
- $C_2 \to A_1 \times U(1)$
- $G_2 \to A_2$
- $G_2 \to A_1 \times A_1$

Projection matrices

$$A_2 \rightarrow A_1 \times U(1) : \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

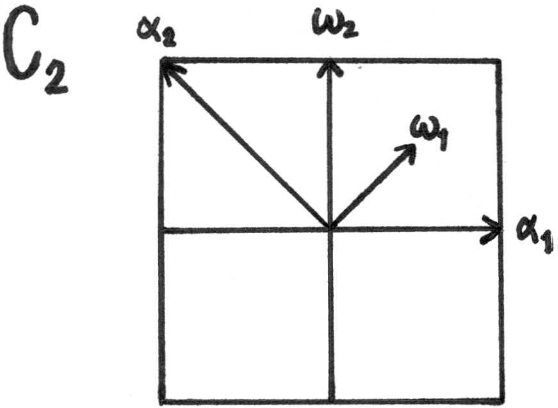
$$C_2 \rightarrow A_1 \times A_1 : \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$C_2 \rightarrow A_1 \times U(1) : \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$$

$$G_2 \rightarrow A_2 : \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$G_2 \rightarrow A_1 \times A_1 : \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}$$

$C_2 \longrightarrow A_1 \times A_1$



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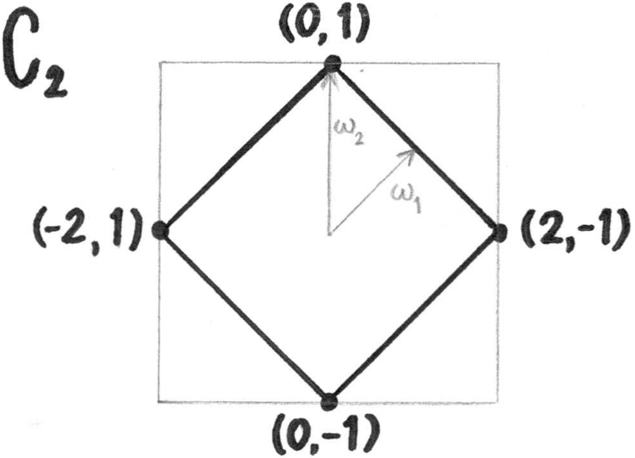


$C_2 \longrightarrow A_1 \times A_1$

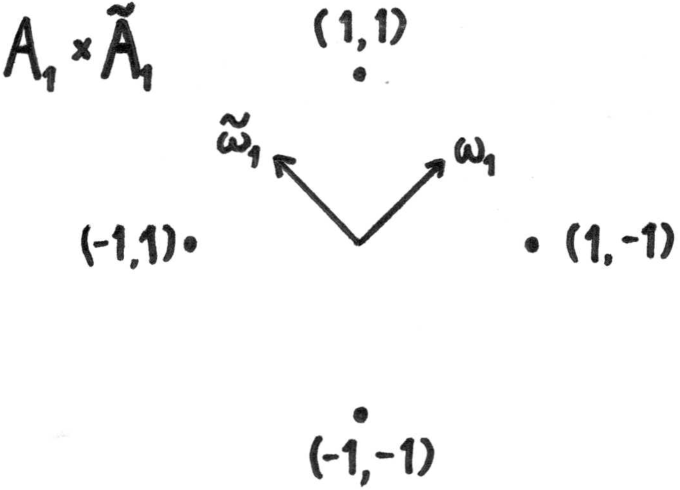
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$C_2 \longrightarrow A_1 \times A_1$



$C_2 \longrightarrow A_1 \times U(1)$