Quantum deformation of the Dirac bracket

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Symmetry in Nonlinear Mathematical Physics

- **Classical second-class constraints systems in phase space**
- **Quantum second-class constraints systems in the Hilbert space**
- **Weyl's association rule and the $\star$-product**
- **Quantum second-class constraints systems in phase space**
- **Application: quantum spherical pendulum in phase space**

Introduction

Gauge symmetries is the mathematical basis for fundamental interactions

Gauge theories $\longrightarrow$ 1-st class constraints systems
upon gauge fixing $\longrightarrow$ 2-nd class constraints systems

Quantum theory:

- **Operator quantization**
- **Path integral method**
- **$\star$-product technique known also as the deformation quantization and quantum dynamics in phase space**

❖ Operator quantization

CLASSICAL WORLD $\longleftrightarrow$ QUANTUM WORLD

phase space	$\longleftrightarrow$	Hilbert space
canonical variables	$\longleftrightarrow$	operators of canonical variables
$\xi^i = (q, p)$	$\longleftrightarrow$	$\mathfrak{x}^i = (\mathfrak{q}, \mathfrak{p})$
real functions	$\longleftrightarrow$	Hermitian operators

$$\xi^i \in T_*\mathbb{R}^n \quad\longleftrightarrow\quad \mathfrak{x}^i \in Op(L^2(\mathbb{R}^n))$$

$$\{\xi^i, \xi^j\} = -I^{ij} \quad\longleftrightarrow\quad [\mathfrak{x}^i, \mathfrak{x}^j] = -i\hbar I^{ij}$$

$$f(\xi) \in C^\infty(T_*\mathbb{R}^n) \quad\longleftrightarrow\quad \mathfrak{f} \in Op(L^2(\mathbb{R}^n))$$

Introduction

Path integral method

$$\text{CLASSICAL WORLD} \longleftrightarrow \text{QUANTUM WORLD}$$

$$\begin{aligned} \text{phase space} &\longleftrightarrow \text{phase space} \\ \text{canonical variables} &\longleftrightarrow \text{canonical variables} \\ \text{phase space trajectories} &\longleftrightarrow \text{transition amplitudes} \end{aligned}$$

$$< q(t')|e^{-i\hat{H}(t'-t)}|q(t) >= \int \prod \frac{dpdq}{2\pi} \exp(-i \int_t^{t'} (p\dot{q} - L) d\tau).$$

Introduction

❖ $\star$-product technique (Groenewold (1946))

$$\left.\left.\begin{array}{ccc} f(\xi) & \longleftrightarrow & \mathfrak{f} \\ g(\xi) & \longleftrightarrow & \mathfrak{g} \\ c \times f(\xi) & \longleftrightarrow & c \times \mathfrak{f} \\ f(\xi) + g(\xi) & \longleftrightarrow & \mathfrak{f} + \mathfrak{g} \end{array}\right\} \text{vector space} \\ \begin{array}{ccc} f(\xi) \star g(\xi) & \longleftrightarrow & \mathfrak{f}\mathfrak{g} \end{array} \right\} \text{algebra}$$

$\star$-PRODUCT

$$f \star g = f \exp(\frac{i\hbar}{2}\mathcal{P})g = f \circ g + \frac{i\hbar}{2} f \wedge g$$

$$\mathcal{P} = -I^{kl} \frac{\overleftarrow{\partial}}{\partial \xi^k} \frac{\overrightarrow{\partial}}{\partial \xi^l} \qquad \lim_{\hbar \to 0} f \wedge g = \{f, g\}$$

$$I^{kl} = \left\| \begin{array}{cc} 0 & -E_n \\ E_n & 0 \end{array} \right\|$$

Introduction

Brackets govern evolution of systems in phase space

Systems:	unconstrained	constrained
classical	$\{f,g\}$	$\{f,g\}_D$
quantum	$f\wedge g$	???

- Poisson bracket $\{f,g\} = f\mathcal{P}g$
- Dirac bracket $\{f,g\}_D = \{f,g\} + \{f,\mathcal{G}^a\}\{\mathcal{G}_a,g\}$
- Moyal bracket $\{f,g\} \overset{\text{quantum deformation}}{\rightrightarrows\rightrightarrows\rightrightarrows} f\wedge g = f\frac{2}{\hbar}\sin(\frac{\hbar}{2}\mathcal{P})g$
- Fourth bracket $\{f,g\}_D \overset{\text{quantum deformation}}{\rightrightarrows\rightrightarrows\rightrightarrows} f_t\wedge g_t =$???

Just making Hamiltonian formalizm complete

Classical second-class constraints systems in phase space

Table 1: Euclidean and symplectic spaces: Similarities and dissimilarities

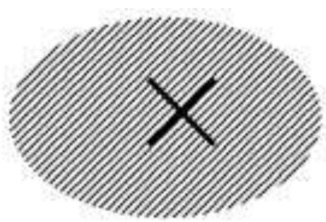
Euclidean space $x, y \in \mathbb{R}^n$	**Symplectic space** $\xi, \zeta \in \mathbb{R}^{2n}$
Metric structure $g_{ij} = g_{ji}$ $g_{ij}g^{jk} = \delta_i^k$	Symplectic structure $I_{ij} = -I_{ji}$ $I_{ij}I^{jk} = \delta_i^k$
Scalar product $(x, y) = g_{ij}x^i y^j$	Skew − scalar product $(\xi, \zeta) = I_{ij}\xi^i\zeta^j$
Distance $L = \sqrt{(x-y, x-y)}$	Area $\mathcal{A} = (\xi, \zeta)$
Gradient $(\bigtriangledown f)^i = g^{ij}\partial f/\partial x^j$	Skew − gradient $(Idf)^i \equiv -I^{ij}\partial f/\partial \xi^j = \{\xi^i, f\}$
Scalar product of gradients of f and g $(\bigtriangledown f, \bigtriangledown g)$	Poisson bracket of f and g $(Idf, Idg) = \{f, g\}$
Orthogonality $g_{ij}x^i y^j = 0$	Skew − orthogonality $I_{ij}\xi^i\zeta^j = 0$

Classical second-class constraints systems in phase space

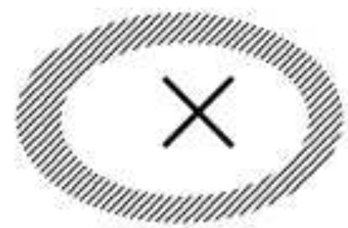
Constraints $\mathcal{G}_a = 0$ **are non-degenerate** ($a = 1, ..., 2m$, $m < n$)**:**

$$\det\{\mathcal{G}_{\mathbf{a}}, \mathcal{G}_{\mathbf{b}}\} \neq \mathbf{0}.$$

Locally, in Riemannian space



Locally, in symplectic space



Locally, in symplectic space one can always find the Darboux basis

$$\{\mathcal{G}_{\mathbf{a}}, \mathcal{G}_{\mathbf{b}}\} = \mathcal{I}_{\mathbf{ab}}$$

where

$$\mathcal{I}_{\mathbf{ab}} = \left\| \begin{array}{ll} 0 & E_m \\ -E_m & 0 \end{array} \right\|,$$

with E_m **being identity** $m \times m$ **matrix.**

Skew-gradient projections $\xi_s(\xi)$

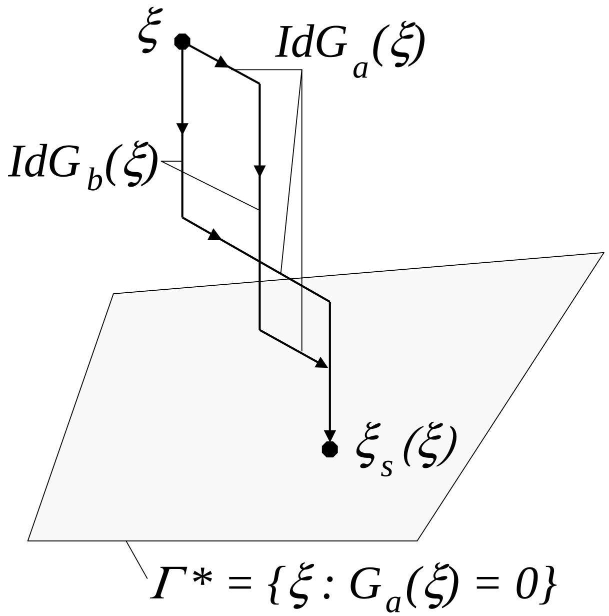
Expanding in the power series of $\mathcal{G}_a$,

$$\xi_s(\xi) = \xi + \mathbf{X}^{a}\mathcal{G}_{a} + \frac{1}{2}\mathbf{X}^{ab}\mathcal{G}_{a}\mathcal{G}_{b} + ...,$$

and requiring

$$\{\xi_s(\xi), \mathcal{G}_a(\xi)\} = 0,$$

one gets:



$$\xi_s(\xi) = \sum_{k=0}^{\infty} \frac{1}{k!}\{...\{\{\xi, \mathcal{G}^{a_1}\}, \mathcal{G}^{a_2}\}, ...\mathcal{G}^{a_k}\}\mathcal{G}_{a_1}\mathcal{G}_{a_2}...\mathcal{G}_{a_k}$$

$$f_s(\xi) = \sum_{k=0}^{\infty} \frac{1}{k!}\{...\{\{f(\xi), \mathcal{G}^{a_1}\}, \mathcal{G}^{a_2}\}, ...\mathcal{G}^{a_k}\}\mathcal{G}_{a_1}\mathcal{G}_{a_2}...\mathcal{G}_{a_k} = f(\xi_s(\xi))$$

Classical second-class constraints systems in phase space

The average of a function $f(\xi)$ is calculated using the probability density distribution $\rho(\xi)$ and the Liouville measure restricted to the constraint submanifold:

$$<\mathbf{f}>=\int \frac{\mathbf{d^{2n}}\xi}{(\mathbf{2}\pi)^{\mathbf{n}}}(\mathbf{2}\pi)^{\mathbf{m}}\prod_{\mathbf{a=1}}^{\mathbf{2m}}\delta(\mathcal{G}_{\mathbf{a}}(\xi))\mathbf{f}(\xi)\rho(\xi).$$

On the constraint submanifold $f(\xi)$ and $\rho(\xi)$ can be replaced with $f_s(\xi)$ and $\rho_s(\xi)$

Equivalence classes of functions in phase space

$$\mathbf{f}(\xi)\sim\mathbf{g}(\xi)\leftrightarrow\mathbf{f_s}(\xi)=\mathbf{g_s}(\xi)$$

In particular, $f(\xi)\sim f_s(\xi)$ and $\mathcal{G}_a\sim 0$.

Evolution of function f

$$\frac{\partial}{\partial\mathbf{t}}\mathbf{f}=\{\mathbf{f},\mathcal{H}\}_{\mathbf{D}}$$

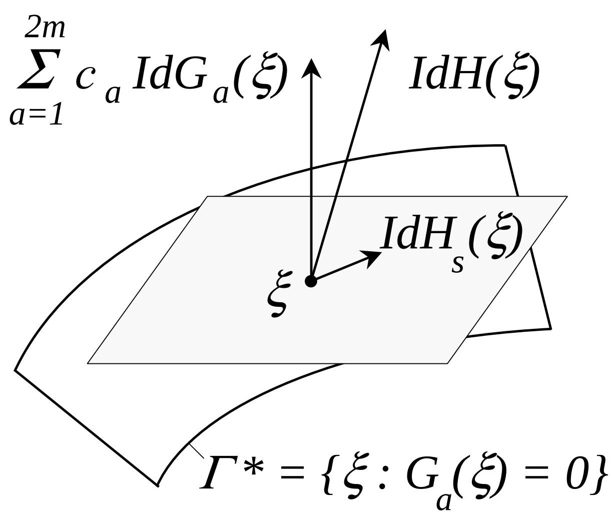
On the constraint submanifold

$$\{\mathbf{f},\mathbf{g}\}_{\mathbf{D}}=\{\mathbf{f},\mathbf{g_s}\}=\{\mathbf{f_s},\mathbf{g}\}=\{\mathbf{f_s},\mathbf{g_s}\}$$

Classical second-class constraints systems in phase space

Replacing $\mathcal{H} \longrightarrow \mathcal{H}_s$, one can rewrite the evolution equation in terms of the Poisson bracket:

$$\frac{\partial}{\partial \mathbf{t}}\mathbf{f} = \{\mathbf{f}, \mathcal{H}_\mathbf{s}\}$$



physical observables

$\Updownarrow$

equivalence classes of functions in phase space

$\Updownarrow$

$$\overbrace{\underbrace{\underbrace{f+g,\ c\times f,}_{\text{vector space}}\ \ fg,}_{\text{algebra}}\ \{f,g\}_D = \{f_s, g_s\}}^{\text{Poisson algebra}}$$

Quantum second-class constraints systems in the Hilbert space

To any function $f(\xi)$ in the unconstrained phase space one may associate an operator $\mathfrak{f}$ in the corresponding Hilbert space

$$\mathcal{H}(\xi) \quad \longleftrightarrow \quad \mathfrak{H}$$

$$\mathcal{G}_a(\xi) \quad \longleftrightarrow \quad \mathfrak{G}_a$$

$$[\mathfrak{G}_\mathbf{a}, \mathfrak{G}_\mathbf{b}] = \mathbf{i}\hbar\mathcal{I}_\mathbf{ab}$$

Projected operator $\mathfrak{f}_s$

$$\mathfrak{f}_\mathbf{s} = \sum_{\mathbf{k}=\mathbf{0}}^{\infty} \frac{(-\mathbf{i}/\hbar)^\mathbf{k}}{\mathbf{k}!}[...[[\mathfrak{f}, \mathfrak{G}^{\mathbf{a_1}}], \mathfrak{G}^{\mathbf{a_2}}], ...\mathfrak{G}^{\mathbf{a_k}}]\mathfrak{G}_{\mathbf{a_1}}\mathfrak{G}_{\mathbf{a_2}}...\mathfrak{G}_{\mathbf{a_k}}.$$

One has

$$[\mathfrak{f}_\mathbf{s}, \mathfrak{G}_\mathbf{a}] = \mathbf{0}$$

$$(\mathfrak{f}\mathfrak{g}_\mathbf{s})_\mathbf{s} = (\mathfrak{f}_\mathbf{s}\mathfrak{g})_\mathbf{s} = \mathfrak{f}_\mathbf{s}\mathfrak{g}_\mathbf{s}$$

$\mathfrak{f}$ and $\mathfrak{g}$ belong to the same equivalence class provided

$$\mathfrak{f} \sim \mathfrak{g} \leftrightarrow \mathfrak{f}_\mathbf{s} = \mathfrak{g}_\mathbf{s}$$

Quantum second-class constraints systems in the Hilbert space

How to calculate the average value of an operator?

Projection operator:

$$\mathfrak{P} = \int \frac{\mathbf{d^{2m}}\lambda}{(\mathbf{2}\pi\hbar)^{\mathbf{m}}} \prod_{\mathbf{a=1}}^{\mathbf{2m}} \exp(\frac{\mathbf{i}}{\hbar}\mathfrak{G}^{\mathbf{a}}\lambda_{\mathbf{a}})$$

Chose basis in the Hilbert space in which the first m constraint operators are diagonal,

$$\mathfrak{G}^{\mathbf{a}}|\mathbf{g}, \mathbf{g}_* >= \mathbf{g}^{\mathbf{a}}|\mathbf{g}, \mathbf{g}_* >,$$

for $a = 1, ..., m$. $\mathfrak{G}^a$ might be taken as momentum operators. The last m constraint operators can be treated as quantal coordinates.

$$\mathfrak{P}|\mathbf{g}, \mathbf{g}_* >= |\mathbf{0}, \mathbf{g}_* >$$

The average value of an operator $\mathfrak{f}$

$$< \mathfrak{f} >= \mathbf{Tr}[\mathfrak{P}\mathfrak{f}_{\mathbf{s}}\mathfrak{r}_{\mathbf{s}}] = \int \frac{\mathbf{d^{n-m}g_*}}{(\mathbf{2}\pi\hbar)^{\mathbf{n-m}}} < \mathbf{0}, \mathbf{g}_*|\mathfrak{f}_{\mathbf{s}}\mathfrak{r}_{\mathbf{s}}|\mathbf{0}, \mathbf{g}_* >.$$

is determined by the physical subspace of the Hilbert space, spanned by $|0, g_* >$.

Quantum second-class constraints systems in the Hilbert space

Physical states satisfy

$$\mathfrak{G}^{\mathbf{a}}|\mathbf{0}, \mathbf{g}_* >= \mathbf{0}$$

Dirac's supplementary condition

for an equivalent gauge system, where

$\mathfrak{G}^a$ **with** $a = 1, ..., m$ **are gauge generators**

$\mathfrak{G}^a$ **with** $a = m + 1, ..., 2m$ **are gauge-fixing operators.**

Quantum evolution equation

$$\mathbf{i}\hbar \frac{\mathbf{d}}{\mathbf{dt}} \mathfrak{f} = [\mathfrak{f}, \mathfrak{H}_{\mathbf{s}}]$$

Evolution does not mix equivalence classes of operators

$$\mathfrak{f}(\mathbf{t}) \sim \mathfrak{g}(\mathbf{t}) \leftrightarrow \mathfrak{f}(\mathbf{0}) \sim \mathfrak{g}(\mathbf{0})$$

Weyl's association rule and the star-product

$$\begin{array}{c}\text{physical observables}\\ \Updownarrow \\ \text{equivalence classes of operators in the Hilbert space}\\ \Updownarrow \\ \overbrace{\underbrace{\underbrace{\mathfrak{f}+\mathfrak{g},\ c\times\mathfrak{f},}_{\text{vector space}}\ \mathfrak{f}\mathfrak{g},}_{\text{algebra}}\ [\mathfrak{f},\mathfrak{g}]}^{\text{Poisson algebra}}\end{array}$$

Vector space? Choose basis! Weyl's basis:

$$\mathfrak{B}(\xi) = (\mathbf{2}\pi\hbar)^{\mathbf{n}}\delta^{\mathbf{2n}}(\xi - \mathfrak{x}) = \int \frac{\mathbf{d}^{\mathbf{2n}}\eta}{(\mathbf{2}\pi\hbar)^{\mathbf{n}}} \exp(-\frac{\mathbf{i}}{\hbar}\eta_{\mathbf{k}}(\xi - \mathfrak{x})^{\mathbf{k}}).$$

The Weyl's association rule

$$\begin{aligned}\mathbf{f}(\xi) &= \mathbf{Tr}[\mathfrak{B}(\xi)\mathfrak{f}],\\ \mathfrak{f} &= \int \frac{\mathbf{d}^{\mathbf{2n}}\xi}{(\mathbf{2}\pi\hbar)^{\mathbf{n}}}\mathbf{f}(\xi)\mathfrak{B}(\xi).\end{aligned}$$

Weyl's association rule and the star-product

$\mathfrak{B}(\xi) \in Op(L^2(\mathbb{R}^n))$ and projection operators $P_i = \mathbf{e}_i \otimes \mathbf{e}_i$ acting in $\mathbb{R}^n$ satisfy

$$\begin{aligned}
\mathfrak{B}(\xi)^+ = \mathfrak{B}(\xi) \quad &\leftrightarrow \quad (P_i)^+ = P_i, \\
Tr[\mathfrak{B}(\xi)] = 1 \quad &\leftrightarrow \quad \mathbf{e}_i \cdot \mathbf{e}_i = 1, \\
\int \frac{d^{2n}\xi}{(2\pi\hbar)^n}\mathfrak{B}(\xi) = \mathbf{1} \quad &\leftrightarrow \quad \sum_i \mathbf{e}_i \otimes \mathbf{e}_i = 1, \\
\int \frac{d^{2n}\xi}{(2\pi\hbar)^n}\mathfrak{B}(\xi)Tr[\mathfrak{B}(\xi)\mathfrak{f}] = \mathfrak{f} \leftrightarrow \sum_i P_i(P_i(\sum_j g_j P_j)) \quad &= \quad \sum_j g_j P_j, \\
Tr[\mathfrak{B}(\xi)\mathfrak{B}(\xi')] = (2\pi\hbar)^n \delta^{2n}(\xi - \xi') \quad &\leftrightarrow \quad \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \\
\mathfrak{B}(\xi)\exp(-\frac{i\hbar}{2}\mathcal{P}_{\xi\xi'})\mathfrak{B}(\xi') = (2\pi\hbar)^n \delta^{2n}(\xi - \xi')\mathfrak{B}(\xi') \quad &\leftrightarrow \quad ?
\end{aligned}$$

and therefore $\xi \leftrightarrow i$.

Quantum second-class constraints systems in phase space

Symplectic basis for constraint functions

$$\mathbf{G_a}(\xi) \wedge \mathbf{G_b}(\xi) = \mathcal{I}_{\mathbf{ab}}$$

Skew-gradient projections for canonical variables and functions in phase space

$$\xi_{\mathbf{t}}(\xi) \wedge \mathbf{G_a}(\xi) = \mathbf{0} \quad \& \quad \mathbf{f_t}(\xi) \wedge \mathbf{G_a}(\xi) = \mathbf{0}$$

Projected canonical variables

$$\xi_{\mathbf{t}}(\xi) = \sum_{\mathbf{k=0}}^{\infty} \frac{\mathbf{1}}{\mathbf{k!}}(...((\xi \wedge \mathbf{G^{a_1}}) \wedge \mathbf{G^{a_2}})... \wedge \mathbf{G^{a_k}}) \circ \mathbf{G_{a_1}} \circ \mathbf{G_{a_2}} ... \circ \mathbf{G_{a_k}}$$

$$\mathbf{f_t}(\xi) = \sum_{\mathbf{k=0}}^{\infty} \frac{\mathbf{1}}{\mathbf{k!}}(...((\mathbf{f}(\xi) \wedge \mathbf{G^{a_1}}) \wedge \mathbf{G^{a_2}})... \wedge \mathbf{G^{a_k}}) \circ \mathbf{G_{a_1}} \circ \mathbf{G_{a_2}} ... \circ \mathbf{G_{a_k}}$$

Classical limit

$$\lim_{\hbar \to \mathbf{0}} \xi_{\mathbf{t}}(\xi) = \xi_{\mathbf{s}}(\xi) \quad \& \quad \lim_{\hbar \to \mathbf{0}} \mathbf{f_t}(\xi) = \mathbf{f_s}(\xi)$$

.

Quantum second-class constraints systems in phase space

Equivalence relations between functions

$$\mathbf{f}(\xi) \sim \mathbf{g}(\xi) \leftrightarrow \mathbf{f_t}(\xi) = \mathbf{g_t}(\xi)$$

$f(\xi) \sim f_t(\xi)$ and $f(\xi) \neq f_t(\xi)$ for $G_a(\xi) = 0$,
therefore $\sim$ and $\approx$ acquire distinct meaning.

The average value of function

$$< \mathbf{f} >= \int \frac{\mathbf{d^{2n}}\xi}{(\mathbf{2}\pi\hbar)^{\mathbf{n}}} \mathbf{P}(\xi) \star \mathbf{f_t}(\xi) \star \mathbf{W_t}(\xi)$$

where $P(\xi)$ is the symbol of the projection operator $\mathfrak{P}$ and $W(\xi)$ is the Wigner function.

EVOLUTION EQUATION:

$$\frac{\partial}{\partial \mathbf{t}} \mathbf{f}(\xi) = \mathbf{f}(\xi) \wedge \mathbf{H_t}(\xi)$$

Quantum spherical pendulum in phase space

Mathematical pendulum on S^{n-1} sphere of unit radius in n-dimensional Euclidean space with coordinates ϕ^α.

The hamiltonian function projected onto the constraint submanifold

$$\mathbf{H_t} = \frac{\mathbf{1}}{\mathbf{2}}(\phi^{\mathbf{2}}\delta^{\alpha\beta} - \phi^\alpha\phi^\beta)\pi^\alpha\pi^\beta$$

Constraint functions

$$\begin{aligned} G^1(\xi) &= \frac{1}{2}\ln\phi^\alpha\phi^\alpha, \\ G^2(\xi) &= \phi^\alpha\pi^\alpha, \end{aligned}$$

where $\xi = (\phi^\alpha, \pi^\alpha)$**, so that**

$$\mathbf{G_a}(\xi) \wedge \mathbf{G_b}(\xi) = \mathcal{I}_{\mathbf{ab}}.$$

the global symplectic basis exists!

Evolution equation for Wigner function

The power series expansion of the Moyal bracket is truncated at $O(\hbar^2)$, since the Hamiltonian $H_t(\xi)$ is a fourth degree polynomial of canonical variables, so we obtain

$$\begin{aligned}\frac{\partial}{\partial \mathbf{t}}\mathbf{W} = & - \{\mathbf{W}, \mathcal{H}_{\mathbf{s}}\} + \frac{\hbar^{\mathbf{2}}}{\mathbf{8}}\Big(\frac{\partial^{\mathbf{3}}\mathbf{W}}{\partial\phi^{\alpha}\partial\phi^{\beta}\partial\pi^{\gamma}}(\mathbf{2}\delta^{\alpha\beta}\phi^{\gamma} - \delta^{\alpha\gamma}\phi^{\beta} - \delta^{\beta\gamma}\phi^{\alpha}) \\ & - \frac{\partial^{\mathbf{3}}\mathbf{W}}{\partial\pi^{\alpha}\partial\pi^{\beta}\partial\phi^{\gamma}}(\mathbf{2}\delta^{\alpha\beta}\pi^{\gamma} - \delta^{\alpha\gamma}\pi^{\beta} - \delta^{\beta\gamma}\pi^{\alpha})\Big).\end{aligned}$$

The first term in the right side is of the classical origin, while the second term represents a quantum correction to the classical Liouville equation and there are no other quantum corrections. Given $W(\xi, 0)$ in the unconstrained phase space, $W(\xi, t)$ can be found by solving the PDE.

Conclusion

Systems:	unconstrained	constrained
classical	$\{f, g\}$	$\{f, g\}_D$
quantum	$f \wedge g$	$f_t \wedge g_t$

- Four types of systems
- Four types of brackets which govern evolution of systems
- The fourth bracket $f_t \wedge g_t$ has been constructed

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