Quantum deformation of the Dirac bracket


M. I. Krivoruchenko
ITEP, Moscow

June 22, 2009, Kiev
Symmetry in Nonlinear Mathematical Physics

- Classical second-class constraints systems in phase space
- Quantum second-class constraints systems in the Hilbert space
- Weyl’s association rule and the \( \star \)-product
- Quantum second-class constraints systems in phase space
- Application: quantum spherical pendulum in phase space
Introduction

Gauge symmetries is the mathematical basis for fundamental interactions

Gauge theories $\rightarrow$ 1-st class constraints systems
upon gauge fixing $\rightarrow$ 2-nd class constraints systems

Quantum theory:

✦ Operator quantization
✦ Path integral method
✦ $\ast$-product technique known also as the deformation quantization and quantum dynamics in phase space
Introducing Operator quantization

CLASSICAL WORLD $\leftrightarrow$ QUANTUM WORLD

- Phase space $\leftrightarrow$ Hilbert space
- Canonical variables $\leftrightarrow$ Operators of canonical variables

\[ \xi^i = (q, p) \leftrightarrow \mathfrak{x}^i = (q, p) \]

- Real functions $\leftrightarrow$ Hermitian operators

\[ \xi^i \in T^* \mathbb{R}^n \leftrightarrow \mathfrak{x}^i \in \text{Op}(L^2(\mathbb{R}^n)) \]
\[ \{\xi^i, \xi^j\} = -I^{ij} \leftrightarrow [\mathfrak{x}^i, \mathfrak{x}^j] = -i\hbar I^{ij} \]
\[ f(\xi) \in C^\infty(T^* \mathbb{R}^n) \leftrightarrow f \in \text{Op}(L^2(\mathbb{R}^n)) \]
Path integral method

\[ \langle q(t') | e^{-i\hat{H}(t'-t)} | q(t) \rangle = \int \prod \frac{dp dq}{2\pi} \exp(-i \int_t^{t'} (p \dot{q} - L) d\tau). \]
**Introduction**

- *-product technique (Groenewold (1946))

\[
\begin{align*}
    f(\xi) & \iff f \\
    g(\xi) & \iff g \\
    c \times f(\xi) & \iff c \times f \\
    f(\xi) + g(\xi) & \iff f + g \\
    f(\xi) \star g(\xi) & \iff fg
\end{align*}
\]

\[
\begin{align*}
\star-\text{PRODUCT} \\
    f \star g & = f \exp\left(\frac{i\hbar}{2}\mathcal{P}\right)g = f \circ g + \frac{i\hbar}{2} f \wedge g
\end{align*}
\]

\[
\mathcal{P} = -I^{kl} \begin{pmatrix} \partial & \partial \\ \partial \xi^k & \partial \xi^l \end{pmatrix} \\
I^{kl} = \begin{vmatrix} 0 & -E_n \\ E_n & 0 \end{vmatrix}
\]

\[
\lim_{\hbar \to 0} f \wedge g = \{f, g\}
\]
## Introduction

Brackets govern evolution of systems in phase space

<table>
<thead>
<tr>
<th>Systems:</th>
<th>unconstrained</th>
<th>constrained</th>
</tr>
</thead>
<tbody>
<tr>
<td>classical</td>
<td>( { f, g } )</td>
<td>( { f, g }_D )</td>
</tr>
<tr>
<td>quantum</td>
<td>( f \wedge g )</td>
<td>???</td>
</tr>
</tbody>
</table>

- Poisson bracket \( \{ f, g \} = f \mathcal{P} g \)
- Dirac bracket \( \{ f, g \}_D = \{ f, g \} + \{ f, \mathcal{G}^a \} \{ \mathcal{G}_a, g \} \)
- Moyal bracket \( \{ f, g \} \implies f \wedge g = f \frac{2}{\hbar} \sin(\frac{\hbar}{2} \mathcal{P}) g \)
- Fourth bracket \( \{ f, g \}_D \implies f_t \wedge g_t = ??? \)

**Just making Hamiltonian formalizm complete**
Table 1: Euclidean and symplectic spaces: Similarities and dissimilarities

<table>
<thead>
<tr>
<th>Euclidean space</th>
<th>Symplectic space</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x, y \in \mathbb{R}^n$</td>
<td>$\xi, \zeta \in \mathbb{R}^{2n}$</td>
</tr>
<tr>
<td><strong>Metric structure</strong></td>
<td><strong>Symplectic structure</strong></td>
</tr>
<tr>
<td>$g_{ij} = g_{ji}$</td>
<td>$I_{ij} = -I_{ji}$</td>
</tr>
<tr>
<td>$g_{ij}g^{jk} = \delta^k_i$</td>
<td>$I_{ij}I^{jk} = \delta^k_i$</td>
</tr>
<tr>
<td><strong>Scalar product</strong></td>
<td><strong>Skew – scalar product</strong></td>
</tr>
<tr>
<td>$(x, y) = g_{ij}x^i y^j$</td>
<td>$(\xi, \zeta) = I_{ij}\xi^i \zeta^j$</td>
</tr>
<tr>
<td><strong>Distance</strong></td>
<td><strong>Area</strong></td>
</tr>
<tr>
<td>$L = \sqrt{(x - y, x - y)}$</td>
<td>$A = (\xi, \zeta)$</td>
</tr>
<tr>
<td><strong>Gradient</strong></td>
<td><strong>Skew – gradient</strong></td>
</tr>
<tr>
<td>$(\nabla f)^i = g^{ij}\partial f / \partial x^j$</td>
<td>$(Id f)^i \equiv -I^{ij}\partial f / \partial \xi^j$</td>
</tr>
<tr>
<td><strong>Scalar product of gradients of $f$ and $g$</strong></td>
<td><strong>Poisson bracket of $f$ and $g$</strong></td>
</tr>
<tr>
<td>$(\nabla f, \nabla g)$</td>
<td>$(Id f, Id g) = {f, g}$</td>
</tr>
<tr>
<td><strong>Orthogonality</strong></td>
<td><strong>Skew – orthogonality</strong></td>
</tr>
<tr>
<td>$g_{ij}x^i y^j = 0$</td>
<td>$I_{ij}\xi^i \zeta^j = 0$</td>
</tr>
</tbody>
</table>
Classical second-class constraints systems in phase space

Constraints $\mathcal{G}_a = 0$ are non-degenerate ($a = 1, \ldots, 2m, m < n$): 

$$\det\{\mathcal{G}_a, \mathcal{G}_b\} \neq 0.$$ 

Locally, in Riemannian space

Locally, in symplectic space

Locally, in symplectic space one can always find the Darboux basis

$$\{\mathcal{G}_a, \mathcal{G}_b\} = \mathcal{I}_{ab}$$

where

$$\mathcal{I}_{ab} = \begin{pmatrix} 0 & E_m \\ -E_m & 0 \end{pmatrix},$$

with $E_m$ being identity $m \times m$ matrix.
Expanding in the power series of $G_a$,

$$\xi_s(\xi) = \xi + X^a G_a + \frac{1}{2} X^{ab} G_a G_b + \ldots,$$

and requiring

$$\{\xi_s(\xi), G_a(\xi)\} = 0,$$

one gets:

$$\xi_s(\xi) = \sum_{k=0}^{\infty} \frac{1}{k!} \{\ldots\{\xi, G^{a_1}\}, G^{a_2}\}, \ldots G^{a_k}\} G_{a_1} G_{a_2} \ldots G_{a_k}$$

$$f_s(\xi) = \sum_{k=0}^{\infty} \frac{1}{k!} \{\ldots\{f(\xi), G^{a_1}\}, G^{a_2}\}, \ldots G^{a_k}\} G_{a_1} G_{a_2} \ldots G_{a_k} = f(\xi_s(\xi))$$
Classical second-class constraints systems in phase space

The average of a function $f(\xi)$ is calculated using the probability density distribution $\rho(\xi)$ and the Liouville measure restricted to the constraint submanifold:

$$\langle f \rangle = \int \frac{d^{2n}\xi}{(2\pi)^n} (2\pi)^m \prod_{a=1}^{2m} \delta(G_a(\xi)) f(\xi) \rho(\xi).$$

On the constraint submanifold $f(\xi)$ and $\rho(\xi)$ can be replaced with $f_s(\xi)$ and $\rho_s(\xi)$

Equivalence classes of functions in phase space

$$f(\xi) \sim g(\xi) \leftrightarrow f_s(\xi) = g_s(\xi)$$

In particular, $f(\xi) \sim f_s(\xi)$ and $G_a \sim 0$.

Evolution of function $f$

$$\frac{\partial}{\partial t} f = \{f, \mathcal{H}\}_D$$

On the constraint submanifold

$$\{f, g\}_D = \{f, g_s\} = \{f_s, g\} = \{f_s, g_s\}$$
Replacing $\mathcal{H} \rightarrow \mathcal{H}_s$, one can rewrite the evolution equation in terms of the Poisson bracket:

$$\frac{\partial}{\partial t} f = \{f, \mathcal{H}_s\}$$
Quantum second-class constraints systems in the Hilbert space

To any function $f(\xi)$ in the unconstrained phase space one may associate an operator $\hat{f}$ in the corresponding Hilbert space

$$\mathcal{H}(\xi) \longleftrightarrow \mathcal{F}$$
$$\mathcal{G}_a(\xi) \longleftrightarrow \mathcal{G}_a$$

$$[\mathcal{G}_a, \mathcal{G}_b] = i\hbar\mathcal{J}_{ab}$$

Projected operator $f_s$

$$f_s = \sum_{k=0}^{\infty} \frac{(-i/\hbar)^k}{k!} [[...[[f, \mathcal{G}^{a_1}], \mathcal{G}^{a_2}], ...\mathcal{G}^{a_k}] \mathcal{G}_{a_1} \mathcal{G}_{a_2} ... \mathcal{G}_{a_k}. $$

One has

$$[f_s, \mathcal{G}_a] = 0$$

$$(f g_s)_s = (f_s g)_s = f_s g_s$$

$f$ and $g$ belong to the same equivalence class provided

$$f \sim g \leftrightarrow f_s = g_s$$
Quantum second-class constraints systems in the Hilbert space

How to calculate the average value of an operator?

Projection operator:

\[ P = \int \frac{d^{2m} \lambda}{(2\pi\hbar)^m} \prod_{a=1}^{2m} \exp\left(\frac{i}{\hbar} G^a \lambda_a\right) \]

Chose basis in the Hilbert space in which the first \( m \) constraint operators are diagonal,

\[ G^a |g, g^*> = g^a |g, g^*> \]

for \( a = 1, ..., m \). \( G^a \) might be taken as momentum operators. The last \( m \) constraint operators can be treated as quantal coordinates.

\[ P |g, g^*> = |0, g^*> \]

The average value of an operator \( f \)

\[ < f > = \text{Tr}[P f_s r_s] = \int \frac{d^{n-m} g^*}{(2\pi\hbar)^{n-m}} < 0, g^* | f_s r_s | 0, g^* >. \]

is determined by the physical subspace of the Hilbert space, spanned by \( |0, g^*> \).
Quantum second-class constraints systems in the Hilbert space

Physical states satisfy
\[ S^a |0, g_* >= 0 \]

**Dirac’s supplementary condition**

for an equivalent gauge system, where
- \( S^a \) with \( a = 1, \ldots, m \) are gauge generators
- \( S^a \) with \( a = m + 1, \ldots, 2m \) are gauge-fixing operators.

**Quantum evolution equation**

\[ i\hbar \frac{d}{dt} f = [f, \mathcal{H}_s] \]

Evolution does not mix equivalence classes of operators

\[ f(t) \sim g(t) \leftrightarrow f(0) \sim g(0) \]
Weyl’s association rule and the star-product

The Weyl’s association rule

\[ f(\xi) = \text{Tr}[\mathcal{B}(\xi)f], \]

\[ f = \int \frac{d^{2n} \xi}{(2\pi \hbar)^n} f(\xi) \mathcal{B}(\xi). \]

Vector space? Choose basis! Weyl’s basis:

\[ \mathcal{B}(\xi) = (2\pi \hbar)^n \delta^{2n}(\xi - \tau) = \int \frac{d^{2n} \eta}{(2\pi \hbar)^n} \exp(\frac{i}{\hbar} \eta_k (\xi - \tau)^k). \]
Weyl’s association rule and the star-product

\[ \mathcal{B}(\xi) \in Op(L^2(\mathbb{R}^n)) \] and projection operators \( P_i = e_i \otimes e_i \) acting in \( \mathbb{R}^n \) satisfy

- \( \mathcal{B}(\xi)^+ = \mathcal{B}(\xi) \iff (P_i)^+ = P_i \),
- \( Tr[\mathcal{B}(\xi)] = 1 \iff e_i \cdot e_i = 1 \),
- \( \int \frac{d^{2n}\xi}{(2\pi\hbar)^n} \mathcal{B}(\xi) = 1 \iff \sum_i e_i \otimes e_i = 1 \),
- \( \int \frac{d^{2n}\xi}{(2\pi\hbar)^n} \mathcal{B}(\xi) Tr[\mathcal{B}(\xi)f] = f \iff \sum_i P_i(P_i(\sum_j g_j P_j)) = \sum_j g_j P_j \),
- \( Tr[\mathcal{B}(\xi)\mathcal{B}(\xi')] = (2\pi\hbar)^n \delta^{2n}(\xi - \xi') \iff e_i \cdot e_j = \delta_{ij} \),
- \( \mathcal{B}(\xi) \exp(-\frac{i\hbar}{2} P_{\xi\xi'}) \mathcal{B}(\xi') = (2\pi\hbar)^n \delta^{2n}(\xi - \xi') \mathcal{B}(\xi') \iff ? \)

and therefore \( \xi \leftrightarrow i \).
Quantum second-class constraints systems in phase space

Symplectic basis for constraint functions

\[ G_a(\xi) \wedge G_b(\xi) = \mathcal{I}_{ab} \]

Skew-gradient projections for canonical variables and functions in phase space

\[ \xi_t(\xi) \wedge G_a(\xi) = 0 \quad \& \quad f_t(\xi) \wedge G_a(\xi) = 0 \]

Projected canonical variables

\[ \xi_t(\xi) = \sum_{k=0}^{\infty} \frac{1}{k!} (\cdots (\xi \wedge G^{a_1} \wedge G^{a_2}) \cdots \wedge G^{a_k}) \circ G_{a_1} \circ G_{a_2} \cdots \circ G_{a_k} \]

\[ f_t(\xi) = \sum_{k=0}^{\infty} \frac{1}{k!} (\cdots (f(\xi) \wedge G^{a_1} \wedge G^{a_2}) \cdots \wedge G^{a_k}) \circ G_{a_1} \circ G_{a_2} \cdots \circ G_{a_k} \]

Classical limit

\[ \lim_{\hbar \to 0} \xi_t(\xi) = \xi_s(\xi) \quad \& \quad \lim_{\hbar \to 0} f_t(\xi) = f_s(\xi) \]
Quantum second-class constraints systems in phase space

Equivalence relations between functions

\[ f(\xi) \sim g(\xi) \leftrightarrow f_t(\xi) = g_t(\xi) \]

\( f(\xi) \sim f_t(\xi) \) and \( f(\xi) \neq f_t(\xi) \) for \( G_\alpha(\xi) = 0 \), therefore \( \sim \) and \( \approx \) acquire distinct meaning.

The average value of function

\[ < f > = \int \frac{d^{2n}\xi}{(2\pi\hbar)^n} P(\xi) \ast f_t(\xi) \ast W_t(\xi) \]

where \( P(\xi) \) is the symbol of the projection operator \( \mathcal{P} \) and \( W(\xi) \) is the Wigner function.

EVOLUTION EQUATION:

\[ \frac{\partial}{\partial t} f(\xi) = f(\xi) \wedge H_t(\xi) \]
Mathematical pendulum on $S^{n-1}$ sphere of unit radius in $n$-dimensional Euclidean space with coordinates $\phi^\alpha$.

The Hamiltonian function projected onto the constraint submanifold

$$H_t = \frac{1}{2} (\phi^2 \delta^{\alpha\beta} - \phi^\alpha \phi^\beta) \pi^\alpha \pi^\beta$$

Constraint functions

$$G^1(\xi) = \frac{1}{2} \ln \phi^\alpha \phi^\alpha, \quad G^2(\xi) = \phi^\alpha \pi^\alpha,$$

where $\xi = (\phi^\alpha, \pi^\alpha)$, so that

$$G_a(\xi) \wedge G_b(\xi) = I_{ab}.$$
The power series expansion of the Moyal bracket is truncated at $O(\hbar^2)$, since the Hamiltonian $H_t(\xi)$ is a fourth degree polynomial of canonical variables, so we obtain

$$\frac{\partial}{\partial t} W = - \{W, H_s\} + \frac{\hbar^2}{8} (\frac{\partial^3 W}{\partial \phi^\alpha \partial \phi^\beta \partial \pi^\gamma} (2\delta^{\alpha\beta} \phi^\gamma - \delta^{\alpha\gamma} \phi^\beta - \delta^{\beta\gamma} \phi^\alpha)$$

$$- \frac{\partial^3 W}{\partial \pi^\alpha \partial \pi^\beta \partial \phi^\gamma} (2\delta^{\alpha\beta} \pi^\gamma - \delta^{\alpha\gamma} \pi^\beta - \delta^{\beta\gamma} \pi^\alpha)).$$

The first term in the right side is of the classical origin, while the second term represents a quantum correction to the classical Liouville equation and there are no other quantum corrections. Given $W(\xi, 0)$ in the unconstrained phase space, $W(\xi, t)$ can be found by solving the PDE.
Four types of systems

Four types of brackets which govern evolution of systems

The fourth bracket $f_t \wedge g_t$ has been constructed
Many thanks to my collaborators:
Apolodor Raduta (Buharest University)
Amand Faessler (Tübigen University)