Hamiltonian formalism for discrete equations.
Symmetries and first integrals.

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1. Symmetries of differential equations

We consider canonical Hamiltonian equations

\[ \dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}, \quad H = H(t, q, p) \]

and their Lie group transformation

\[ \bar{t} = \bar{t}(t, q, p, a) \approx t + \xi(t, q, p)a \]
\[ \bar{q} = \bar{q}(t, q, p, a) \approx q + \eta(t, q, p)a \]
\[ \bar{p} = \bar{p}(t, q, p, a) \approx p + \zeta(t, q, p)a \]

Lie group transformations in the space \((t, q, p)\) are generated by operators of the form

\[ X = \xi(t, q, p) \frac{\partial}{\partial t} + \eta(t, q, p) \frac{\partial}{\partial q} + \zeta(t, q, p) \frac{\partial}{\partial p} \]

Symmetries \(\longleftrightarrow\) transformed equations have the same form
Example: Harmonic oscillator $H = \frac{1}{2}(p^2 + q^2)$.

The canonical Hamiltonian equations

$$\dot{q} = p, \quad \dot{p} = -q$$

are invariant, for example, for

1. Translation in time

$$\bar{t} = t + a, \quad \bar{q} = q, \quad \bar{p} = p,$$

generated by the operator

$$X_1 = \frac{\partial}{\partial t}$$

2. Scaling

$$\bar{t} = t, \quad \bar{q} = e^a q \approx q + qa, \quad \bar{p} = e^a p \approx p + pa,$$

generated by the operator

$$X_2 = q \frac{\partial}{\partial q} + p \frac{\partial}{\partial p}$$
Infinitesimal criterion of invariance

We prolong the operator on $\dot{q}$ and $\dot{p}$:

$$X = \xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial q} + \zeta \frac{\partial}{\partial p} + (D(\eta) - \dot{q}D(\xi)) \frac{\partial}{\partial q} + (D(\zeta) - \dot{p}D(\xi)) \frac{\partial}{\partial p}$$

The equations are invariant with respect to operator $X$ if

$$X \left( \dot{q} - \frac{\partial H}{\partial p} \right) \bigg|_{\frac{\partial}{\partial q}, \frac{\partial}{\partial p}, \dot{q} = \frac{\partial H}{\partial q}, \dot{p} = -\frac{\partial H}{\partial q}} = 0, \quad X \left( \dot{p} + \frac{\partial H}{\partial q} \right) \bigg|_{\frac{\partial}{\partial q}, \frac{\partial}{\partial p}, \dot{q} = \frac{\partial H}{\partial q}, \dot{p} = -\frac{\partial H}{\partial q}} = 0$$

Example: Harmonic oscillator.

1. Translation in time

$$X_1 = \frac{\partial}{\partial t}, \quad X_1(\dot{q} - p) \equiv 0, \quad X_1(\dot{p} + q) \equiv 0.$$

2. Scaling

$$X_2 = q \frac{\partial}{\partial q} + p \frac{\partial}{\partial p} + \dot{q} \frac{\partial}{\partial q} + \dot{p} \frac{\partial}{\partial p}$$

$$X_2(\dot{q} - p) = \dot{q} - p, \quad X_2(\dot{p} + q) = \dot{p} + q.$$
2. Canonical Hamiltonian equations

2.a. Hamiltonian symmetries and first integrals

Hamiltonian symmetries have the form

\[ X = 0 \frac{\partial}{\partial t} + \eta(t, q, p) \frac{\partial}{\partial q} + \zeta(t, q, p) \frac{\partial}{\partial p}, \]

where

\[ \eta = \frac{\partial I}{\partial p}, \quad \zeta = -\frac{\partial I}{\partial q}, \quad I = I(t, q, p). \]

They generate transformations which preserve the canonical Hamiltonian form of the equations, i.e. generate \textbf{canonical transformations}, which are usually known as provided by generating functions \( S_1(q, \bar{q}), S_2(\bar{p}, q), S_3(p, \bar{q}) \) and \( S_4(p, \bar{p}) \). For example,

\[ S_1(q, \bar{q}) : \quad p = \frac{\partial S_1}{\partial q}(q, \bar{q}), \quad \bar{p} = -\frac{\partial S_1}{\partial \bar{q}}(q, \bar{q}). \]
Invariance of the equation $\dot{q} = \frac{\partial H}{\partial p}$ with respect to a Hamiltonian symmetry

$$\eta_t + \eta_q \dot{q} + \eta_p \dot{p} = \eta \frac{\partial}{\partial q} \left( \frac{\partial H}{\partial p} \right) + \zeta \frac{\partial}{\partial p} \left( \frac{\partial H}{\partial p} \right), \quad \text{where} \quad \eta = \frac{\partial I}{\partial p}, \quad \zeta = -\frac{\partial I}{\partial q}$$

on the solutions $\dot{q} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial q}$ yields

$$\frac{\partial^2 I}{\partial t \partial p} + \frac{\partial H}{\partial p} \frac{\partial^2 I}{\partial q \partial p} - \frac{\partial H}{\partial q} \frac{\partial^2 I}{\partial p \partial p} = \frac{\partial I}{\partial p} \frac{\partial^2 H}{\partial q \partial p} - \frac{\partial I}{\partial q} \frac{\partial^2 H}{\partial p \partial p}.$$ 

This can be rewritten as

$$\frac{\partial}{\partial p} \left( \frac{\partial I}{\partial t} + \frac{\partial H}{\partial p} \frac{\partial I}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial I}{\partial p} \right) = 0.$$ 

Similarly, invariance of $\dot{p} = -\frac{\partial H}{\partial q}$ leads to

$$\frac{\partial}{\partial q} \left( \frac{\partial I}{\partial t} + \frac{\partial H}{\partial p} \frac{\partial I}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial I}{\partial p} \right) = 0.$$
Therefore,

\[
\frac{\partial I}{\partial t} + \frac{\partial H}{\partial p} \frac{\partial I}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial I}{\partial p} = f(t).
\]

Since

\[
\frac{\partial I}{\partial t} + \frac{\partial H}{\partial p} \frac{\partial I}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial I}{\partial p} = \frac{\partial I}{\partial t} + \frac{\partial I}{\partial q} \dot{q} + \frac{\partial I}{\partial p} \dot{p} \bigg|_{\dot{q} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial q}} = D(I)\bigg|_{\dot{q} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial q}}
\]

we get a first integral \(I(t) - F(t)\). Function \(F(t)\) is to be found from the Hamiltonian equations.

- **PROBLEM:** How to consider discrete case?
2.b. Variational formulation

Canonical Hamiltonian equations

\[ \dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} \]

can be obtained by the variational principle from the action functional

\[ \int_{t_1}^{t_2} (p\dot{q} - H(t, q, p)) \, dt, \quad \delta q(t_1) = \delta q(t_2) = 0 \]

Indeed,

\[ \delta \int_{t_1}^{t_2} (p\dot{q} - H(t, q, p)) \, dt = \int_{t_1}^{t_2} \left( \delta p\dot{q} + p\delta \dot{q} - \frac{\partial H}{\partial q} \delta q - \frac{\partial H}{\partial p} \delta p \right) \, dt \]

\[ = \int_{t_1}^{t_2} \left[ \left( \dot{q} - \frac{\partial H}{\partial p} \right) \delta p - \left( \dot{p} + \frac{\partial H}{\partial q} \right) \delta q \right] \, dt + [p\delta q]_{t_1}^{t_2}. \]
2.c. Variational symmetries and first integrals

Invariance of **elementary action**

\[
(p \dot{q} - H) dt = pdq - H dt
\]

**Theorem.** The elementary Hamiltonian action (we say a Hamiltonian) is invariant with respect to a symmetry operator if and only if

\[
\zeta \dot{q} + pD(\eta) - X(H) - HD(\xi) = 0.
\]

**Proof.** Application of prolonged \( X \) yields:

\[
X (pdq - H dt) = (\zeta \dot{q} + pD(\eta) - X(H) - HD(\xi)) dt = 0.
\]
Lemma. (The Hamiltonian identity) The identity

\[ \zeta \dot{q} + pD(\eta) - X(H) - HD(\xi) \equiv \xi \left( D(H) - \frac{\partial H}{\partial t} \right) \]

\[ -\eta \left( \dot{p} + \frac{\partial H}{\partial q} \right) + \zeta \left( \dot{q} - \frac{\partial H}{\partial p} \right) + D[p\eta - \xi H] \]

is true for any smooth function \( H = H(t, q, p) \).

Theorem. (Noether theorem) The canonical Hamiltonian equations possess a first integral

\[ I = p\eta - \xi H \]

if and only if the Hamiltonian function is invariant with respect to the corresponding symmetry operator on the solutions of Hamiltonian equations.

Remark. If the Hamiltonian is divergence invariant, i.e.

\[ \zeta \dot{q} + pD(\eta) - X(H) - HD(\xi) = D(V), \quad V = V(t, q, p), \]

then there is a first integral

\[ I = p\eta - \xi H - V. \]
Invariance of canonical Hamiltonian equations

Let us consider variation operators

$$\frac{\delta}{\delta p} = \frac{\partial}{\partial p} - D \frac{\partial}{\partial \dot{p}}, \quad \frac{\delta}{\delta q} = \frac{\partial}{\partial q} - D \frac{\partial}{\partial \dot{q}}, \quad D = \frac{\partial}{\partial t} + \dot{q} \frac{\partial}{\partial q} + \dot{p} \frac{\partial}{\partial p} + ...$$

Lemma. Application of variational operators to invariance condition yields

$$\frac{\delta}{\delta p} (\zeta \dot{q} + p D(\eta) - X(H) - H D(\xi)) = \left[ D(\eta) - \dot{q} D(\xi) - X \left( \frac{\partial H}{\partial p} \right) \right]$$

$$+ \xi_p \left( D(H) - \frac{\partial H}{\partial t} \right) - \eta_p \left( \dot{p} + \frac{\partial H}{\partial q} \right) + (\zeta + D(\xi)) \left( \dot{q} - \frac{\partial H}{\partial p} \right).$$

and

$$\frac{\delta}{\delta q} (\zeta \dot{q} + p D(\eta) - X(H) - H D(\xi)) = - \left[ D(\zeta) - \dot{p} D(\xi) + X \left( \frac{\partial H}{\partial q} \right) \right]$$

$$+ \xi_q \left( D(H) - \frac{\partial H}{\partial t} \right) - (\eta_q + D(\xi)) \left( \dot{p} + \frac{\partial H}{\partial q} \right) + \zeta_q \left( \dot{q} - \frac{\partial H}{\partial p} \right).$$
**Theorem.** If a Hamiltonian is invariant with respect to a symmetry operator, then the canonical Hamiltonian equations are also invariant.

**Remark.** The same is true for divergence symmetries of the Hamiltonian, because the term $D(V)$ belongs to the kernel of the variational operators.

**Theorem.** Canonical Hamiltonian equations are invariant with respect to an operator $X$ if and only if the following conditions are true (on the solutions of the canonical Hamiltonian equations):

\[
\frac{\delta}{\delta p} \left( \zeta \dot{q} + pD(\eta) - X(H) - HD(\xi) \right) \bigg|_{\dot{q} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial q}} = 0,
\]

\[
\frac{\delta}{\delta q} \left( \zeta \dot{q} + pD(\eta) - X(H) - HD(\xi) \right) \bigg|_{\dot{q} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial q}} = 0.
\]
Example

The Hamiltonian equations

\[ \dot{q} = p, \quad \dot{p} = \frac{1}{q^3}, \]

provided by the Hamiltonian function

\[ H(t, q, p) = \frac{1}{2} \left( p^2 + \frac{1}{q^2} \right), \]

admit symmetries

\[ X_1 = \frac{\partial}{\partial t}, \quad X_2 = 2t \frac{\partial}{\partial t} + q \frac{\partial}{\partial q} - p \frac{\partial}{\partial p}, \quad X_3 = t^2 \frac{\partial}{\partial t} + tq \frac{\partial}{\partial q} + (q - tp) \frac{\partial}{\partial p}. \]
1. Variational symmetries

Variational symmetry operators $X_1$ and $X_2$ provide first integrals

$$I_1 = -H = -\frac{1}{2} \left( p^2 + \frac{1}{q^2} \right), \quad I_2 = pq - t \left( p^2 + \frac{1}{q^2} \right).$$

Operator $X_3$ is a divergence symmetry with $V_3 = q^2/2$. It yields the following conserved quantity

$$I_3 = -\frac{1}{2} \left( \frac{t^2}{q^2} + (q - tp)^2 \right).$$

Putting $I_1 = A/2$ and $I_2 = B$, we find the solution as

$$Aq^2 + (At - B)^2 + 1 = 0, \quad p = \frac{B - At}{q}.$$
2. Hamiltonian symmetries

We rewrite symmetry operators in the evolutionary form

\[ \bar{X}_1 = -q \frac{\partial}{\partial q} - \dot{p} \frac{\partial}{\partial p}, \quad \bar{X}_2 = (q - 2t \dot{q}) \frac{\partial}{\partial q} - (p + 2t \dot{p}) \frac{\partial}{\partial p}, \]

\[ \bar{X}_3 = (tq - t^2 \dot{q}) \frac{\partial}{\partial q} + (q - tp - t^2 \dot{p}) \frac{\partial}{\partial p}. \]

On the solutions of the canonical Hamiltonian equations \( \dot{q} = p, \ \dot{p} = \frac{1}{q^3} \) these operators are equivalent to the set

\[ \tilde{X}_1 = -p \frac{\partial}{\partial q} - \frac{1}{q^3} \frac{\partial}{\partial p}, \quad \tilde{X}_2 = (q - 2tp) \frac{\partial}{\partial q} - \left( p + \frac{2t}{q^3} \right) \frac{\partial}{\partial p}, \]

\[ \tilde{X}_3 = (tq - t^2 p) \frac{\partial}{\partial q} + \left( q - tp - \frac{t^2}{q^3} \right) \frac{\partial}{\partial p}. \]

To find first integrals one should integrate the equations

\[ \eta = \frac{\partial I}{\partial p}, \quad \zeta = -\frac{\partial I}{\partial q}, \quad \tilde{X} = \eta \frac{\partial}{\partial q} + \zeta \frac{\partial}{\partial p} \]

for each symmetry. Integration provides us with the same first integrals.
3. Discrete Hamiltonian equations

3.a. Discrete variational equations in Lagrangian framework

We consider a finite–difference functional

\[ \mathbb{L}_h = \sum_{\Omega} \mathcal{L}(t, t_+, q, q_+) h_+, \]

defined on some one-dimensional lattice \( \Omega \) with step \( h_+ = t_+ - t \).

Let us take a variation of the functional along some curve \( q = \phi(t) \) at some point \( (t, q) \). The variation will effect only two terms in the sum:

\[ \mathbb{L}_h = \ldots + \mathcal{L}(t_-, t, q_-, q) h_- + \mathcal{L}(t, t_+, q, q_+) h_+ + \ldots, \]

so we get the following expression for the variation of the difference functional

\[ \delta \mathbb{L}_h = \frac{\delta \mathcal{L}}{\delta q} \delta q + \frac{\delta \mathcal{L}}{\delta t} \delta t, \]

where \( \delta q = \phi' \delta t \) and
\[ \frac{\delta \mathcal{L}}{\delta q} = h_+ \frac{\partial \mathcal{L}}{\partial q} + h_- \frac{\partial \mathcal{L}^-}{\partial q} , \quad \frac{\delta \mathcal{L}}{\delta t} = h_+ \frac{\partial \mathcal{L}}{\partial t} + h_- \frac{\partial \mathcal{L}^-}{\partial t} + \mathcal{L}^- - \mathcal{L}, \]

where \( \mathcal{L} = \mathcal{L}(t, t_+, q, q_+) \) and \( \mathcal{L}^- = S_h(\mathcal{L}) = \mathcal{L}(t-, t, q-, q) \).

Thus, for an arbitrary curve the stationary value of difference functional is given by any solution of the 2 equations, called \textbf{quasiextremal equations},

\[ \frac{\delta \mathcal{L}}{\delta q} = 0, \quad \frac{\delta \mathcal{L}}{\delta t} = 0. \]

These equations represent the entire difference scheme (approximation of ODE and mesh) and could be called "the discrete Euler–Lagrange system".

- Noether theorem links variational symmetries and first integrals.
3.b. Discrete Legendre transform and discrete Hamiltonian equations

We consider discrete Legendre transform \((t, t+, q, q+) \rightarrow (t, t+, q, p+)
\):

\[
p_+ = h_+ \frac{\partial L}{\partial q_+}(t, t+, q, q+)
\]

\[
\mathcal{H}(t, t+, q, p_+) = p_+ D(q) - L(t, t+, q, q+),
\]

where \(D(q) = \frac{q+ - q}{t+ - t}\),

which is a slightly modified version of the transform proposed in

Lall S, West M, Discrete variational Hamiltonian mechanics, 

where the discrete Hamiltonian equations were developed as the dual, in the
sense of optimization, to discrete Euler–Lagrange equations.

Alternatively, one can use discrete Legendre transform \((t, t+, q, q+) \rightarrow (t, t+, p, q+)
\):

\[
p = -h_+ \frac{\partial L}{\partial q}(t, t+, q, q+)
\]

\[
\mathcal{H}(t, t+, q+, p) = p D(q) - L(t, t+, q, q+)
\]

where \(D(q) = \frac{q+ - q}{t+ - t}\).
Relations for derivatives of the Lagrangian and Hamiltonian:

\[ h_+ \frac{\partial H}{\partial t} - H = -h_+ \frac{\partial L}{\partial t} + L, \quad h_+ \frac{\partial H}{\partial t_+} + H = -h_+ \frac{\partial L}{\partial t_+} - L, \]

\[ h_+ \frac{\partial H}{\partial q} = -p_+ - h_+ \frac{\partial L}{\partial q}, \quad h_+ \frac{\partial H}{\partial p_+} = q_+ - q. \]

Transforming 2 quasiextremal equations (discrete Euler–Lagrange equations) into Hamiltonian framework, we obtain discrete Hamiltonian equations.

\[
\begin{cases}
   D_+ (q) = \frac{\partial H}{\partial p_+}, & D_+ (p) = -\frac{\partial H}{\partial q}, \\
   h_+ \frac{\partial H}{\partial t} - H + h_- \frac{\partial H^-}{\partial t} + H^- = 0,
\end{cases}
\]

where \( H = H(t, t_+, q, p_+) \) and \( H^- = H(t_-, t, q_-, p) \).
3.c. Variational formulation

We consider the finite-difference functional

$$H_h = \sum_{\Omega} (p_+ (q_+ - q) - \mathcal{H}(t, t_+, q, p+) h_+).$$

A variation of this functional along a curve \( q = \phi(t), p = \psi(t) \) at some point \((t, q, p)\) will effect only two term of the sum:

$$H_h = \ldots + p(q - q_-) - \mathcal{H}(t_-, t, q_-, p) h_- + p_+ (q_+ - q) - \mathcal{H}(t, t_+, q_+, p_+) h_+ + \ldots$$

Therefore, we get the following expression for the variation

$$\delta H_h = \frac{\delta \mathcal{H}}{\delta p} \delta p + \frac{\delta \mathcal{H}}{\delta q} \delta q + \frac{\delta \mathcal{H}}{\delta t} \delta t,$$

where \( \delta q = \phi' \delta t, \delta p = \psi' \delta t \) and
\[ \frac{\delta \mathcal{H}}{\delta p} = q - q_- - h_- \frac{\partial \mathcal{H}^-}{\partial p}, \quad \frac{\delta \mathcal{H}}{\delta q} = -\left( p_+ - p + h_+ \frac{\partial \mathcal{H}}{\partial q} \right), \]

\[ \frac{\delta \mathcal{H}}{\delta t} = -\left( h_+ \frac{\partial \mathcal{H}}{\partial t} - \mathcal{H} + h_- \frac{\partial \mathcal{H}^-}{\partial t} + \mathcal{H}^- \right), \]

where \( \mathcal{H} = \mathcal{H}(t, t_+, q, p_+) \) and \( \mathcal{H}^- = \mathcal{H}(t_-, t, q_-, p) \).

For the stationary value of the finite-difference functional we obtain the system of 3 equations

\[ \frac{\delta \mathcal{H}}{\delta p} = 0, \quad \frac{\delta \mathcal{H}}{\delta q} = 0, \quad \frac{\delta \mathcal{H}}{\delta t} = 0, \]

which are equivalent to the discrete Hamiltonian equations.
3.d. Variational symmetries and first integrals

Discrete prolongation of the operator \( X \):

\[
X = \xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial q} + \zeta \frac{\partial}{\partial p} + \xi_- \frac{\partial}{\partial t_-} + \eta_- \frac{\partial}{\partial q_-} + \zeta_- \frac{\partial}{\partial p_-} + \xi_+ \frac{\partial}{\partial t_+} + \eta_+ \frac{\partial}{\partial q_+} + \zeta_+ \frac{\partial}{\partial p_+}
\]

where

\[
\begin{align*}
\xi_- &= \xi(t_-, q_-, p_-), & \eta_- &= \eta(t_-, q_-, p_-), & \zeta_- &= \zeta(t_-, q_-, p_-), \\
\xi_+ &= \xi(t_+, q_+, p_+), & \eta_+ &= \eta(t_+, q_+, p_+), & \zeta_+ &= \zeta(t_+, q_+, p_+).
\end{align*}
\]
Let us consider the finite–difference functional

$$H_h = \sum_\Omega \left( p_+ (q_+ - q) - \mathcal{H}(t, t_+, q, p_+) h_+ \right).$$

on some lattice given by equation

$$\Omega(t, h_+, h_-, q, p) = 0.$$ 

The lattice is provided by the discrete Hamiltonian equations.

**Theorem.** The discrete action functional (we say a Hamiltonian function) considered together with the mesh is invariant with respect to a group generated by operator $X$ if and only if the conditions

$$\left. \left( \zeta_+ D^h (q) + p_+ D^h (\eta) - X(\mathcal{H}) - \mathcal{H} D^h (\xi) \right) \right|_{\Omega=0} = 0,$$

$$X \Omega(t, h_+, h_-, q, p) \big|_{\Omega=0} = 0$$

hold on the solutions of the discrete Hamiltonian equations.

- Since the mesh is provided by discrete Hamiltonian equations we need their invariance
**Lemma.** (Discrete Hamiltonian identity) The following identity is true for any smooth function $\mathcal{H} = \mathcal{H}(t, t_+, q, p_+)$:

$$
\zeta_+ D_+^h(q) + p_+ D_+^h(\eta) - X(\mathcal{H}) - \mathcal{H} D_+^h(\xi) \equiv \xi \left( \frac{h_-}{h_+} D_+^h(\mathcal{H}) - \frac{\partial \mathcal{H}}{\partial t} - \frac{h_-}{h_+} \frac{\partial \mathcal{H}^-}{\partial t} \right)
$$

$$
-\eta \left( D_+^h(p) + \frac{\partial \mathcal{H}}{\partial q} \right) + \zeta_+ \left( D_+^h(q) - \frac{\partial \mathcal{H}}{\partial p_+} \right) + D_+^h \left[ \eta p - \xi \left( \mathcal{H}^- + h_- \frac{\partial \mathcal{H}^-}{\partial t} \right) \right]
$$

**Theorem.** (Noether theorem) The invariant with respect to symmetry operator $X$ discrete Hamiltonian equations possess a first integral

$$
\mathcal{I} = \eta p - \xi \left( \mathcal{H}^- + h_- \frac{\partial \mathcal{H}^-}{\partial t} \right)
$$

if and only if the Hamiltonian function is invariant with respect to the same symmetry on the solutions of the equations.
Remark 1. If the operator $X$ is a divergence symmetry of the Hamiltonian action, i.e.

$$\zeta_+ D(q) + p_+ D(\eta) - X(\mathcal{H}) - \mathcal{H} D(\xi) = D(V), \quad V = V(t, q, p),$$

then there is a first integral

$$\mathcal{I} = \eta p - \xi \left( \mathcal{H}^- + h_- \frac{\partial \mathcal{H}^-}{\partial t} \right) - V.$$

Remark 2. If Hamiltonian is invariant with respect to time translations, i.e. $\mathcal{H} = \mathcal{H}(h_+, q, p^+)$, where $h_+ = t_+ - t$, then there is a conservation of energy

$$\mathcal{E} = \mathcal{H}^- + h_- \frac{\partial \mathcal{H}^-}{\partial h_-} = \mathcal{H} + h_+ \frac{\partial \mathcal{H}}{\partial h_+}.$$
Example: Discrete harmonic oscillator.

Let us consider the one-dimensional harmonic oscillator

\[ \dot{q} = p, \quad \dot{p} = -q, \]

which is generated by the Hamiltonian function

\[ H(t, q, p) = \frac{1}{2}(p^2 + q^2). \]

As a discretization we consider the application of the midpoint rule

\[ \frac{q_+ - q}{h_+} = \frac{p + p_+}{2}, \quad \frac{p_+ - p}{h_+} = -\frac{q + q_+}{2} \]

on a uniform mesh \( h_+ = h_- = h \).

- The midpoint rule conserves quadratic first integral. Therefore, \( H \) is conserved.
This discretization can be rewritten as the system

\[ D(q) = \frac{4}{4 - h_+^2} \left( p_+ + \frac{h_+}{2} q \right), \quad D(p) = -\frac{4}{4 - h_+^2} \left( q + \frac{h_+}{2} p_+ \right), \quad h_+ = h_- = h. \]

It can be shown that this system provides discrete Hamiltonian equations

\[
\begin{align*}
D(q) &= \frac{\partial H}{\partial p_+}, \\
D(p) &= -\frac{\partial H}{\partial q}, \\
\frac{h_+}{\partial t} - H + h_- \frac{\partial H}{\partial t} + H^- &= 0,
\end{align*}
\]

generated by the discrete Hamiltonian

\[ H(t, t_+, q, p_+) = \frac{2}{4 - h_+^2} (q^2 + p_+^2 + h_+ q p_+). \]
The system admits, in particular, symmetries

\[ X_1 = \sin(\omega t) \frac{\partial}{\partial q} + \cos(\omega t) \frac{\partial}{\partial p}, \quad X_2 = \cos(\omega t) \frac{\partial}{\partial q} - \sin(\omega t) \frac{\partial}{\partial p}, \]

\[ X_3 = \frac{\partial}{\partial t}, \quad X_4 = q \frac{\partial}{\partial q} + p \frac{\partial}{\partial p}, \quad X_5 = p \frac{\partial}{\partial q} - q \frac{\partial}{\partial p}, \]

where

\[ \omega = \frac{\arctan(h/2)}{h/2}. \]

Operators \( X_1 \) and \( X_2 \) are divergence symmetries with functions \( V_1 = q \cos(\omega t) \) and \( V_2 = -q \sin(\omega t) \) respectively. Therefore, we obtain two first integrals

\[ I_1 = p \sin(\omega t) - q \cos(\omega t), \quad I_2 = p \cos(\omega t) + q \sin(\omega t). \]

From the first integrals \( I_1 \) and \( I_2 \) we have conservation

\[ I_1^2 + I_2^2 = q^2 + p^2 = \text{const}. \]
Operator $X_3$ is a variational symmetry. It provides the first integral

$$I_3 = -\frac{4}{4 - h_-^2} \left( \frac{4 + h_-^2 q_-^2 + p_-^2}{2} + \frac{4h_-}{4 - h_-^2} q_- p_- \right).$$

Using the equations, we can simplify it as

$$I_3 = -\frac{4}{4 + h_-^2} \frac{q_-^2 + p_-^2}{2}$$

Using $q_-^2 + p_-^2 = \text{const}$, we can take the third first integrals equivalently as

$$\tilde{I}_3 = h_-.$$  

Finally, we have three first integrals $I_1, I_2, \tilde{I}_3$, which are sufficient for integration of the discrete system. We obtain the solution

$$q = I_2 \sin(\omega t) - I_1 \cos(\omega t), \quad p = I_1 \sin(\omega t) + I_2 \cos(\omega t)$$

on the lattice

$$t_i = t_0 + ih, \quad i = 0, \pm 1, \pm 2, ..., \quad h = \tilde{I}_3.$$
Example.
The discrete Hamiltonian
\[ \mathcal{H}(t, t+, q, p+) = \frac{1}{2} \left( p^2 + \frac{1}{q^2} \right) \]
yields the discrete Hamiltonian equations:
\[ D_{+h}(q) = p_+, \quad D_{+h}(p) = \frac{1}{q^3}, \quad p_+^2 + \frac{1}{q^2} = p^2 + \frac{1}{q_-^2}. \]

Operators
\[ X_1 = \frac{\partial}{\partial t}, \quad X_2 = 2t \frac{\partial}{\partial t} + q \frac{\partial}{\partial q} - p \frac{\partial}{\partial p} \]
are variational symmetries. They provide first integrals
\[ \mathcal{I}_1 = -\frac{1}{2} \left( p^2 + \frac{1}{q_-^2} \right), \quad \mathcal{I}_2 = qp - t \left( p^2 + \frac{1}{q_-^2} \right). \]

Therefore, the solution satisfies the relation
\[ \mathcal{I}_2 = qp + 2t\mathcal{I}_1 \]
in all points of the lattice.
4. Concluding remarks

1. For canonical Hamiltonian equations and discrete Hamiltonian equations there is a relation:

   Variational (divergence) symmetries $\longleftrightarrow$ first integrals

2. The same holds for $n$ degrees of freedom

   $q = (q_1, \ldots, q_n), \quad p = (p_1, \ldots, p_n).$

3. Two discrete versions of discrete Legendre transform

   $(t, t_+, q, q_+) \rightarrow (t, t_+, q, p_+), \quad \text{and} \quad (t, t_+, q, q_+) \rightarrow (t, t_+, p, q_+)$

   let us obtain $2n + 1$ discrete Hamiltonian equations from $n + 1$ discrete Euler–Lagrangian equations.
4. For discrete Hamiltonian equations with Hamiltonian functions invariant with respect to time translations, i.e. \( \mathcal{H} = \mathcal{H}(h_+, q, p_+) \), where \( h_+ = t_+ - t \), there is a conservation of energy

\[
E = \mathcal{H}^- + h_- \frac{\partial \mathcal{H}^-}{\partial t}.
\]

Note that \( \mathcal{H} \) is not the discrete energy, it has a meaning of a generating function for discrete Hamiltonian flow.

This is related to

Kane C., Marsden J.E., Ortiz M.,
Symplectic–energy–momentum preserving variational integrators,
5. It is possible to consider complete discrete Legendre transform.

Given a discrete Lagrangian \( \mathcal{L}(t, t_+, q, q_+) \), we can consider, for example, a discrete Legendre transform \( (t, t_+, q, q_+) \rightarrow (t, E_+, q, p_+) \):

\[
\begin{align*}
p_+ &= \frac{\partial \mathcal{L}}{\partial q_+}, \\
E_+ &= -\frac{\partial \mathcal{L}}{\partial t_+},
\end{align*}
\]

\[
S(t, E_+, q, p_+) = p_+ (q_+ - q) - E_+ (t_+ - t) - \mathcal{L}(t, t_+, q, q_+).
\]

In this case \( n + 1 \) discrete Euler–Lagrange equations are transformed into the system of \( 2n + 2 \) equations

\[
\begin{align*}
q_+ - q &= \frac{\partial S}{\partial p_+}, \\
p_+ - p &= -\frac{\partial S}{\partial q}, \\
E_+ - E &= \frac{\partial S}{\partial t}, \\
t_+ - t &= -\frac{\partial S}{\partial E_+}.
\end{align*}
\]

- Stepsize \( h_+ = t_+ - t \) becomes a complicated expression.