Plancherel measure for the quantum matrix ball - the 'radial part'.

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Recall that $SU_{n,n}$ is a subgroup of $SL_{2n}(\mathbb{C})$ of matrices preserving the form $\langle t\,,\,t\rangle = -\sum\limits_{k=1}^{n}|t_k|^2 + \sum\limits_{k=n+1}^{2n}|t_k|^2$.

$$\mathbb{D} = S(U_n\times U_n)\backslash\widetilde{X}$$

is $SU_{n,n}$-homogeneous space, where

$$\widetilde{X} = \{g\in SL_{2n}(\mathbb{C})\,|\,\langle tg^{-1},tg^{-1}\rangle = -\langle t,t\rangle\ , \quad \text{for all } t\in\mathbb{C}^4\}.$$

$\mathbb{D}$ is a bounded symmetric domain of type AI, i.e. it is isomorphic to the unit ball in $\mathrm{Mat}_n(\mathbb{C})$.

Using $S(U_n\times U_n)$-biinvariant functions

$$x_k \;=\; \sum_{\substack{I\subset\{1,2,\ldots,n\},\,J\subset\{n+1,n+2,\ldots,2n\}\\ \mathrm{card}(I)=\mathrm{card}(J)=k}} t^{\wedge k}_{I\,J}\,t^{\wedge(2n-k)}_{I^c\,J^c},$$

one obtain an injective map

$$(x_1,\ldots x_n): S(U_n\times U_n)\backslash\widetilde{X}/S(U_n\times U_n)\to\mathbb{R}^n.$$

The image is

$$\{(e_1(u_1,\ldots,u_n),\ldots,e_n(u_1,\ldots,u_n))\in\mathbb{R}^n\,|\,1\le u_1\le\ldots\le u_n\},$$

where e_k are elementary symmetric polynomials in n variables.

An invariant integral of a $S(U_n\times U_n)$-invariant function f on $\mathbb{D}$ becomes

$$\mathrm{const}\int\limits_{\Delta_{\mathbb{D}}} f(e_1(u_1,\ldots,u_n),\ldots,e_n(u_1,\ldots,u_n))\Delta(u)^2du_1\ldots du_n,$$

where

$$\Delta(u) = \prod_{1\le i<j\le n}(u_i-u_j).$$

The algebra of invariant differential operators is generated by

$$\mathcal{L}_k = \frac{1}{\Delta(u)}\, e_k(\Box_1, \dots, \Box_n)\, \Delta(u),$$

where

$$\Box_i = \frac{\partial}{\partial u_i} u_i(1-u_i) \frac{\partial}{\partial u_i}.$$

Unitary spherical representations of $SU_{n,n}$ can be enumerated by

$$\widetilde{\Lambda} = \{\overline{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \,|\, \lambda_1 \geq \dots \geq \lambda_n \geq 0\}.$$

Let $\Phi_{\overline{\lambda}}$ be a corresponding spherical function. Than (Berezin, Karpelevich)

$$\Phi_{\overline{\lambda}} = \mathrm{const}(\overline{\lambda})\, \frac{\sum_{\sigma \in S_n} \mathrm{sign}(\sigma)\, \Phi_{\lambda_1}(u_{\sigma_1}) \dots \Phi_{\lambda_n}(u_{\sigma_n})}{\Delta(u)},$$

where $\Phi_a(u) = {}_2F_1\left(\frac{1+ia}{2}, \frac{1-ia}{2}; 1; 1-u\right)$.

Consider the quantum universal enveloping algebra $U_q\mathfrak{sl}_{2n}$. Equip the Hopf algebra $U_q\mathfrak{sl}_{2n}$ with an involution $*$:

$$(K_j^{\pm 1})^* = K_j^{\pm 1}, \quad E_j^* = \begin{cases} K_jF_j, & j \neq n \\ -K_jF_j, & j = n \end{cases}, \quad F_j^* = \begin{cases} E_jK_j^{-1}, & j \neq n \\ -E_jK_j^{-1}, & j = n \end{cases}.$$

Then $U_q\mathfrak{su}_{n,n} \stackrel{\text{def}}{=} (U_q\mathfrak{sl}_{2n}, *)$ is a $*$-Hopf algebra. It is a quantum analogue of the algebra $U\mathfrak{su}_{n,n} \otimes_{\mathbb{R}} \mathbb{C}$.

Consider the well known Hopf algebra $\mathbb{C}[SL_{2n}]_q$ with generators $\{t_{ij}\}_{i,j=1,\dots,n}$. Denote by $\mathbf{t}_{ji}$ the matrix derived from $\mathbf{t} = (t_{ij})$ by discarding its j-th row and its i-th column.

Equip $\mathbb{C}[SL_{2n}]_q$ with an involution given by

$$t_{ij}^* = \operatorname{sign}[(i - n - 1/2)(n - j + 1/2)](-q)^{j-i} \det\nolimits_q \mathbf{t}_{ij}. \quad (1)$$

Than it is easy to prove that $\mathbb{C}[w_0SU_{n,n}]_q \stackrel{\text{def}}{=} (\mathbb{C}[SL_{2n}]_q, *)$ is a $U_q\mathfrak{su}_{n,n}$-module $*$-algebra (this means that $\mathbb{C}[w_0SU_{n,n}]_q$ is a $U_q\mathfrak{su}_{n,n}$-module algebra and

$$(\xi f)^* = S(\xi)^* f^* \quad (2)$$

for all $\xi \in U_q\mathfrak{su}_{n,n}$, $f \in \mathbb{C}[w_0SU_{n,n}]_q$) .

It is a q-analogue of the regular functions on the real affine variety $\widetilde{X}$.

Let $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n) \subset U_q\mathfrak{sl}_{2n}$ denote the Hopf subalgebra generated by E_j, F_j, $j \neq n$, and $K_i, K_i^{-1}, i = 1, ..., 2n-1$. The corresponding Hopf $*$-subalgebra in $U_q\mathfrak{su}_{n,n}$ is denoted by $U_q\mathfrak{s}(\mathfrak{u}_n \times \mathfrak{u}_n)$.

The elements of $\mathbb{C}[w_0 SU_{n,n}]_q$

$$x_k = q^{k(k-1)} \sum_{\substack{I\subset\{1,2,\dots,n\},\, J\subset\{n+1,n+2,\dots,2n\}\\ \mathrm{card}(I)=\mathrm{card}(J)=k}} q^{-2\sum\limits_{m=1}^{k}(n-i_m)}(-q)^{\sum\limits_{m=1}^{k}(j_m-i_m-n)} t^{\wedge k}_{I\,J}\, t^{\wedge(2n-k)}_{I^c\,J^c},$$

are pairwise commuting self-adjoint $U_q\mathfrak{s}(\mathfrak{u}_n\times\mathfrak{u}_n)$-biinvariants. These elements generate the subalgebra of all $U_q\mathfrak{s}(\mathfrak{u}_n\times\mathfrak{u}_n)$-biinvariant elements in $\mathbb{C}[w_0 SU_{n,n}]$.

There is exist a unique (up to unitary equivalence) faithful representation T of $\mathbb{C}[w_0 SU_{n,n}]$. The set of common eigenvalues $\Sigma_{\mathbb{D}}$ of the operators $T(x_1)$, …, $T(x_n)$ is

$$\{(e_1(q^{-2(\lambda+\delta)}), q^2 e_2(q^{-2(\lambda+\delta)}), \dots, q^{n(n-1)} e_n(q^{-2(\lambda+\delta)})) \mid \lambda\in\Lambda_n\}.$$

Consider the algebra $\mathbb{C}[u_1,\dots,u_n]$ and the injection

$$\mathbb{C}[w_0 SU_{n,n}]_q^{(U_q\mathfrak{s}(\mathfrak{u}_n\times\mathfrak{u}_n))^{\mathrm{op}}\otimes U_q\mathfrak{s}(\mathfrak{u}_n\times\mathfrak{u}_n)} \hookrightarrow \mathbb{C}[u_1,\dots,u_n],$$

$$x_k \mapsto q^{k(k-1)} e_k(u_1,\dots,u_n).$$

Thus

$$\mathbb{C}[w_0 SU_{n,n}]_q^{(U_q\mathfrak{s}(\mathfrak{u}_n\times\mathfrak{u}_n))^{\mathrm{op}}\otimes U_q\mathfrak{s}(\mathfrak{u}_n\times\mathfrak{u}_n)} \cong \mathbb{C}[u_1,\dots,u_n]^{S_n}.$$

Specify

$$\Delta_{\mathbb{D}} = \{(q^{-2(\lambda+\delta)} \mid \lambda\in\Lambda_n\}.$$

Let also $\mathcal{D}(\Delta_{\mathbb{D}})$ be a vector space of functions $f(u_1,\dots,u_n)$ with finite support on the set $\Delta_{\mathbb{D}}$. Thereby,

$$\mathcal{D}(\Delta_{\mathbb{D}}) \cong \mathcal{D}(\Sigma_{\mathbb{D}}) \cong \mathcal{D}(\mathbb{D})_q^{U_q\mathfrak{s}(\mathfrak{gl}_n\times\mathfrak{gl}_n)}.$$

Recall the definition of multiple Jackson integral with 'base' q^{-2}

$$\int_{q^{-2(n-1)}}^{\infty}\int_{q^{-2(n-2)}}^{q^2u_n}\cdots\int_1^{q^2u_2}\phi(u)d_{q^{-2}}u_1\ldots d_{q^{-2}}u_n=$$

$$(1-q^2)^n\sum_{\lambda\in\Lambda_n}\phi(q^{-2(\lambda+\delta)})q^{-2|\lambda+\delta|}.$$

Proposition 1 *The restriction of the invariant integral onto the space* $\mathcal{D}(\mathbb{D})_q^{U_q\mathfrak{s}(\mathfrak{gl}_n\times\mathfrak{gl}_n)}$ *is*

$$\mathcal{N}\int_{q^{-2(n-1)}}^{\infty}\int_{q^{-2(n-2)}}^{q^2u_n}\cdots\int_1^{q^2u_2}$$

$$f(e_1(u),\ldots,q^{n(n-1)}e_n(u))\Delta(u)^2\,d_{q^{-2}}u_1\ldots d_{q^{-2}}u_n,$$

where

$$\Delta(u)=\prod_{1\le i<j\le n}(u_i-u_j),$$

$$\mathcal{N}=(1-q^2)^{n(n-1)}q^{n(n-1)}\Delta(q^{-2\delta})^{-2}$$

is a positive constant.

As in the classical case, let us define a simple finite-dimensional weight $U_q\mathfrak{sl}_{2n}$-module to be spherical, if

$$\dim V^{U_q\mathfrak{s}(\mathfrak{gl}_n\times\mathfrak{gl}_n)} = 1.$$

Note 1 *(J. V. Stokman, 1999) A simple finite-dimensional weight $U_q\mathfrak{sl}_{2n}$-module is $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-spherical if and only if its highest weight has the following form:*

$$(\lambda_1-\lambda_2,\dots,\lambda_{n-1}-\lambda_n,2\lambda_n,\lambda_{n-1}-\lambda_n,\dots,\lambda_1-\lambda_2),$$

where λ is a partition.

A scalar product $(\cdot,\cdot)$ in V is called $U_q\mathfrak{su}_{2n}$-invariant if for any $\xi \in U_q\mathfrak{sl}_{2n}$ and for any $v_1, v_2 \in V$

$$(\xi v_1, v_2) = (v_1, \xi^\star v_2).$$

Any spherical $U_q\mathfrak{sl}_{2n}$-module V can be equipped with a $U_q\mathfrak{su}_{2n}$-invariant scalar product. Fix $v \in V^{U_q\mathfrak{s}(\mathfrak{gl}_n\times\mathfrak{gl}_n)}$ by the requirement $(v,v) = 1$. Recall that $\varphi_V(\xi) = (\xi v, v)$ is the spherical function on the quantum group SU_{2n} corresponding to V.

Let $\widetilde{P}_\lambda$ be a polynomial such that

$$P_\lambda(z) = \widetilde{P}_\lambda(e_1(z), q^2e_2(z),\dots,q^{n(n-1)}e_n(z)),$$

where $P_\lambda(z)$ is a specialization $P_\lambda(z;0,0;q^2)$ of Little q-Jacobi polynomials.

Theorem 1 *(J. V. Stokman, 1999) A spherical function φ_λ is equal (up to a multiplicative constant) to*

$$\widetilde{P}_\lambda(x_1,x_2,\dots,x_n).$$

Let for $\lambda \in \mathbb{C}^n$

$$a(\lambda+\delta) = (a(\lambda_1+n-1), a(\lambda_2+n-2), \ldots, a(\lambda_n)),$$

where

$$a(l) = \frac{(1-q^{-2l})(1-q^{2l+2})}{(1-q^2)^2}, \qquad l \in \mathbb{C}$$

Proposition 2 *There exist elements* $C_k \in Z(U_q^{\mathrm{ext}}\mathfrak{sl}_{2n})$, $k = 1, 2, \ldots, n$ *such that*

$$C_k \varphi_\lambda = e_k(a(\lambda+\delta))\varphi_\lambda.$$

Let $\mathcal{L}_k$ be a linear operator in $\mathbb{C}[SL_{2n}]_q$, defined with $\mathcal{L}_k f = C_k f$.

Let us define the difference operator $\Box_{u_i}$ in the space $\mathbb{C}[u_1, u_2, \ldots, u_n]^{S_n}$ with

$$\Box_{u_i} f(u_1, \ldots, u_n) = D_{u_i} u_i (1 - q^{-1} u_i) D_{u_i} f(u_1, \ldots, u_n),$$

where $D_{u_i} f(u_1, \ldots, u_n) = \frac{f(u_1,\ldots,u_{i-1},q^{-1}u_i,u_{i+1},\ldots,u_n) - f(u_1,\ldots,u_{i-1},qu_i,u_{i+1},\ldots,u_n)}{q^{-1}u_i - qu_i}$.

Proposition 3

$$\mathcal{L}_k|_{\mathbb{C}[u_1,u_2,\ldots,u_n]^{S_n}} = \frac{1}{\Delta(u)}\, e_k(\Box_{u_1}, \ldots, \Box_{u_n})\, \Delta(u). \qquad (3)$$

One may extend $\mathcal{L}_k$ to $\mathcal{D}(\mathbb{D})_q$ and then restrict to $\mathcal{D}(\mathbb{D})_q^{U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)}$ by using (3). We denote this restrictions by $\mathcal{L}_k^{\mathrm{radial}}$.

Let us introduce the Hilbert space $L^2(\Delta_{\mathbb{D}}, d\nu_q)$ of square integrable functions on $\Delta_{\mathbb{D}}$, where

$$d\nu_q(u) = \mathcal{N}\ \Delta(u)^2 d_{q^{-2}}u_1 d_{q^{-2}}u_2 \dots d_{q^{-2}}u_n. \tag{4}$$

Introduce some notation:

$$\Phi_l(x) = {}_3\Phi_2\left(\begin{matrix} q^{-2l}, q^{2(l+1)}, x \\ q^2, 0 \end{matrix}; q^2, q^2\right), \qquad l \in \mathbb{C}$$

for basic hypergeometric function,

$$c(l) = \frac{\Gamma_{q^2}(2l+1)}{(\Gamma_{q^2}(l+1))^2}$$

for a q-analogue of Harish-Chandra's c-function.

$$d\sigma(\rho) = \frac{1}{2\pi} \cdot \frac{h}{1-q^2} \cdot \frac{d\rho}{c(-\frac{1}{2}+i\rho)c(-\frac{1}{2}-i\rho)} \tag{5}$$

for the measure on the interval $[0, \pi/h]$, where $h = -2\ln q$.

Proposition 4 *(Shklyarov, Vaksman)*

$$\Box_u \Phi_l(u) = a(l)\Phi_l(u).$$

Proposition 5 *i) There exists an unitary operator*

$$\mathcal{F} : L^2(\Delta_{\mathbb{D}}, d\nu_q) \to L^2(\mathcal{R}, d\Sigma),$$

$$\mathcal{F} : \ f(u) \mapsto \widehat{f}(\rho_1, \rho_2, \ldots, \rho_n) = \frac{1}{\kappa(\rho_1, \rho_2, \ldots, \rho_n)} \int\limits_{\Delta_{\mathbb{D}}} \Phi_{-\frac{1}{2}+i\rho_1, -\frac{1}{2}+i\rho_2, \ldots, -\frac{1}{2}+i\rho_n}(u) f(u) d\nu_q(u),$$

where

$$\kappa(\rho_1, \ldots, \rho_n) = \mathcal{N}\Delta(q^{-2\delta})^{-1} \left(\prod_{j=0}^{n-1} \frac{(q^{-2j}; q^2)_j}{(q^2; q^2)_j^2} q^{(j+1)^2-1} \right) \prod_{1 \le k < j \le n} (q^{-2i\rho_j} + q^{2i\rho_j} - q^{-2i\rho_k} - q^{2i\rho_k}),$$

$$\Phi_{l_1, \ldots, l_n}(u) = \frac{\sum\limits_{\sigma \in S_n} \operatorname{sign}(\sigma) \ \Phi_{l_1}(u_{\sigma_1}) \ldots \Phi_{l_n}(u_{\sigma_n})}{\Delta(u)},$$

$$\mathcal{R} = \{(\rho_1, \ldots, \rho_n) \in [0, \pi/h]^n, \quad \rho_1 > \ldots > \rho_n\},$$

$$d\Sigma(\rho_1, \ldots, \rho_n) = \kappa(\rho_1, \ldots, \rho_n)^2 (n!) \mathcal{N}(d\sigma(\rho_1) \ldots d\sigma(\rho_n))|_{\mathcal{R}}$$

ii) The inverse operator is

$$\mathcal{F}^{-1} : \widehat{f}(\rho_1, \ldots, \rho_n) \mapsto \int\limits_{\mathcal{R}} \widehat{f}(\rho_1, \ldots, \rho_n) \Phi_{-\frac{1}{2}+i\rho_1, \ldots, -\frac{1}{2}+i\rho_n}(u) \ d\Sigma(\rho_1, \ldots, \rho_n).$$

Proposition 6 *Pairwise commuting bounded self-adjoint operators* $\mathcal{L}_k^{\mathrm{radial}}$, $k = 1, 2, \ldots, n$ *are unitary equivalent to the operators of multiplication by*

$$\frac{e_k(a(-\frac{1}{2}+i\rho_1), a(-\frac{1}{2}+i\rho_2), \ldots, a(-\frac{1}{2}+i\rho_n))}{\kappa(\rho_1, \rho_2, \ldots, \rho_n)},$$

(respectively) in the Hilbert space $L^2(\mathcal{R}, d\Sigma)$. *The unitary equivalence is provided by the operator* $\mathcal{F}$.