Plancherel measure for the quantum matrix ball - the ’radial part’.

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Recall that $SU_{n,n}$ is a subgroup of $SL_{2n}(\mathbb{C})$ of matrices preserving the form $\langle t, t \rangle = -\sum_{k=1}^{n} |t_k|^2 + \sum_{k=n+1}^{2n} |t_k|^2$.

$$\mathbb{D} = S(U_n \times U_n) \backslash \tilde{X}$$

is $SU_{n,n}$-homogeneous space, where

$$\tilde{X} = \{ g \in SL_{2n}(\mathbb{C}) | \langle tg^{-1}, tg^{-1} \rangle = -\langle t, t \rangle, \text{ for all } t \in \mathbb{C}^4 \}.$$ 

$\mathbb{D}$ is a bounded symmetric domain of type $AI$, i.e. it is isomorphic to the unit ball in $\text{Mat}_n(\mathbb{C})$.

Using $S(U_n \times U_n)$-biinvariant functions

$$x_k = \sum_{\substack{I \subset \{1,2,\ldots,n\}, J \subset \{n+1,n+2,\ldots,2n\} \atop \text{card}(I) = \text{card}(J) = k}} t_I^{\wedge k} t_{I^c J^c}^{\wedge (2n-k)},$$

one obtain an injective map

$$(x_1, \ldots x_n) : S(U_n \times U_n) \backslash \tilde{X} / S(U_n \times U_n) \to \mathbb{R}^n.$$ 

The image is

$$\{(e_1(u_1, \ldots, u_n), \ldots, e_n(u_1, \ldots, u_n)) \in \mathbb{R}^n | 1 \leq u_1 \leq \ldots \leq u_n \},$$

where $e_k$ are elementary symmetric polynomials in $n$ variables.

An invariant integral of a $S(U_n \times U_n)$-invariant function $f$ on $\mathbb{D}$ becomes

$$\text{const} \int_{\Delta_{\mathbb{D}}} f(e_1(u_1, \ldots, u_n), \ldots, e_n(u_1, \ldots, u_n)) \Delta(u)^2 du_1 \ldots du_n,$$

where

$$\Delta(u) = \prod_{1 \leq i < j \leq n} (u_i - u_j).$$
The algebra of invariant differential operators is generated by
\[ \mathcal{L}_k = \frac{1}{\Delta(u)} e_k(\Box_1, \ldots, \Box_n) \Delta(u), \]
where
\[ \Box_i = \frac{\partial}{\partial u_i} u_i (1 - u_i) \frac{\partial}{\partial u_i}. \]

Unitary spherical representations of \( SU_{n,n} \) can be enumerated by
\[ \tilde{\Lambda} = \{ \tilde{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n | \lambda_1 \geq \ldots \geq \lambda_n \geq 0 \}. \]

Let \( \Phi_{\tilde{\lambda}} \) be a corresponding spherical function. Than (Berezin, Karpelevich)
\[ \Phi_{\tilde{\lambda}} = \text{const}(\tilde{\lambda}) \sum_{\sigma \in S_n} \text{sign}(\sigma) \Phi_{\lambda_1}(u_{\sigma_1}) \ldots \Phi_{\lambda_n}(u_{\sigma_n}) \frac{\Delta(u)}{\Delta(u)}, \]
where \( \Phi_a(u) = \mathbf{2F1} \left( \frac{1+ia}{2}, \frac{1-ia}{2}; 1; 1 - u \right). \)
Consider the quantum universal enveloping algebra $U_q\mathfrak{sl}_{2n}$. Equip the Hopf algebra $U_q\mathfrak{sl}_{2n}$ with an involution $*$:

$$(K_j^{\pm 1})^* = K_j^{\pm 1}, \quad E_j^* = \begin{cases} K_jF_j, & j \neq n \\ -K_jF_j, & j = n \end{cases}, \quad F_j^* = \begin{cases} E_jK_j^{-1}, & j \neq n \\ -E_jK_j^{-1}, & j = n \end{cases}.$$

Then $U_q\mathfrak{su}_{n,n} \overset{\text{def}}{=} (U_q\mathfrak{sl}_{2n}, *)$ is a $*$-Hopf algebra. It is a quantum analogue of the algebra $U\mathfrak{su}_{n,n} \otimes_{\mathbb{R}} \mathbb{C}$.

Consider the well known Hopf algebra $\mathbb{C}[SL_{2n}]_q$ with generators $\{t_{ij}\}_{i,j=1,\ldots,n}$. Denote by $t_{ji}$ the matrix derived from $t = (t_{ij})$ by discarding its $j$-th row and its $i$-th column.

Equip $\mathbb{C}[SL_{2n}]_q$ with an involution given by

$$t_{ij}^* = \text{sign}[(i - n - 1/2)(n - j + 1/2)](-q)^{j-i} \det_q t_{ij}. \quad (1)$$

Then it is easy to prove that $\mathbb{C}[w_0SU_{n,n}]_q \overset{\text{def}}{=} (\mathbb{C}[SL_{2n}]_q, *)$ is a $U_q\mathfrak{su}_{n,n}$-module $*$-algebra (this means that $\mathbb{C}[w_0SU_{n,n}]_q$ is a $U_q\mathfrak{su}_{n,n}$-module algebra and

$$(\xi f)^* = S(\xi)^* f^* \quad (2)$$

for all $\xi \in U_q\mathfrak{su}_{n,n}$, $f \in \mathbb{C}[w_0SU_{n,n}]_q$).

It is a $q$-analogue of the regular functions on the real affine variety $\tilde{X}$.

Let $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n) \subset U_q\mathfrak{sl}_{2n}$ denote the Hopf subalgebra generated by $E_j, F_j, j \neq n$, and $K_i, K_i^{-1}, i = 1, \ldots, 2n - 1$. The corresponding Hopf $*$-subalgebra in $U_q\mathfrak{su}_{n,n}$ is denoted by $U_q\mathfrak{s}(\mathfrak{u}_n \times \mathfrak{u}_n)$. 
The elements of $\mathbb{C}[w_0SU_{n,n}]_q$

$$x_k = q^{k(k-1)} \sum_{I \subseteq \{1,2,\ldots,n\}, J \subseteq \{n+1,n+2,\ldots,2n\}} \left( \sum_{m=1}^k (q^{-2(n-i_m)}(-q)^{m-i_m-n} t_{I,I}^\wedge t_{J,J}^\wedge (2n-k) \right),$$

are pairwise commuting self-adjoint $U_q\mathfrak{g}(\mathfrak{u}_n \times \mathfrak{u}_n)$-biinvariants. These elements generate the subalgebra of all $U_q\mathfrak{g}(\mathfrak{u}_n \times \mathfrak{u}_n)$-biinvariant elements in $\mathbb{C}[w_0SU_{n,n}]$.

There is exist a unique (up to unitary equivalence) faithful representation $T$ of $\mathbb{C}[w_0SU_{n,n}]$. The set of common eigenvalues $\Sigma_D$ of the operators $T(x_1), \ldots, T(x_n)$ is

$$\{(e_1(q^{-2(\lambda+\delta)}), q^2e_2(q^{-2(\lambda+\delta)}), \ldots, q^{n(n-1)}e_n(q^{-2(\lambda+\delta)}) | \lambda \in \Lambda_n\}.$$

Consider the algebra $\mathbb{C}[u_1, \ldots, u_n]$ and the injection

$$\mathbb{C}[w_0SU_{n,n}]_q^{(U_q\mathfrak{g}(\mathfrak{u}_n \times \mathfrak{u}_n))^{op} \otimes U_q\mathfrak{g}(\mathfrak{u}_n \times \mathfrak{u}_n)} \hookrightarrow \mathbb{C}[u_1, \ldots, u_n],

x_k \mapsto q^{k(k-1)}e_k(u_1, \ldots, u_n).$$

Thus

$$\mathbb{C}[w_0SU_{n,n}]_q^{(U_q\mathfrak{g}(\mathfrak{u}_n \times \mathfrak{u}_n))^{op} \otimes U_q\mathfrak{g}(\mathfrak{u}_n \times \mathfrak{u}_n)} \cong \mathbb{C}[u_1, \ldots, u_n]^{S_n}.$$

Specify

$$\Delta_D = \{(q^{-2(\lambda+\delta)} | \lambda \in \Lambda_n\}.$$

Let also $\mathcal{D}(\Delta_D)$ be a vector space of functions $f(u_1, \ldots, u_n)$ with finite support on the set $\Delta_D$. Thereby, $\mathcal{D}(\Delta_D) \cong \mathcal{D}(\Sigma_D) \cong \mathcal{D}(\mathbb{D}_q^{U_q\mathfrak{g}^{\mathfrak{gl}_n \times \mathfrak{gl}_n}}).$
Recall the definition of multiple Jackson integral with 'base' $q^{-2}$

$$
\int_{q^{-2(n-1)}}^{\infty} \int_{q^{-2(n-2)}}^{q^{-2}} \ldots \int_{1}^{q^{-2}} \phi(u) d_{q^{-2}}u_{1} \ldots d_{q^{-2}}u_{n} =

(1 - q^{2})^{n} \sum_{\lambda \in \Lambda_{n}} \phi(q^{-2(\lambda+\delta)}) q^{-2|\lambda+\delta|}.

**Proposition 1** The restriction of the invariant integral onto the space $\mathcal{D}(\mathcal{D})_{q}^{U_{q}\mathfrak{g}_{n} \times \mathfrak{g}_{n}}$ is

$$
\mathcal{N} \int_{q^{-2(n-1)}}^{\infty} \int_{q^{-2(n-2)}}^{q^{-2}} \ldots \int_{1}^{q^{-2}}

f(e_{1}(u), \ldots, q^{n(n-1)}e_{n}(u)) \Delta(u)^{2} d_{q^{-2}}u_{1} \ldots d_{q^{-2}}u_{n},

where

$$
\Delta(u) = \prod_{1 \leq i < j \leq n} (u_{i} - u_{j}),

\mathcal{N} = (1 - q^{2})^{n(n-1)} q^{n(n-1)} \Delta(q^{-2\delta})^{-2}

is a positive constant.
As in the classical case, let us define a simple finite-dimensional weight $U_q\mathfrak{sl}_{2n}$-module to be spherical, if
\[
\dim V^{U_q(\mathfrak{gl}_n \times \mathfrak{gl}_n)} = 1.
\]

**Note 1** (J. V. Stokman, 1999) A simple finite-dimensional weight $U_q\mathfrak{sl}_{2n}$-module is $U_q(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-spherical if and only if its highest weight has the following form:
\[
(\lambda_1 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_n, 2\lambda_n, \lambda_{n-1} - \lambda_n, \ldots, \lambda_1 - \lambda_2),
\]
where $\lambda$ is a partition.

A scalar product $(\cdot, \cdot)$ in $V$ is called $U_q\mathfrak{su}_{2n}$-invariant if for any $\xi \in U_q\mathfrak{sl}_{2n}$ and for any $v_1, v_2 \in V$
\[
(\xi v_1, v_2) = (v_1, \xi^* v_2).
\]

Any spherical $U_q\mathfrak{sl}_{2n}$-module $V$ can be equipped with a $U_q\mathfrak{su}_{2n}$-invariant scalar product. Fix $v \in V^{U_q(\mathfrak{gl}_n \times \mathfrak{gl}_n)}$ by the requirement $(v, v) = 1$. Recall that $\varphi_V(\xi) = (\xi v, v)$ is the spherical function on the quantum group $SU_{2n}$ corresponding to $V$.

Let $\tilde{P}_\lambda$ be a polynomial such that
\[
P_\lambda(z) = \tilde{P}_\lambda(e_1(z), q^2 e_2(z), \ldots, q^{n(n-1)} e_n(z)),
\]
where $P_\lambda(z)$ is a specialization $P_\lambda(z; 0, 0; q^2)$ of Little $q$-Jacobi polynomials.

**Theorem 1** (J. V. Stokman, 1999) A spherical function $\varphi_\lambda$ is equal (up to a multiplicative constant) to
\[
\tilde{P}_\lambda(x_1, x_2, \ldots, x_n).
\]
Let for $\lambda \in \mathbb{C}^n$
\[ a(\lambda + \delta) = (a(\lambda_1 + n - 1), a(\lambda_2 + n - 2), \ldots, a(\lambda_n)), \]
where
\[ a(l) = \frac{(1 - q^{-2l})(1 - q^{2l+2})}{(1 - q^2)^2}, \quad l \in \mathbb{C} \]

**Proposition 2** There exist elements $C_k \in Z(U_q^{\text{ext}} \mathfrak{sl}_{2n})$, $k = 1, 2, \ldots, n$ such that
\[ C_k \varphi_\lambda = e_k(a(\lambda + \delta)) \varphi_\lambda. \]

Let $\mathcal{L}_k$ be a linear operator in $\mathbb{C}[SL_{2n}]_q$, defined with $\mathcal{L}_k f = C_k f$.

Let us define the difference operator $\Box_{u_i}$ in the space $\mathbb{C}[u_1, u_2, \ldots, u_n]^{S_n}$ with
\[ \Box_{u_i} f(u_1, \ldots, u_n) = D_{u_i} u_i (1 - q^{-1} u_i) D_{u_i} f(u_1, \ldots, u_n), \]
where $D_{u_i} f(u_1, \ldots, u_n) = \frac{f(u_1, \ldots, u_{i-1}, q^{-1} u_i, u_{i+1}, \ldots, u_n) - f(u_1, \ldots, u_{i-1}, q u_i, u_{i+1}, \ldots, u_n)}{q^{-1} u_i - q u_i}$.

**Proposition 3**
\[ \mathcal{L}_k |_{\mathbb{C}[u_1, u_2, \ldots, u_n]^{S_n}} = \frac{1}{\Delta(u)} e_k(\Box_{u_1}, \ldots, \Box_{u_n}) \Delta(u). \quad (3) \]

One may extend $\mathcal{L}_k$ to $\mathcal{D}(\mathbb{D})_q$ and then restrict to $\mathcal{D}(\mathbb{D})_q^{U_q \mathfrak{sl}(\mathfrak{g}_n \times \mathfrak{g}_n)}$ by using (3). We denote this restrictions by $\mathcal{L}_k^{\text{radial}}$. 

Let us introduce the Hilbert space $L^2(\Delta_\mathbb{D}, d\nu_q)$ of square integrable functions on $\Delta_\mathbb{D}$, where

$$d\nu_q(u) = \mathcal{N} \Delta(u)^2 d_{q-2}u_1 d_{q-2}u_2 \cdots d_{q-2}u_n. \quad (4)$$

Introduce some notation:

$$\Phi_l(x) = 3\Phi_2\left(\frac{q^{-2l}, q^{2(l+1)}, x}{q^2, 0}; q^2, q^2\right), \quad l \in \mathbb{C}$$

for basic hypergeometric function,

$$c(l) = \frac{\Gamma_q(2l + 1)}{(\Gamma_q(l + 1))^2}$$

for a $q$-analogue of Harish-Chandra’s $c$-function.

$$d\sigma(\rho) = \frac{1}{2\pi} \cdot \frac{h}{1-q^2} \cdot \frac{d\rho}{c(-\frac{1}{2} + i\rho)c(-\frac{1}{2} - i\rho)} \quad (5)$$

for the measure on the interval $[0, \pi/h]$, where $h = -2 \ln q$.

**Proposition 4** (Shklyarov, Vaksman)

$$\Box_u \Phi_l(u) = a(l)\Phi_l(u).$$
Proposition 5 i) There exists an unitary operator

\[ \mathcal{F} : L^2(\Delta_{D}, d\nu_q) \to L^2(\mathcal{R}, d\Sigma), \]

\[ \mathcal{F} : f(u) \mapsto \hat{f}(\rho_1, \rho_2, \ldots, \rho_n) = \]

\[ \frac{1}{\kappa(\rho_1, \rho_2, \ldots, \rho_n)} \int_{\Delta_{D}} \Phi_{-\frac{1}{2}+i\rho_1,-\frac{1}{2}+i\rho_2,\ldots,-\frac{1}{2}+i\rho_n}(u) f(u) d\nu_q(u), \]

where

\[ \kappa(\rho_1, \ldots, \rho_n) = \]

\[ \mathcal{N} \Delta(q^{-2\delta})^{-1} \left( \prod_{j=0}^{n-1} \frac{(q^{-2j}; q^2)^j (q^{j+1})^2 - 1}{(q^2; q^2)^2_j} \right) \prod_{1 \leq k < j \leq n} (q^{-2i\rho_j} + q^{2i\rho_j} - q^{-2i\rho_k} - q^{2i\rho_k}), \]

\[ \Phi_{l_1, \ldots, l_n}(u) = \frac{\sum_{\sigma \in S_n} \text{sign}(\sigma) \Phi_{l_1}(u_{\sigma_1}) \cdots \Phi_{l_n}(u_{\sigma_n})}{\Delta(u)} , \]

\[ \mathcal{R} = \{ (\rho_1, \ldots, \rho_n) \in [0, \pi/h]^n, \quad \rho_1 > \ldots > \rho_n \}, \]

\[ d\Sigma(\rho_1, \ldots, \rho_n) = \kappa(\rho_1, \ldots, \rho_n)^2(n!) \mathcal{N}(d\sigma(\rho_1) \ldots d\sigma(\rho_n)) |_{\mathcal{R}} \]

ii) The inverse operator is

\[ \mathcal{F}^{-1} : \hat{f}(\rho_1, \ldots, \rho_n) \mapsto \]

\[ \int_{\mathcal{R}} \hat{f}(\rho_1, \ldots, \rho_n) \Phi_{-\frac{1}{2}+i\rho_1,\ldots,-\frac{1}{2}+i\rho_n}(u) d\Sigma(\rho_1, \ldots, \rho_n). \]
Proposition 6 Pairwise commuting bounded self-adjoint operators $\mathcal{L}^{\text{radial}}_k$, $k = 1, 2, \ldots, n$ are unitary equivalent to the operators of multiplication by

$$e_k(a(-\frac{1}{2} + i\rho_1), a(-\frac{1}{2} + i\rho_2), \ldots, a(-\frac{1}{2} + i\rho_n))$$

$$\kappa(\rho_1, \rho_2, \ldots, \rho_n),$$

(respectively) in the Hilbert space $L^2(\mathcal{R}, d\Sigma)$. The unitary equivalence is provided by the operator $\mathcal{F}$. 