On the hidden supersymmetry of the reflectionless Pöschl-Teller model

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Nonlinear $N = 2$ supersymmetry

Hamiltonian H with two self-adjoint supercharges Q_1 and Q_2 satisfy

$$[H, Q_a] = 0, \quad \{Q_a, Q_b\} = \delta_{ab} P(H)$$

where $P(H)$ is polynomial in H.
Γ is the grading operator of the superalgebra - classifies bosonic and fermionic operators
- an operator is bosonic when it commutes with Γ
- an operator is fermionic when it anticommutes with Γ
Hamiltonian is supposed to be bosonic

Explicit supersymmetry

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0 & q_1 \\ q_1^\dagger & 0 \end{pmatrix}, \quad Q_2 = i\Gamma Q_1 = \begin{pmatrix} 0 & q_2 \\ q_2^\dagger & 0 \end{pmatrix}$$

$H_\pm$ are second-order differential operators, q_a can be of higher order
Grading operator $\Gamma = \sigma_3$

$$\{Q_a, \Gamma\} = 0, \quad [H, \Gamma] = 0$$

Hidden supersymmetry

$$[H, Z] = 0, \quad Z = Z^\dagger,$$

H is just second order differential operator, Z is differential operator of higher order, Γ can be nonlocal operator (parity,...)

Intertwining relations of the explicit supersymmetry

$$[Q_a, H] = 0 \Rightarrow \qquad H_+ q_a = q_a H_-, \quad H_- q_a^\dagger = q_a^\dagger H_+$$

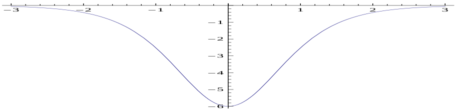
Crum-Darboux theorem:
If a differential operator A_n of order n annihilates n eigenstates ψ_i of a Hamiltonian $\mathcal{H}$, one can construct another Hamiltonian $\tilde{H} = H - 2(\ln \mathcal{W}(\psi_1, \ldots, \psi_n))''$, and these three operators are related by the identities

$$\tilde{\mathcal{H}} A_n = A_n \mathcal{H}, \qquad A_n^\dagger \tilde{\mathcal{H}} = \mathcal{H} A_n^\dagger.$$

Here, $\mathcal{W}$ is the Wronskian of not obligatorly to be physical states ψ_i, $i = 1, \ldots, n$, such that $\mathcal{W} \neq 0$.

Pöschl-Teller model

$$H_m = -\frac{d^2}{dx^2} - \frac{m(m+1)}{\cosh^2 x}\,.$$



-finite number of bound states, doubly degenerated energies of scattering states
- symmetry with respect to the substitution $m \to -m-1$ (!!!)

- nuclear physics
- used in investigation of the tachyon condensation phenomenon in gauge field string dynamics
- ...

Poschl-Teller Hamiltonians satisfy

$$\mathcal{D}_m H_m = H_{m-1}\mathcal{D}_m,$$

where the transformation

$$\mathcal{D}_m = \frac{d}{dx} + m \tanh x, \qquad \mathcal{D}_{-m} = -\mathcal{D}_m^\dagger,$$

annihilates just the ground state of H_m
-H_m and H_{m-1} are superpartner Hamiltonians in the realm of standard (linear) supersymmetry - effect of shape invariance

Using of the intertwining relations n-times, we get

$$\mathcal{D}_{m-n}..\mathcal{D}_m H_m = H_{m-n-1}\mathcal{D}_{m-n}..\mathcal{D}_m$$

Two special cases:
$n = m - 1$:
$\left(\mathcal{D}_{-m}\mathcal{D}_{-(m-1)}\dots\mathcal{D}_{-1}\right)H_0 = H_m\left(\mathcal{D}_{-m}\mathcal{D}_{-(m-1)}\dots\mathcal{D}_{-1}\right)$,
!!! when m is integer, H_m is intertwined with free particle Hamiltonian H_0 and is reflectionless!!!

$n = 2m + 1$: using the symmetry $H_m = H_{-m-1}$, we get

$$\left(\mathcal{D}_{-m}\mathcal{D}_{-(m-1)}\dots\mathcal{D}_{m}\right)H_m = H_m\left(\mathcal{D}_{-m}\mathcal{D}_{-(m-1)}\dots\mathcal{D}_{m}\right),$$

we can define hermitian operator which commutes with H_m

$$Z_m = i^{2m+1}\mathcal{D}_{-m}..\mathcal{D}_m, \quad [H_m, Z_m] = 0$$

!!!when $m \in \mathbb{N}$, $\{Z, R\} = 0$ where $RxR = -x$ - hidden susy!!!

In general case

$$H_l X_{m,l} = X_{m,l} H_m, \qquad H_l Y_{m,l} = Y_{m,l} H_m$$

where

$$\begin{aligned} X_{m,l} &= -i^{m-l}\mathcal{D}_{l+1}\mathcal{D}_{l+2}..\mathcal{D}_m \\ Y_{m,l} &= i^{2l+1}\mathcal{D}_{-m}\mathcal{D}_{-m+1}..\mathcal{D}_m \end{aligned}$$

and

$$H_l \tilde{Z}_{m,l} = \tilde{Z}_{m,l} H_l, \qquad H_m Z_{m,l} = Z_{m,l} H_m$$

where

$$Z_{m,l} = X_{m,l}^\dagger Y_{m,l}, \quad \tilde{Z}_{m,l} = X_{m,l} Y_{m,l}^\dagger$$

order of $|X_{m,l}| = m - l$, of $|Y_{m,l}| = m + l + 1$ and of $|Z_{m,l}| = |\tilde{Z}_{m,l}| = 2m + 1$

Non-linear supersymmetry of reflectionless PT model

Superextended system

$$\mathcal{H}_{m,l} = \begin{pmatrix} H_l & 0 \\ 0 & H_m \end{pmatrix}$$

Integrals of motion

$$\mathcal{X}_{m,l} = \begin{pmatrix} 0 & X_{m,l} \\ X^\dagger_{m,l} & 0 \end{pmatrix}, \quad \mathcal{Y}_{m,l} = \begin{pmatrix} 0 & Y m, l \\ Y^\dagger_{m,l} & 0 \end{pmatrix},$$

$$\mathcal{Z}_{m,l} = \mathcal{X}_{m,l}\mathcal{Y}_{m,l} = \mathcal{Y}_{m,l}\mathcal{X}_{m,l} = \begin{pmatrix} \tilde{Z}_{m,l} & 0 \\ 0 & Z_{m,l} \end{pmatrix}$$

$$[\mathcal{H}_{m,l}, \mathcal{X}_{m,l}] = [\mathcal{H}_{m,l}, \mathcal{Y}_{m,l}] = [\mathcal{H}_{m,l}, \mathcal{Z}_{m,l}] = 0$$

Algebra graded to superalgebra by one of the three operators grading operators σ_3, R or $\sigma_3 R$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad RxR = -x$$

the bosonic and fermionic operators (for $m - l$ odd):

grading operator	fermionic operators	bosonic operators
σ_3	$\mathcal{X}_{m,l}$, $\mathcal{Y}_{m,l}$	$\mathcal{H}_{m,l}$, $\mathcal{Z}_{m,l}$
R	$\mathcal{X}_{m,l}$, $\mathcal{Z}_{m,l}$	$\mathcal{H}_{m,l}$, $\mathcal{Y}_{m,l}$
$\sigma_3 R$	$\mathcal{Y}_{m,l}$, $\mathcal{Z}_{m,l}$	$\mathcal{H}_{m,l}$, $\mathcal{X}_{m,l}$

$\longrightarrow$ structure of tri-supersymmetry

Other relations of the superalgebra

$$[\mathcal{X}_{m,l},\mathcal{Y}_{m,l}]=[\mathcal{Y}_{m,l},\mathcal{Z}_{m,l}]=[\mathcal{X}_{m,l},\mathcal{Z}_{m,l}]=0,$$

$$\begin{aligned}
\mathcal{X}_{m,l}^2 &= P_{\mathcal{X}}(\mathcal{H}_{m,l})=\prod_{n=0}^{m-l-1}(\mathcal{H}_{m,l}-E_{m;n}),\\
\mathcal{Y}_{m,l}^2 &= P_{\mathcal{Y}}(\mathcal{H}_{m,l})=P_{\mathcal{X}}(\mathcal{H}_{m,l})\cdot(\mathcal{H}_{m,l}-E_{m;m})\prod_{n=m-l}^{m-1}(\mathcal{H}_{m,l}-E_{m;n})^2.\\
\mathcal{Z}_{m,l}^2 &= P_{\mathcal{Z}}(\mathcal{H}_{m,l})=P_{\mathcal{X}}(\mathcal{H}_{m,l})P_{\mathcal{Y}}(\mathcal{H}_{m,l})
\end{aligned}$$

where $E_{m;n}$ are energies of bound states of H_m

Geometry and external field

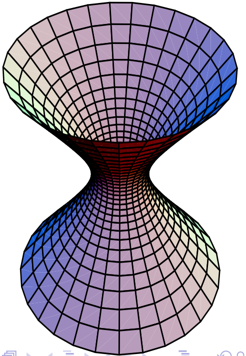
One-sheeted hyperboloid in Minkowski space with $SO(2,1)$ symmetry group: $x^\mu x_\mu = -x_1^2 - x_2^2 + x_3^2 = -\mathcal{R}^2$

$$\begin{aligned} x^1 &= \mathcal{R}\cosh\chi\cos\varphi \\ x^2 &= \mathcal{R}\cosh\chi\sin\varphi, \quad x^3 = \mathcal{R}\sinh\chi, \end{aligned}$$

External Aharonov-Bohm field:

$$\begin{aligned} A_1 &= -\frac{\Phi}{2\pi}\frac{x_2}{x_1^2+x_2^2}, \quad A_2 = \frac{\Phi}{2\pi}\frac{x_1}{x_1^2+x_2^2} \\ A_3 &= 0 \end{aligned}$$

$$B_\mu = \epsilon_{\mu\nu\lambda}\partial^\nu A^\lambda = (0,0,\Phi\delta^2(x_1,x_2))$$



The system and its algebraic properties

Hamiltonian $\hat{H}$ coincides with Casimir operator of $so(2,1)$

$$\hat{H} = \hat{J}_+\hat{J}_- - (\hat{J}_3 - 1/2)^2 = \hat{J}_-\hat{J}_+ - (\hat{J}_3 + 1/2)^2,$$

- "free" particle on the hyperboloid - analog to the Laplace operator in Euclidean space
- ladder operators $\hat{J}_\pm$ and $\hat{J}_3$ satisfy

$$[\hat{J}_3, \hat{J}_\pm] = \pm\hat{J}_\pm, \quad [\hat{J}_+, \hat{J}_-] = -2\hat{J}_3$$

-$\hat{J}_3$ is "shifted" due to the magnetic flux

Explicitly in curvilinear coordinates

$$\hat{J}_+ = e^{i\varphi}\left(\frac{\partial}{\partial\chi} - \left(\hat{J}_3 + \frac{1}{2}\right)\tanh\chi\right),$$
$$\hat{J}_- = e^{-i\varphi}\left(-\frac{\partial}{\partial\chi} - \left(\hat{J}_3 - \frac{1}{2}\right)\tanh\chi\right).$$
$$\hat{J}_3 = -i\partial_\varphi + \alpha, \quad \alpha \equiv \frac{e\Phi}{2\pi c}$$

Hamiltonian

$$\hat{H} = -\partial_\chi^2 - \frac{\hat{J}_3^2 - \frac{1}{4}}{\cosh^2\chi}$$

!!! corresponds to PT system for fixed values of $\hat{J}_3$!!!

Spectrum of the $2D$ system

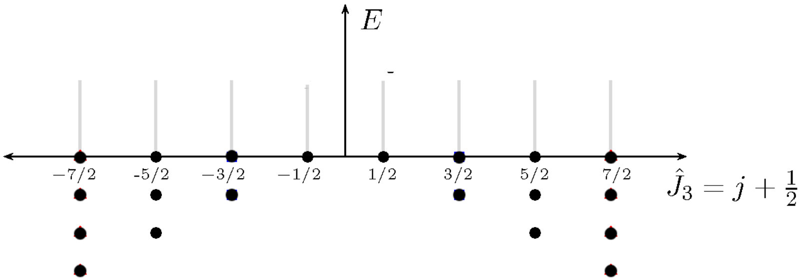
For half-integer values of $\hat{J}_3 \sim \mathbb{Z} + \frac{1}{2}$, $\hat{H}$ corresponds to reflectionless PT Hamiltonian

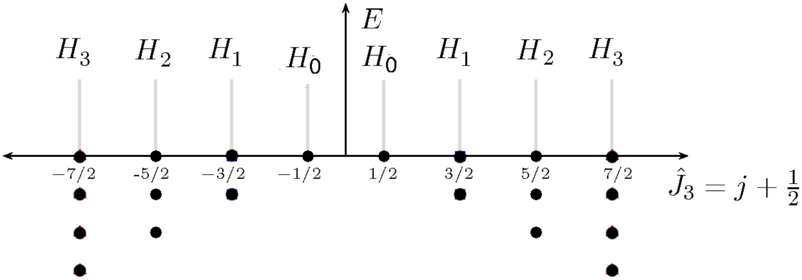
$$\hat{H}|_{J_3=m+\frac{1}{2}} = H_m.$$

We fix $\alpha = 1/2$
Then

$$\hat{H}|_{J_3\sim m+\frac{1}{2}} = H_m = H_{-m-1} = \hat{H}|_{J_3\sim -m-1+\frac{1}{2}}$$

$\rightarrow$ restrictions $\hat{J}_3 \sim j_3$ and $\hat{J}_3 \sim -j_3$ gives the same system
The spectral properties $\hat{H}$ can be deduced from the spectrum of H_m


E
−7/2
-5/2
−3/2
−1/2
1/2
3/2
5/2
7/2
$\hat{J}_3 = j + \frac{1}{2}$


E
H3
H2
H1
H0
H0
H1
H2
H3
−7/2
-5/2
−3/2
−1/2
1/2
3/2
5/2
7/2
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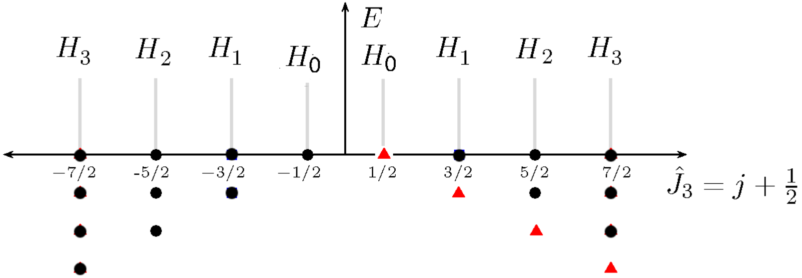
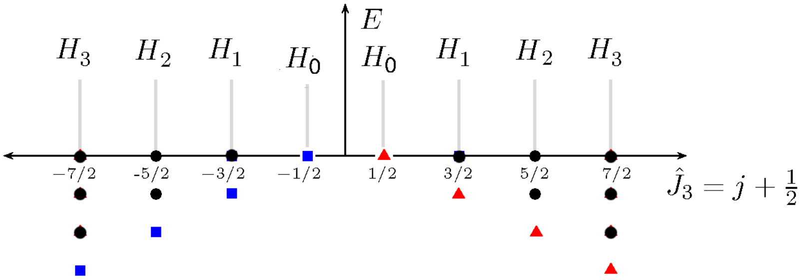


wave functions of fixed energy form semi-bounded (negative and zero energy) and unbounded (positive energies) infinite dimensional representation of $so(2,1)$
-red triangles annihilated by $\hat{J}_-$



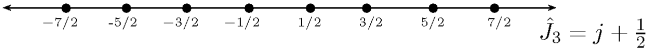
wave functions of fixed energy form semi-bounded (negative and zero energy) and unbounded (positive energies) infinite dimensional representation of $so(2,1)$
-red triangles annihilated by $\hat{J}_-$
-blue squares annihilated by $\hat{J}_+$

The idea:

$\hat{H}$ is Casimir operator and hence commutes with any function $f = f(\hat{J}_3, \hat{J}_+, \hat{J}_-)$ and with $J_\pm^n$ in particular.

We will find the relation of $J_\pm^n$ with supercharges $\mathcal{X}_{m,l}$, $\mathcal{Y}_{m,l}$ and $\mathcal{Z}_{m,l}$ of Pöschl-Teller system. Commutation relation $[H, J_\pm^n] = 0$ will correspond to intertwining relations of H_m and H_l mediated by $X_{m,l}$, $Y_{m,l}$ and $Z_{m,l}$.

Non-linear susy of PT system in terms of ladder operators



Subspaces with fixed value of $\hat{J}_3$: $\hat{J}_3|j\rangle = (j_3 + \frac{1}{2})|j\rangle$

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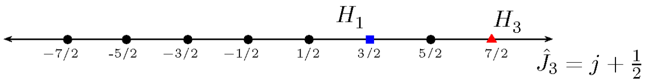

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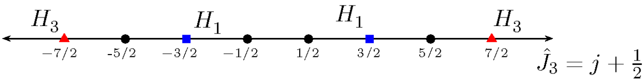

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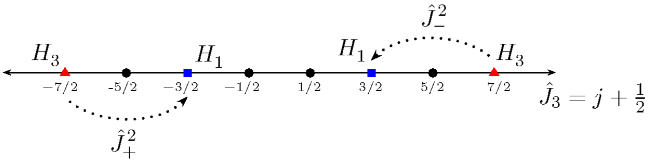

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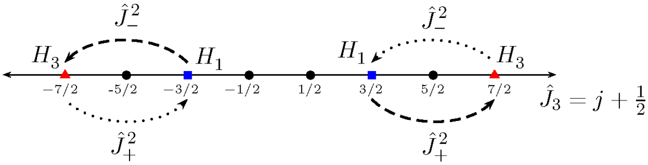

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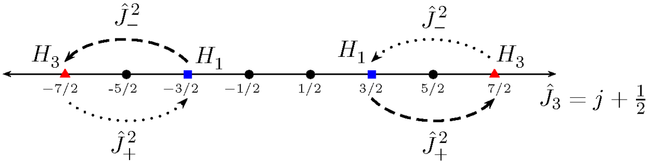
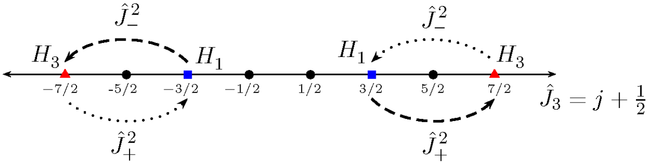


Subspaces with fixed value of $\hat{J}_3$: $\hat{J}_3|j\rangle = (j_3 + \frac{1}{2})|j\rangle$

$$[\hat{H}, J_\pm^2] = 0 \quad \Rightarrow \quad H_1 X_{3,1} = X_{3,1} H_3$$

$$X_{3,1} = \langle 1|\hat{J}_-^2|3\rangle = \langle -2|\hat{J}_+^2| - 4\rangle$$

Non-linear susy of PT system in terms of ladder operators

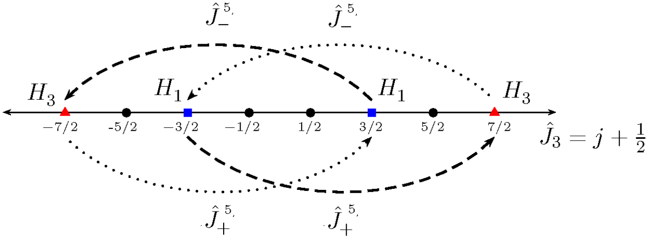


Subspaces with fixed value of $\hat{J}_3$: $\hat{J}_3|j\rangle = (j_3 + \frac{1}{2})|j\rangle$

$$[\hat{H}, J_\pm^2] = 0 \quad \Rightarrow \quad H_1 X_{3,1} = X_{3,1} H_3$$

$$X_{3,1} = \langle 1|\hat{J}_-^2|3\rangle = \langle -2|\hat{J}_+^2| - 4\rangle$$

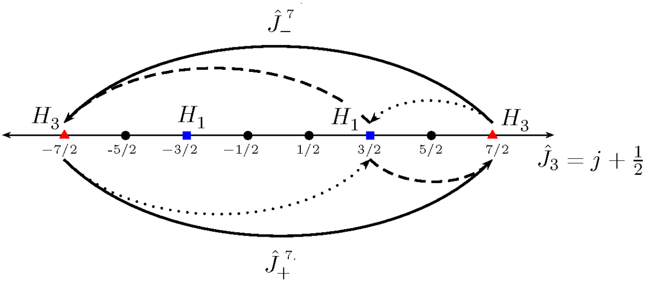
$$X_{m,l} = -(-i)^{m-l}\langle l|\hat{J}_-^{m-l}|m\rangle = -i^{m-l}\langle -l-1|\hat{J}_+^{m-l}| - m - 1\rangle$$



$$[\hat{H}, J_\pm^5] = 0 \quad \Rightarrow \quad H_1 Y_{3,1} = Y_{3,1} H_3$$

$$Y_{3,1} = \langle -2|\hat{J}_-^5|3\rangle = \langle 1|\hat{J}_+^5| - 4\rangle$$

$$Y_{m,l} = (-i)^{m+l+1}\langle -l-1|\hat{J}_-^{m+l+1}|m\rangle = i^{m+l+1}\langle l|\hat{J}_+^{m+l+1}| - m - 1\rangle$$



$$[\hat{H}, J^7_\pm] = 0 \quad \Rightarrow \quad H_3 Z_{3,1} = Z_{3,1} H_3$$

$$Z_{3,1} = \langle -4|\hat{J}^7_-|3\rangle = \langle 3|\hat{J}^7_+|-4\rangle$$

$$Z_{m,l} = -(-i)^{2m+1}\langle -m-1|\hat{J}^{2m+1}_-|m\rangle = -i^{2m+1}\langle m|\hat{J}^{2m+1}_+|-m-1\rangle$$

Summary

Restrictions of AB system in AdS_2 to fixed values of $\hat{J}_3 \rightarrow$ Pöschl-Teller model

- $[\hat{H}, \hat{J}_{\pm}^n] = 0 \rightarrow$ intertwining relations between H_m and H_{m-n}
- $\alpha \in \mathbb{Z} + \frac{1}{2}$ hidden supersymmetry represented by nontrivial supercharge $Z_{m,l}$
  $\rightarrow$ subsystem with fixed angular momentum coincides with reflectionless PT model
  $\rightarrow$ coexistence of two different Crum-Darboux transformations between two PT systems of different coupling parameter, existence of non-trivial integral of motion $Z_{m,l}$

Outlook

- the algebraic properties of PT system explained in a simple way in the framework of more-dimensional system properties of the system explained by its merging into the higher-dimensional setting
- superextended reflectionless Pöschl-Teller system is limit of tri-supersymmetric associated Lamé system $\rightarrow$ could we merge this periodic system into the higher-dimensional setting as well?

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