Nekhoroshev theorem for the periodic Toda lattice

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Outline

Periodic Toda lattice

Hamiltonian of Toda lattice with N particles:

$$H_{Toda} = \frac{1}{2}\sum_{n=1}^{N} p_n^2 + \alpha^2 \sum_{n=1}^{N} e^{q_n - q_{n+1}},$$

with periodic boundary conditions $(q_{i+N}, p_{i+N}) = (q_i, p_i) \quad \forall i \in \mathbb{N}$.

The Toda lattice is a special case of a *Fermi-Pasta-Ulam chain*, a system with a Hamiltonian similar to H_{Toda} but with potential $V(x) = \frac{1}{2}x^2 - \frac{\alpha_{FPU}}{3!}x^3 + \frac{\beta_{FPU}}{4!}x^4 + \ldots$ instead of $V(x) = e^{-x}$.

Since the total momentum $\sum_{n=1}^{N} p_n$ is conserved, we only consider the motion of the $N-1$ relative coordinates $(q_{N+1} - q_n)$; the corresponding phase space is then $\mathbb{R}^{2N-2}$, and we denote by $H_{\beta,\alpha}$ the Hamiltonian with respect to these relative coordinates for the total momentum $\beta = \frac{1}{N}\sum_{n=1}^{N} p_n$.

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Kolmogorov, Arnol’d, Moser, Nekhoroshev

Figure: Kolmogorov, Arnol’d, Moser

- Kolmogorov (1954), Arnol’d (1963), Moser (1962): Stability of the motion of nondegenerate integrable systems under small perturbations for a “majority” of initial conditions
- Nekhoroshev (1977): Stability for all initial conditions, under stronger assumptions (convexity) on the unperturbed system
- In the sequel: Many refinements and generalizations

The classical KAM theorem

Assumptions:

- Perturbed Hamiltonian $H = H_0(I) + \varepsilon H_1(x, y)$, where $(x, y) = (x_j, y_j)_{1 \leq j \leq n} \in D \subseteq \mathbb{R}^{2n}$, $I = (I_1, \ldots, I_n)$ are the *actions*, and $I_j = \frac{1}{2}(x_j^2 + y_j^2)$ for $1 \leq j \leq n$.
- The unperturbed Hamiltonian $H_0(I)$ is an *integrable system*, i.e. the $(I_j)_{1 \leq j \leq n}$ are functionally independent integrals in involution. Therefore the phase space of the unperturbed system is foliated into tori of dimension d with $0 \leq d \leq n$.
- *Kolmogorov condition:* The unperturbed integrable Hamiltonian $H_0(I)$ is *nondegenerate*, i.e. for all $(x, y) \in D$

$$\det \left(\frac{\partial^2 H_0}{\partial I_i \partial I_j} \right)_{1 \leq i,j \leq n} \neq 0.$$

Conclusions:

- There exists an $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$, a "majority" of all tori of maximal dimension of the unperturbed system H_0 *persist* as tori of the perturbed system $H = H_0(I) + \varepsilon H_1(x, y)$.

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The Nekhoroshev theorem

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- Perturbed Hamiltonian $H = H_0(I) + \varepsilon H_1(x, y)$ (as above).
- The unperturbed Hamiltonian $H_0(I)$ is an *integrable system* (as above).
- *Nekhoroshev condition:* The unperturbed integrable Hamiltonian $H_0(I)$ is *convex*, i.e. for all $(x, y) \in D$

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- There exists an $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$, the trajectory of the perturbed system $H = H_0(I) + \varepsilon H_1(x, y)$ stays "close" to the trajectory of H_0 for an "exponentially" long time.

Problem with the KAM and (especially) Nekhoroshev theorems:

- The assumptions are difficult to check. For the Nekhoroshev theorem, they require deriving an explicit formula for the Hamiltonian in terms of the action variables.

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Birkhoff normal form up to order m

- The application of the KAM and Nekhoroshev theorems requires introducing canonical coordinates $(q_i, p_i)_{1 \leq i \leq n}$ such that the Hamiltonian of a given system depends up to order 4 only on the *action variables* $\frac{1}{2}(q_i^2 + p_i^2)$; the terms of higher order will then be considered as perturbation.
- Assume that H is expressed in canonical coordinates (q, p) near an isolated equilibrium of a Hamiltonian system on some symplectic manifold with coordinates $(q, p) = (0, 0)$.

Definition

A Hamiltonian H is in *Birkhoff normal form up to order* 4, if it is of the form

$$H = N_2 + N_4 + H_5 + \dots,$$

where N_2 und N_4 are homogeneous polynomials of order 2 and 4, respectively, which are actually functions of $q_1^2 + p_1^2, \dots, q_n^2 + p_n^2$, and where $H_5 + \dots$ stands for (arbitrary) terms of order strictly greater than 4. The coordinates $(q_i, p_i)_{1 \leq i \leq n}$ are *Birkhoff coordinates*.

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Overview over the results for the periodic Toda lattice with N particles:

Normal form results

- Global Birkhoff coordinates, i.e. Birkhoff coordinates in the entire phase space
- Explicit computation of the Birkhoff normal form around the equilibrium point up to order 4
- Nondegeneracy around the equilibrium point and hence, by principles of complex analysis, almost everywhere in phase space
- Convexity around the the equilibrium point
- By an argument from Riemann surface theory, convexity in an open dense subset of the phase space

Perturbation theory results

- By (iii), KAM theorem almost everywhere in phase space
- By (iv), Nekhoroshev theorem, locally around the equilibrium point
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Global Birkhoff normal form for the periodic Toda lattice

Global Birkhoff coordinates with expansion up to order 4 at the origin for the Toda Hamiltonian $H_{\beta,\alpha}$ with respect to the $2N-2$ relative coordinates:

Theorem

For any fixed $\beta \in \mathbb{R}$, $\alpha > 0$, and $N \geq 2$, the periodic Toda lattice admits a Birkhoff normal form. More precisely, there are (globally defined) canonical coordinates $(x_k, y_k)_{1\leq k\leq N-1}$ so that $H_{\beta,\alpha}$, when expressed in these coordinates, takes the form $H_{\beta,\alpha}(I) := \frac{N\beta^2}{2} + H_\alpha(I)$, where $H_\alpha(I)$ is a real analytic function of the action variables $I_k = (x_k^2 + y_k^2)/2$ $(1 \leq k \leq N-1)$. Moreover, near $I = 0$, $H_\alpha(I)$ has an expansion of the form

$$N\alpha^2 + 2\alpha \sum_{k=1}^{N-1} \sin\frac{k\pi}{N}\, I_k + \frac{1}{4N}\sum_{k=1}^{N-1} I_k^2 + O(I^3). \tag{1}$$

Hessian at the origin

Corollary

Let $\alpha > 0$ and $\beta \in \mathbb{R}$ be arbitrary. Then the Hessian of $H_{\beta,\alpha}(I)$ at $I = 0$ is given by

$$d_I^2 H_{\beta,\alpha}|_{I=0} = \frac{1}{2N} Id_{N-1}.$$

In particular, the frequency map $I \mapsto \nabla_I H_{\beta,\alpha}$ is nondegenerate at $I = 0$ and hence, by analyticity, nondegenerate on an open dense subset of $(\mathbb{R}_{\geq 0})^{N-1}$.

Consequently, the KAM theorem can be applied on an *open dense subset* of the phase space, and the Nekhoroshev theorem can be applied *locally around the fixed point*.

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Convexity of the frequency map

Theorem

The Hamiltonian of $H_{\beta,\alpha}$, when expressed in the globally defined action variables $(I_k)_{1\leq k\leq N-1}$, is a (strictly) convex function. More precisely, for any bounded set $U \subseteq \mathbb{R}^{N-1}_{>0}$ and any $0 < \alpha_1 < \alpha_2$ there exists $m > 0$, $m = m(U_{\alpha_1\alpha_2})$, such that

$$\langle \partial_I^2 H_{\beta,\alpha}(I)\xi, \xi\rangle \geq m\|\xi\|^2, \qquad \forall \xi \in \mathbb{R}_{N-1} \tag{2}$$

for any $I \in U$, any $\beta \in \mathbb{R}$, and any $\alpha_1 \leq \alpha \leq \alpha_2$.

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General remarks

Motivation for our work: Previous results for the periodic KdV equation, an infinite-dimensional system closely related to the Toda lattice (via the Lax pair formalism), by

- Kappeler & Pöschel ("KAM & KdV"), establishing an inifinite-dimensional KAM theorem
- Krichever, Bikbaev & Kuksin, discussing the parametrization of certain solutions of KdV by suitable quantities on an associated Riemann surface

Main steps in the proof of our results: Imitation of the steps of the above mentioned work, namely

- construction of global action-angle variables and Birkhoff coordinates for the Toda lattice exactly following the method used for the KdV equation by Kappeler & Pöschel, and computation of the BNF of order 4 by standard methods
- proof of the global convexity of the Toda Hamiltonian by following the method of Bikbaev & Kuksin

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Proof of the theorem on global convexity

- The Hessian of the Toda Hamiltonian is the Jacobian of the frequency map $\omega_{\beta,\alpha} := \partial H_{\beta,\alpha}/\partial I$.
- Using tools from the theory of Riemann surfaces and tridiagonal Jacobi matrices, the frequencies $(\omega_k)_{1\leq k\leq N-1} = \partial H_{\beta,\alpha}/\partial I_k$ can be shown to be identical to the integrals of certain Abelian differentials on the Riemann surface associated to the spectrum of the matrix L associated to the Toda lattice.
- Following a method of Bikbaev & Kuksin, we show that these contour integrals are a globally nondegenerate function of the eigenvalues of L.
- The convexity of the Toda Hamiltonian at the origin together with its global nondegeneracy imply the global convexity.

The spectrum of the matrix $L(b, a)$

- The Toda equations can be put into the *Lax pair* formulation $\dot{L} = [L, B]$ with the Jacobi matrix $L = L(b, a)$, a periodic tridiagonal matrix with the b_j's as diagonal and the a_j's as offdiagonal entries.
- Associated to $L(b, a)$ is the eigenvalue equation

$$a_{k-1}y_(k-1) + b_k y(k) + a_k y(k+1) = \lambda y(k) \qquad (3)$$

  and its fundamental solutions $y_1(\cdot, \lambda)$ and $y_2(\cdot, \lambda)$.
- The *discriminant* $\Delta(\lambda) \equiv \Delta(\lambda, b, a)$ of (3) is defined by

$$\Delta(\lambda) \equiv \Delta_\lambda := y_1(N, \lambda) + y_2(N+1, \lambda)$$

- It follows from Floquet theory that we have the product representation

$$\Delta_\lambda^2 - 4 = \alpha^{-2N} \prod_{j=1}^{2N} (\lambda - \lambda_j),$$

  where $(\lambda_j)_{1 \leq j \leq 2N}$ is the combined sequence of the eigenvalues of $L = L^+$ and L^- (antiperiodic version of L^+).

Asymptotic expansion of $\operatorname{arcosh} \Delta_\lambda(b, a)$

Lemma

$$arcosh\, \frac{\Delta_\lambda}{2} = N \log \lambda - N \log \alpha + \frac{N\beta}{\lambda} - \frac{H_{Toda}}{\lambda^2} + O(\lambda^{-3}).$$

Proof.

- Consider the difference equation $L(b, a)y = \lambda y$ and the associated Floquet multiplier $w(\lambda)$.
- Note that $\log w(\lambda) = \operatorname{arcosh} \frac{\Delta_\lambda}{2}$.
- For an associated nonzero solution $u(\cdot, \lambda)$ of $L(b, a)y = \lambda y$, we define $\phi(n) = \frac{u(n+1)}{u(n)}$.
- Note that $\phi(\cdot)$ satisfies the discrete Riccati equation $a_n\phi(n)\phi(n-1) + (b_n - \lambda)\phi(n-1) + a_{n-1} = 0$.
- By substituting an expansion of $\phi(n, \lambda) \equiv \phi(n)$ into the Riccati equation, comparing coefficients and comparing the above formulas, we obtain the desired identity.

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Riemann surface $\Sigma_{b,a}$

Consider the Riemann surface

$$\Sigma_{b,a} = \{(\lambda, z) \in \mathbb{C}^2 : z^2 = \Delta_\lambda^2(b, a) - 4\} \cup \{\infty^\pm\}.$$

Pairwise disjoint cycles $(c_k)_{1\le k\le N-1}$, $(d_k)_{1\le k\le N-1}$ on $\Sigma_{b,a}$:

- $(c_k)_{1\le k\le N-1}$: the projection of c_n onto $\mathbb{C}$ is a closed curve around $[\lambda_{2k}, \lambda_{2k+1}]$.
- $(d_k)_{1\le k\le N-1}$: the intersection indices with $(c_k)_{1\le k\le N-1}$ are given by $c_n \circ d_k = \delta_{nk}$.

Abelian differentials Ω_1, Ω_2 on $\Sigma_{b,a}$:

- Ω_1, Ω_2 holomorphic on $\Sigma_{b,a}$
- Prescribed expansions at infinity
- Normalization conditions $\int_{c_k} \Omega_i = 0 (i = 1, 2)$ for any $1 \le k \le N - 1$

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For any $1 \leq n \leq N-1$,

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For any $(b,a) \in \mathcal{M}^\bullet$ and any $1 \leq n \leq N-1$,

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Krichever's theorem

Define

$$U_k := \int_{d_k} \Omega_1, \qquad V_k := \int_{d_k} \Omega_2$$

and consider the map

$$\mathcal{F} : (\lambda_1 < \ldots < \lambda_{2N}) \mapsto ((U_i, V_i)_{1 \leq i \leq N-1}, e_1, e_0),$$

where $\int_{\lambda_{2N}}^{\lambda} \Omega_1 = -\left(\log \lambda + e_0 + e_1 \frac{1}{\lambda} + \ldots\right)$ near ∞^+.

Theorem

At each point $\lambda = (\lambda_1 < \ldots < \lambda_{2N})$, the map $\mathcal{F}$ is a local diffeomorphism, i.e. the differential $d_\lambda \mathcal{F} : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ is a linear isomorphism.

The proof follows the scheme by Bikbaev & Kuksin to prove a similar theorem by Krichever; it mainly consists of couting the zeroes and poles of various auxiliary differentials.

Krichever's theorem

Define

$$U_k := \int_{d_k} \Omega_1, \qquad V_k := \int_{d_k} \Omega_2$$

and consider the map

$$\mathcal{F} : (\lambda_1 < \ldots < \lambda_{2N}) \mapsto ((U_i, V_i)_{1 \le i \le N-1}, e_1, e_0),$$

where $\int_{\lambda_{2N}}^{\lambda} \Omega_1 = -\left(\log \lambda + e_0 + e_1 \frac{1}{\lambda} + \ldots\right)$ near ∞^+.

Theorem

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Results for the periodic Toda lattice

- **Global Birkhoff normal form**
- **Global convexity of the frequency map**
- **Applications of the KAM and Nekhoroshev theorems**

Ongoing projects:

- Extension to the entire phase space, i.e. the parts of the phase space where some of the action variables vanish
- Extension to the Toda lattice with Dirichlet boundary conditions
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