Nekhoroshev theorem for the periodic Toda lattice

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Outline

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   - Perturbed integrable systems
   - Birkhoff normal form

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Hamiltonian of Toda lattice with $N$ particles:

$$H_{\text{Toda}} = \frac{1}{2} \sum_{n=1}^{N} p_n^2 + \alpha^2 \sum_{n=1}^{N} e^{q_n - q_{n+1}},$$

with periodic boundary conditions $(q_{i+N}, p_{i+N}) = (q_i, p_i)$ $\forall i \in \mathbb{N}$.

The Toda lattice is a special case of a Fermi-Pasta-Ulam chain, a system with a Hamiltonian similar to $H_{\text{Toda}}$ but with potential $V(x) = \frac{1}{2} x^2 - \frac{\alpha_{FPU}}{3!} x^3 + \frac{\beta_{FPU}}{4!} x^4 + \ldots$ instead of $V(x) = e^{-x}$.

Since the total momentum $\sum_{n=1}^{N} p_n$ is conserved, we only consider the motion of the $N - 1$ relative coordinates $(q_{N+1} - q_n)$; the corresponding phase space is then $\mathbb{R}^{2N-2}$, and we denote by $H_{\beta,\alpha}$ the Hamiltonian with respect to these relative coordinates for the total momentum $\beta = \frac{1}{N} \sum_{n=1}^{N} p_n$. 
Periodic Toda lattice

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Kolmogorov, Arnol’d, Moser, Nekhoroshev

**Figure:** Kolmogorov, Arnol’d, Moser

- Kolmogorov (1954), Arnol’d (1963), Moser (1962): Stability of the motion of nondegenerate integrable systems under small perturbations for a “majority” of initial conditions
- Nekhoroshev (1977): Stability for all initial conditions, under stronger assumptions (convexity) on the unperturbed system
- In the sequel: Many refinements and generalizations
The classical KAM theorem

Assumptions:

- Perturbed Hamiltonian $H = H_0(I) + \varepsilon H_1(x, y)$, where $(x, y) = (x_j, y_j)_{1 \leq j \leq n} \in D \subseteq \mathbb{R}^{2n}$, $I = (I_1, \ldots, I_n)$ are the actions, and $I_j = \frac{1}{2} (x_j^2 + y_j^2)$ for $1 \leq j \leq n$.
- The unperturbed Hamiltonian $H_0(I)$ is an integrable system, i.e. the $(I_j)_{1 \leq j \leq n}$ are functionally independent integrals in involution. Therefore the phase space of the unperturbed system is foliated into tori of dimension $d$ with $0 \leq d \leq n$.
- Kolmogorov condition: The unperturbed integrable Hamiltonian $H_0(I)$ is nondegenerate, i.e. for all $(x, y) \in D$

$$\det \left( \frac{\partial^2 H_0}{\partial I_i \partial I_j} \right)_{1 \leq i, j \leq n} \neq 0.$$  

Conclusions:

- There exists an $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$, a “majority” of all tori of maximal dimension of the unperturbed system $H_0$ persist as tori of the perturbed system $H = H_0(I) + \varepsilon H_1(x, y)$. 
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- **Nekhoroshev condition:** The unperturbed integrable Hamiltonian $H_0(I)$ is convex, i.e. for all $(x, y) \in D$

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is positive definite.

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**Problem with the KAM and (especially) Nekhoroshev theorems:**
- The assumptions are difficult to check. For the Nekhoroshev theorem, they require deriving an explicit formula for the Hamiltonian in terms of the action variables.
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**Birkhoff normal form up to order** \( m \)

- The application of the KAM and Nekhoroshev theorems requires introducing canonical coordinates \((q_i, p_i)_{1 \leq i \leq n}\) such that the Hamiltonian of a given system depends up to order 4 only on the *action variables* \( \frac{1}{2}(q_i^2 + p_i^2) \); the terms of higher order will then be considered as perturbation.

- Assume that \( H \) is expressed in canonical coordinates \((q, p)\) near an isolated equilibrium of a Hamiltonian system on some symplectic manifold with coordinates \((q, p) = (0, 0)\).

**Definition**

A Hamiltonian \( H \) is in *Birkhoff normal form up to order* 4, if it is of the form

\[
H = N_2 + N_4 + H_5 + \ldots,
\]

where \( N_2 \) und \( N_4 \) are homogeneous polynomials of order 2 and 4, respectively, which are actually functions of \( q_1^2 + p_1^2, \ldots, q_n^2 + p_n^2 \), and where \( H_5 + \ldots \) stands for (arbitrary) terms of order strictly greater than 4. The coordinates \((q_i, p_i)_{1 \leq i \leq n}\) are *Birkhoff coordinates.*
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Overview over the results for the periodic Toda lattice with $N$ particles:

**Normal form results**
- Global Birkhoff coordinates, i.e. Birkhoff coordinates in the entire phase space
- Explicit computation of the Birkhoff normal form around the equilibrium point up to order 4
- Nondegeneracy around the equilibrium point and hence, by principles of complex analysis, almost everywhere in phase space
- Convexity around the equilibrium point
- By an argument from Riemann surface theory, convexity in an open dense subset of the phase space

**Perturbation theory results**
- By (iii), KAM theorem almost everywhere in phase space
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Global Birkhoff normal form for the periodic Toda lattice

Global Birkhoff coordinates with expansion up to order 4 at the origin for the Toda Hamiltonian $H_{\beta,\alpha}$ with respect to the $2N - 2$ relative coordinates:

**Theorem**

*For any fixed $\beta \in \mathbb{R}$, $\alpha > 0$, and $N \geq 2$, the periodic Toda lattice admits a Birkhoff normal form. More precisely, there are (globally defined) canonical coordinates $(x_k, y_k)_{1 \leq k \leq N-1}$ so that $H_{\beta,\alpha}$, when expressed in these coordinates, takes the form $H_{\beta,\alpha}(I) := \frac{N\beta^2}{2} + H_\alpha(I)$, where $H_\alpha(I)$ is a real analytic function of the action variables $I_k = (x_k^2 + y_k^2)/2$ ($1 \leq k \leq N-1$). Moreover, near $I = 0$, $H_\alpha(I)$ has an expansion of the form*

$$N\alpha^2 + 2\alpha \sum_{k=1}^{N-1} \frac{k\pi}{N} I_k + \frac{1}{4N} \sum_{k=1}^{N-1} I_k^2 + O(I^3).$$
Corollary

Let $\alpha > 0$ and $\beta \in \mathbb{R}$ be arbitrary. Then the Hessian of $H_{\beta,\alpha}(I)$ at $I = 0$ is given by

$$d^2 H_{\beta,\alpha} \bigg|_{I=0} = \frac{1}{2N} Id_{N-1}.$$  

In particular, the frequency map $I \mapsto \nabla_I H_{\beta,\alpha}$ is nondegenerate at $I = 0$ and hence, by analyticity, nondegenerate on an open dense subset of $(\mathbb{R}_{\geq 0})^{N-1}$.

Consequently, the KAM theorem can be applied on an open dense subset of the phase space, and the Nekhoroshev theorem can be applied locally around the fixed point.
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Convexity of the frequency map

**Theorem**

*The Hamiltonian of \( H_{\beta, \alpha} \), when expressed in the globally defined action variables \((I_k)_{1 \leq k \leq N-1}\), is a (strictly) convex function. More precisely, for any bounded set \( U \subseteq \mathbb{R}^{N-1}_+ \) and any \( 0 < \alpha_1 < \alpha_2 \) there exists \( m > 0, m = m(U_{\alpha_1 \alpha_2}) \), such that*

\[
\langle \partial^2_l H_{\beta, \alpha}(l) \xi, \xi \rangle \geq m \|\xi\|^2, \quad \forall \xi \in \mathbb{R}^{N-1}
\]

(2)

*for any \( l \in U \), any \( \beta \in \mathbb{R} \), and any \( \alpha_1 \leq \alpha \leq \alpha_2 \).*

Consequently, the Nekhoroshev theorem holds in the *an open dense subset phase space.*
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Motivation for our work: Previous results for the periodic KdV equation, an infinite-dimensional system closely related to the Toda lattice (via the Lax pair formalism), by

- Kappeler & Pöschel ("KAM & KdV"), establishing an infinite-dimensional KAM theorem
- Krichever, Bikbaev & Kuksin, discussing the parametrization of certain solutions of KdV by suitable quantities on an associated Riemann surface

Main steps in the proof of our results: Imitation of the steps of the above mentioned work, namely

- construction of global action-angle variables and Birkhoff coordinates for the Toda lattice exactly following the method used for the KdV equation by Kappeler & Pöschel, and computation of the BNF of order 4 by standard methods
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Proof of the theorem on global convexity

- The Hessian of the Toda Hamiltonian is the Jacobian of the frequency map \( \omega_{\beta,\alpha} := \frac{\partial H_{\beta,\alpha}}{\partial I} \).
- Using tools from the theory of Riemann surfaces and tridiagonal Jacobi matrices, the frequencies \((\omega_k)_{1 \leq k \leq N-1} = \frac{\partial H_{\beta,\alpha}}{\partial l_k}\) can be shown to be identical to the integrals of certain Abelian differentials on the Riemann surface associated to the spectrum of the matrix \( L \) associated to the Toda lattice.
- Following a method of Bikbaev & Kuksin, we show that these contour integrals are a globally nondegenerate function of the eigenvalues of \( L \).
- The convexity of the Toda Hamiltonian at the origin together with its global nondegeneracy imply the global convexity.
The spectrum of the matrix $L(b, a)$

- The Toda equations can be put into the *Lax pair* formulation $\dot{L} = [L, B]$ with the Jacobi matrix $L = L(b, a)$, a periodic tridiagonal matrix with the $b_j$'s as diagonal and the $a_j$'s as offdiagonal entries.
- Associated to $L(b, a)$ is the eigenvalue equation
  \[ a_{k-1}y(k-1) + b_ky(k) + a_ky(k+1) = \lambda y(k) \]  
  (3)
  and its fundamental solutions $y_1(\cdot, \lambda)$ and $y_2(\cdot, \lambda)$.
- The *discriminant* $\Delta(\lambda) \equiv \Delta(\lambda, b, a)$ of (3) is defined by
  \[ \Delta(\lambda) \equiv \Delta_\lambda := y_1(N, \lambda) + y_2(N+1, \lambda) \]
- It follows from Floquet theory that we have the product representation
  \[ \Delta^2_\lambda - 4 = \alpha^{-2N} \prod_{j=1}^{2N}(\lambda - \lambda_j), \]
  where $(\lambda_j)_{1 \leq j \leq 2N}$ is the combined sequence of the eigenvalues of $L = L^+$ and $L^-$ (antiperiodic version of $L^+$).
Asymptotic expansion of arcosh $\Delta_\lambda(b, a)$

**Lemma**

$$arcosh \frac{\Delta_\lambda}{2} = N \log \lambda - N \log \alpha + \frac{N\beta}{\lambda} - \frac{H_{\text{Toda}}}{\lambda^2} + O(\lambda^{-3}).$$

**Proof.**

- Consider the difference equation $L(b, a)y = \lambda y$ and the associated Floquet multiplier $w(\lambda)$.
- Note that $\log w(\lambda) = arcosh \frac{\Delta_\lambda}{2}$.
- For an associated nonzero solution $u(\cdot, \lambda)$ of $L(b, a)y = \lambda y$, we define $\phi(n) = \frac{u(n+1)}{u(n)}$.
- Note that $\phi(\cdot)$ satisfies the discrete Riccati equation $a_n\phi(n)\phi(n-1) + (b_n - \lambda)\phi(n-1) + a_{n-1} = 0$.
- By substituting an expansion of $\phi(n, \lambda) \equiv \phi(n)$ into the Riccati equation, comparing coefficients and comparing the above formulas, we obtain the desired identity.
Asymptotic expansion of $\text{arcosh} \Delta_{\lambda}(b, a)$

**Lemma**

$$\text{arcosh} \Delta_{\lambda} \frac{2}{2} = N \log \lambda - N \log \alpha + \frac{N\beta}{\lambda} - \frac{H_{\text{Toda}}}{\lambda^2} + O(\lambda^{-3}).$$

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Consider the Riemann surface

\[ \Sigma_{b,a} = \{(\lambda, z) \in \mathbb{C}^2 : z^2 = \Delta_{\lambda}^2(b,a) - 4\} \cup \{\infty^\pm\} \].

Pairwise disjoint cycles \((c_k)_{1 \leq k \leq N-1}, (d_k)_{1 \leq k \leq N-1}\) on \(\Sigma_{b,a}\):
- \((c_k)_{1 \leq k \leq N-1}\): the projection of \(c_n\) onto \(\mathbb{C}\) is a closed curve around \([\lambda_{2k}, \lambda_{2k+1}]\).
- \((d_k)_{1 \leq k \leq N-1}\): the intersection indices with \((c_k)_{1 \leq k \leq N-1}\) are given by \(c_n \circ d_k = \delta_{nk}\).

Abelian differentials \(\Omega_1, \Omega_2\) on \(\Sigma_{b,a}\):
- \(\Omega_1, \Omega_2\) holomorphic on \(\Sigma_{b,a}\)
- Prescribed expansions at infinity
- Normalization conditions \(\int_{c_k} \Omega_i = 0 (i = 1, 2)\) for any \(1 \leq k \leq N - 1\)
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Differentials on $\Sigma_{b,a}$

We consider for any $1 \leq n \leq N - 1$ the following holomorphic one-forms on $\Sigma_{b,a}$:

$$\eta_n := \partial_{ln} \left( \text{arcosh} \frac{\Delta \lambda}{2} \right) d\lambda, \quad \zeta_n := \frac{\psi_n(\lambda)}{\sqrt{\Delta_{\lambda}^2 - 4}} d\lambda$$

**Lemma**

For any $1 \leq n \leq N - 1$,

$$\eta_n = \zeta_n.$$

**Corollary**

For any $(b, a) \in \mathcal{M}^*$ and any $1 \leq n \leq N - 1$,

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Define
\[ U_k := \int d_k \Omega_1, \quad V_k := \int d_k \Omega_2 \]
and consider the map
\[ F : (\lambda_1 < \ldots < \lambda_{2N}) \mapsto ((U_i, V_i)_{1 \leq i \leq N-1}, e_1, e_0), \]
where \( \int_{\lambda_{2N}}^{\lambda} \Omega_1 = - (\log \lambda + e_0 + e_1 \frac{1}{\lambda} + \ldots) \text{ near } \infty^+ \).

**Theorem**

At each point \( \lambda = (\lambda_1 < \ldots < \lambda_{2N}) \), the map \( F \) is a local diffeomorphism, i.e. the differential \( d_\lambda F : \mathbb{R}^{2N} \to \mathbb{R}^{2N} \) is a linear isomorphism.

The proof follows the scheme by Bikbaev & Kuksin to prove a similar theorem by Krichever; it mainly consists of counting the zeroes and poles of various auxiliary differentials.
Krichever’s theorem

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\[ U_k := \int_{d_k} \Omega_1, \quad V_k := \int_{d_k} \Omega_2 \]

and consider the map

\[ F : (\lambda_1 < \ldots < \lambda_{2N}) \mapsto ((U_i, V_i)_{1 \leq i \leq N-1}, e_1, e_0), \]

where \( \int_{\lambda_{2N}}^\lambda \Omega_1 = - (\log \lambda + e_0 + e_1 \frac{1}{\lambda} + \ldots) \) near \( \infty^+ \).

**Theorem**

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The proof follows the scheme by Bikbaev & Kuksin to prove a similar theorem by Krichever; it mainly consists of counting the zeroes and poles of various auxiliary differentials.
Summary and Discussion

Results for the periodic Toda lattice

- Global Birkhoff normal form
- Global convexity of the frequency map
- Applications of the KAM and Nekhoroshev theorems

Ongoing projects:

- Extension to the entire phase space, i.e. the parts of the phase space where some of the action variables vanish
- Extension to the Toda lattice with Dirichlet boundary conditions
- Related projects for general Fermi-Pasta-Ulam chains
- Perturbation theory for the infinite Toda lattice
- ...
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