Multi-component NLS and MKdV models on symmetric spaces and generalized Fourier transforms

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Plan of the talk:

1. Introduction

2. NLS and MKdV over symmetric spaces: algebraic and analytic aspects

3. Direct and the inverse scattering problem for $L$

4. The Generalized Fourier Transforms for Non-regular $J$

5. Hamiltonian formulation

6. Conclusions
1. Introduction

• In the one-dimensional approximation the dynamics of spinor BEC (in the $F = 1$ hyperfine state) is described by the following three-component nonlinear Schrödinger (MNLS) system in (1D) $x$-space [Ieda, Miyakawa, Wadati; 2004]:

$$
i \partial_t \Phi_1 + \partial_x^2 \Phi_1 + 2(|\Phi_1|^2 + 2|\Phi_0|^2)\Phi_1 + 2\Phi_{-1}^* \Phi_0^2 = 0,
$$

$$
i \partial_t \Phi_0 + \partial_x^2 \Phi_0 + 2(|\Phi_{-1}|^2 + |\Phi_0|^2 + |\Phi_1|^2)\Phi_0 + 2\Phi_0^* \Phi_1 \Phi_{-1} = 0,
$$

$$
i \partial_t \Phi_{-1} + \partial_x^2 \Phi_{-1} + 2(|\Phi_{-1}|^2 + 2|\Phi_0|^2)\Phi_{-1} + 2\Phi_1^* \Phi_0^2 = 0.
$$

• This model is integrable by means of inverse scattering transform method [Ieda, Miyakawa, Wadati; 2004].
  – It also allows an exact description of the dynamics and interaction of bright solitons with spin degrees of freedom.
Matter-wave solitons are expected to be useful in atom laser, atom interferometry and coherent atom transport.

- Lax pairs and geometric interpretation of our 3-component MNLS type model are given in [Fordy,Kulish;1983].

- Darboux transformation for this special integrable model is developed in [Li,Li, Malomed, Mihalache, Liu;2005].

- We will show that our system is related to the symmetric space $\text{BD.I} \simeq \text{SO}(2r + 1)/\text{SO}(2) \times \text{SO}(2r − 1)$ (in the Cartan classification [Helgasson;2001]) with canonical $\mathbb{Z}_2$-reduction and has a natural Lie algebraic interpretation.

- The model allows also a special class of soliton solutions.
● MKdV over symmetric spaces [Athorne, Fordy]:

\[
\frac{\partial Q}{\partial t} + \frac{\partial^3 Q}{\partial x^3} + 3 \left( Q_x Q^2 + Q^2 Q_x \right) = 0.
\]
2. NLS and MKdV over symmetric spaces: algebraic and analytic aspects

- Our model belongs to the class of multi-component NLS equations that can be solved by the inverse scattering method.

It is a particular case of the MNLS related to the BD.I type symmetric space \( \text{SO}(2r + 1)/\text{SO}(2) \times \text{SO}(2r - 1) \) [Fordy,Kulish;1983].

MNLS over symmetric spaces

- These MNLS systems allow Lax representation with the generalized Zakharov–Shabat system as the Lax operator:

\[
L\psi(x, t, \lambda) \equiv i \frac{\partial \psi}{\partial x} + (Q(x, t) - \lambda J)\psi(x, t, \lambda) = 0.
\]
$M\psi(x, t, \lambda) \equiv i\frac{\partial\psi}{\partial t} + (V_0(x, t) + \lambda V_1(x, t) - \lambda^2 J)\psi(x, t, \lambda) = 0,$

$V_1(x, t) = Q(x, t), \quad V_0(x, t) = i{\text{ad}}_J^{-1}dQ + \frac{1}{2}[{\text{ad}}_J^{-1}Q, Q(x, t)].$

where

$Q = \begin{pmatrix} 0 & \vec{q}^T & 0 \\ \vec{p} & 0 & s_0\vec{q} \\ 0 & \vec{p}^T s_0 & 0 \end{pmatrix}, \quad J = \text{diag}(1, 0, \ldots 0, -1).$

$\vec{q} = (q_2, \ldots, q_r, q_{r+1}, q_{r+2}, \ldots, q_{2r})^T, \quad \vec{p} = (p_2, \ldots, p_r, p_{r+1}, p_{r+2}, \ldots, p_{2r})^T,$

$S_0 = \sum_{k=1}^{2r+1} (-1)^{k+1} E_{k, 2r+2-k} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -s_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (E_{kn})_{ij} = \delta_{ik}\delta_{nj}$
\[ E_1^{\pm} = (E_{\pm(e_1-e_2)}, \ldots, E_{\pm(e_1-e_r)}, E_{\pm e_1}, E_{\pm(e_1+e_r)}, \ldots, E_{\pm(e_1+e_2)}) , \]

\[ (\vec{q} \cdot \vec{E}_1^+) = \sum_{k=2}^{r} (q_k(x, t) E_{e_1-e_k} + q_{2r-k+2}(x, t) E_{e_1+e_k}) + q_{r+1}(x, t) E_{e_1} . \]

- Then the generic form of the potentials \( Q(x, t) \) related to these type of symmetric spaces is

\[ Q(x, t) = (\vec{q}(x, t) \cdot \vec{E}_1^+) + (\vec{p}(x, t) \cdot \vec{E}_1^-) , \]

\( E_\alpha \) – Weyl generators;
\( \Delta_1^+ \) is the set of all positive roots of \( so(2r + 1) \) such that \( (\alpha, e_1) = 1 \):

\[ \Delta_1^+ = \{ e_1, \ e_1 \pm e_k, \ k = 2, \ldots, r \} . \]
• The generic MNLS type equations on $\text{BD.I.}$ symmetric spaces:

\[
\begin{align*}
    i\vec{q}_t + \vec{q}_{xx} + 2(\vec{q}, \vec{p})\vec{q} - (\vec{q}, s_0\vec{q})s_0\vec{p} &= 0, \\
    i\vec{p}_t - \vec{p}_{xx} - 2(\vec{q}, \vec{p})\vec{p} - (\vec{p}, s_0\vec{p})s_0\vec{q} &= 0,
\end{align*}
\]

$r = 2 \rightarrow \mathcal{F} = 1$ spinor BEC;
$r = 3 \rightarrow \mathcal{F} = 2$ spinor BEC;
\[\vdots\]
$r \rightarrow \mathcal{F} = r - 1$ spinor BEC.

**Example:** $\mathcal{F} = 2$ spinor BEC

Introduce the variables: $\Phi_2 = q_2$, $\Phi_1 = q_3$, $\Phi_0 = q_4$, $\Phi_{-1} = q_5$, $\Phi_{-2} = q_6$.

The assembly of atoms in the $F = 2$ hyperfine state can be described by a normalized spinor wave vector

\[
\Phi(x, t) = (\Phi_2(x, t), \Phi_1(x, t), \Phi_0(x, t), \Phi_{-1}(x, t), \Phi_{-2}(x, t))^T,
\]
whose components are labelled by the values of $m_F = 2, 1, 0, -1, -2$.

- The model equations read:

$$i\vec{\Phi}_t + \vec{\Phi}_{xx} = -2\epsilon (\vec{\Phi}, \vec{\Phi}^*) \vec{\Phi} + \epsilon (\vec{\Phi}, s_0 \vec{\Phi}) s_0 \vec{\Phi}^*,$$

or in explicit form by components:

$$i\partial_t \Phi_{\pm 2} + \partial_{xx} \Phi_{\pm 2} = -2\epsilon (\vec{\Phi}, \vec{\Phi}^*) \Phi_{\pm 2} + \epsilon (2\Phi_2 \Phi_{-2} - 2\Phi_1 \Phi_{-1} + \Phi_0^2) \Phi_{\mp 2}^*,$$
$$i\partial_t \Phi_{\pm 1} + \partial_{xx} \Phi_{\pm 1} = -2\epsilon (\vec{\Phi}, \vec{\Phi}^*) \Phi_{\pm 1} - \epsilon (2\Phi_2 \Phi_{-2} - 2\Phi_1 \Phi_{-1} + \Phi_0^2) \Phi_{\mp 1}^*,$$
$$i\partial_t \Phi_0 + \partial_{xx} \Phi_0 = -2\epsilon (\vec{\Phi}, \vec{\Phi}^*) \Phi_{\pm 0} + \epsilon (2\Phi_2 \Phi_{-2} - 2\Phi_1 \Phi_{-1} + \Phi_0^2) \Phi_0^*. $$
MKdV over symmetric spaces

- Lax representation

\[
L\psi \equiv \left( i \frac{d}{dx} + Q(x, t) - \lambda J \right) \psi(x, t, \lambda) = 0,
\]

\[
Q(x, t) = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 11 & 0 \\ 0 & -11 \end{pmatrix},
\]

\[
M\psi \equiv \left( i \frac{d}{dt} + V_0(x, t) + \lambda V_1(x, t) + \lambda^2 V_2(x, t) - 4\lambda^3 J \right) \psi(x, t, \lambda) = \psi(x, t, \lambda)C(\lambda),
\]

\[
V_2(x, t) = 4Q(x, t), \quad V_1(x, t) = 2iJQ_x + 2JQ^2, \quad V_0(x, t) = -Q_{xx} - 2Q^3,
\]

\(J\) and \(Q(x, t)\) – \(2r \times 2r\) matrices, \(J\) – block diagonal;

\(Q(x, t)\) – block-off-diagonal matrix.
• The MMKdV equations take the form

\[
\frac{\partial Q}{\partial t} + \frac{\partial^3 Q}{\partial x^3} + 3 \left( Q_x Q^2 + Q^2 Q_x \right) = 0.
\]
3. Direct and the inverse scattering problem for $L$

- Jost solutions $\phi = (\phi^+, \phi^-)$ and $\psi = (\psi^-, \psi^+)$:

$$\lim_{x \to -\infty} \phi(x, t, \lambda)e^{i\lambda Jx} = 1, \quad \lim_{x \to \infty} \psi(x, t, \lambda)e^{i\lambda Jx} = 1$$

- These definitions are compatible with the class of smooth potentials $Q(x, t)$ vanishing sufficiently rapidly at $x \to \pm \infty$.
- It can be shown that $\phi^+$ and $\psi^+$ (resp. $\phi^-$ and $\psi^-$) composed by 4 rows and 2 columns are analytic in the upper (resp. lower) half plane of $\lambda$. 
The scattering matrix:

\[
T(\lambda, t) = \begin{pmatrix}
    m_1^+ & -b^-T & c_1^- \\
    \vec{b}^+ & T_{22} & -s_0\vec{B}^- \\
    c_1^+ & \vec{B}^+T s_0 & m_1^-
\end{pmatrix},
\]

\(\vec{b}^\pm(\lambda, t)\) - 2r - 1-component vectors,
\(T_{22}(\lambda)\) - 2r - 1 \times 2r - 1 block
\(m_1^\pm(\lambda), c_1^\pm(\lambda)\) - scalar functions satisfying

\[
c_1^+ = \frac{(\vec{b}^+ \cdot s_0\vec{b}^+)}{2m_1^+} = \frac{(\vec{B}^+ \cdot s_0\vec{B}^+)}{2m_1^-}, \quad c_1^- = \frac{(\vec{B}^- \cdot s_0\vec{B}^-)}{2m_1^-} = \frac{(\vec{b}^- \cdot s_0\vec{b}^-)}{2m_1^+}.
\]

The fundamental analytic solutions (FAS) \(\chi^\pm(x, t, \lambda)\) of \(L(\lambda)\) are analytic
functions of $\lambda$ for $\text{Im} \lambda \gtrless 0$ and are related to the Jost solutions by:

$$
\chi^\pm(x, t, \lambda) = \phi(x, t, \lambda)S^\pm_J(t, \lambda) = \psi(x, t, \lambda)T^\mp_J(t, \lambda).
$$

Here $S^\pm_J$, $T^\pm_J$ upper- and lower- block-triangular matrices:

$$
S^\pm_J(t, \lambda) = \exp\left(\pm(\vec{\tau}^\pm(\lambda, t) \cdot \vec{E}_1^\pm)\right), \quad T^\pm_J(t, \lambda) = \exp\left(\mp(\vec{\rho}^\pm(\lambda, t) \cdot \vec{E}_1^\pm)\right),
$$

$$
D^+_J = \begin{pmatrix} m^+_1 & 0 & 0 \\ 0 & m^+_2 & 0 \\ 0 & 0 & 1/m^+_1 \end{pmatrix}, \quad D^-_J = \begin{pmatrix} 1/m^-_1 & 0 & 0 \\ 0 & m^-_2 & 0 \\ 0 & 0 & m^-_1 \end{pmatrix},
$$

where

$$
\vec{\tau}^+(\lambda, t) = \frac{\vec{b}^-}{m^+_1}, \quad \vec{\rho}^+(\lambda, t) = \frac{\vec{b}^+}{m^+_1}, \quad \vec{\tau}^-(\lambda, t) = \frac{\vec{B}^+}{m^-_1}, \quad \vec{\rho}^-,(\lambda, t) = \frac{\vec{B}^-}{m^-_1},
$$
and

\[ m^+_2 = T_{22} + \frac{\vec{b} + \vec{b}^* - T}{2m^+_1}, \quad m^-_2 = T_{22} + \frac{s_0 \vec{b} - \vec{b}^* + T s_0}{2m^-_1}. \]

\[ T^\pm_J(t, \lambda) \dot{S}^\pm_J(t, \lambda) = T(t, \lambda) \]

\[ \rightarrow T^\pm_J(t, \lambda) \text{ and } S^\pm_J(t, \lambda) \text{ and can be viewed as the factors of a generalized Gauss decompositions of } T(t, \lambda) \text{ [Gerdjikov;1994].} \]

- If \( Q(x,t) \) evolves according to our MNLS model then \( \vec{b}^\pm(\lambda), m^\pm_1(t, \lambda) \) and \( m^\pm_2(t, \lambda) \) satisfy the following **linear evolution equations**:

\[ i \frac{d\vec{b}^\pm}{dt} \pm \lambda^2 \vec{b}^\pm(t, \lambda) = 0, \quad i \frac{dm^\pm_1}{dt} = 0, \quad i \frac{dm^\pm_2}{dt} = 0, \]

so the block-matrices \( D^\pm(\lambda) \) can be considered as generating functionals of the integrals of motion.
The fact that all $(2r - 1)^2$ matrix elements of $m_2^+(\lambda)$ for $\lambda \in \mathbb{C}_+$ (resp. of $m_2^-(\lambda)$ for $\lambda \in \mathbb{C}_-$) generate integrals of motion reflect the superintegrability of the model and are due to the degeneracy of the dispersion law of our model.

The FAS for real $\lambda$ are linearly related

$$\chi^+(x, t, \lambda) = \chi^-(x, t, \lambda)G_J(\lambda, t), \quad G_{0,J}(\lambda, t) = S_J^-(\lambda, t)S_J^+(\lambda, t).$$

So, the sewing function $G_J(x, \lambda, t)$ is uniquely determined by the Gauss factors $S_J^{\pm}(\lambda, t)$. 
4. The Generalized Fourier Transforms for Non-regular $J$

- Wronskian relations

\[ \langle (\hat{\chi}^\pm J \chi^\pm (x, \lambda) - J) E_\beta \rangle|_{x=\infty}^{\infty} = i \int_{-\infty}^{\infty} dx \langle \left[ J, Q(x) \right] e_\beta \pm (x, \lambda) \rangle, \]

\[ \langle (\hat{\chi}'^\pm J \chi'^\pm (x, \lambda) - J) E_\beta \rangle|_{x=-\infty}^{\infty} = i \int_{-\infty}^{\infty} dx \langle \left[ J, Q(x) \right] e'_\beta \pm (x, \lambda) \rangle, \]

- ‘squared solutions’: 

\[ e_\beta \pm (x, \lambda) = \chi^\pm E_\beta \hat{\chi}^\pm (x, \lambda), \quad e_\beta \pm (x, \lambda) = P_0 J (\chi^\pm E_\beta \hat{\chi}^\pm (x, \lambda)), \]

\[ e'_\beta \pm (x, \lambda) = \chi'^\pm E_\beta \hat{\chi}'^\pm (x, \lambda), \quad e'_\beta \pm (x, \lambda) = P_0 J (\chi'^\pm E_\beta \hat{\chi}'^\pm (x, \lambda)), \]
• Skew-scalar product in the “spectral space”:

\[
[[X, Y]] = \int_{-\infty}^{\infty} dx \langle X(x), [J, Y(x)] \rangle,
\]

\[\langle X, Y \rangle \] – the Killing form;

We assume that the Cartan-Weyl generators satisfy

\[\langle E_\alpha, E_{-\beta} \rangle = \delta_{\alpha, \beta} \quad \langle H_j, H_k \rangle = \delta_{jk}.\]

[[X, Y]] is non-degenerate on the space of allowed potentials \(\mathcal{M}\).

\[
\rho_\beta^+ = -i [[Q(x), e_\beta^+]], \quad \rho_\beta^- = -i [[Q(x), e_{-\beta}^+]],
\]

\[
\tau_\beta^+ = -i [[Q(x), e_\beta^-]], \quad \tau_\beta^- = -i [[Q(x), e_{-\beta}^-]],
\]
Thus the mappings $\mathcal{F} : Q(x, t) \rightarrow T_i$ can be viewed as generalized Fourier transform in which $e^{\pm \beta}(x, \lambda)$ and $e^{\prime \pm \beta}(x, \lambda)$ can be viewed as generalizations of the standard exponentials.

- In order to work out the contributions from the discrete spectrum of $L$ we will need the explicit form of the singularities that the ‘squared solutions’ can develop in the vicinity of the discrete eigenvalues $\lambda_{\pm}^j$. 
Lemma: If all principal minors $m_{k}^{\pm}(\lambda)$ of $T(\lambda)$ only $m_{1}^{\pm}(\lambda)$ have zeroes, i.e.:

$$m_{1}^{\pm}(\lambda) = m_{1,k}^{\pm}(\lambda - \lambda_{k}^{\pm}) + \frac{1}{2} \ddot{m}_{1,k}^{\pm}(\lambda - \lambda_{k}^{\pm})^{2} + \mathcal{O}(\lambda - \lambda_{k}^{\pm})^{3}.$$ 

then the structure of the singularities of $e_{\alpha}^{\pm}(x,\lambda)$ with $\alpha \in \Delta_{1}^{+} \cup \Delta_{1}^{-}$ simplifies to:

$$e_{\alpha}^{+}(x,\lambda) = e_{\alpha:j}^{+}(x) + \dot{e}_{\alpha:j}^{+}(x)(\lambda - \lambda_{j}^{+}) + \mathcal{O}((\lambda - \lambda_{j}^{+})^{2}),$$

$$e_{-\alpha}^{+}(x,\lambda) = \frac{e_{-\alpha:j}^{+}(x)}{(\lambda - \lambda_{j}^{+})^{2}} + \frac{\dot{e}_{-\alpha:j}^{+}(x)}{\lambda - \lambda_{j}^{+}} + \mathcal{O}(1),$$

$$e_{\alpha}^{-}(x,\lambda) = \frac{e_{-\alpha:j}^{-}(x)}{(\lambda - \lambda_{j}^{-})^{2}} + \frac{\dot{e}_{-\alpha:j}^{-}(x)}{\lambda - \lambda_{j}^{-}} + \mathcal{O}(1),$$

$$e_{-\alpha}^{-}(x,\lambda) = e_{-\alpha:j}^{-}(x) + \dot{e}_{-\alpha:j}^{-}(x)(\lambda - \lambda_{j}^{-}) + \mathcal{O}((\lambda - \lambda_{j}^{-})^{2}).$$
• One more type of Wronskian relations (relating the potential $\delta Q(x)$ to the corresponding variations of the scattering data:

$$\hat{\chi}^+ \delta \chi^+(x, \lambda)|_{x=\mp\infty} = \hat{D}^+(\delta \rho^+, \vec{E}^-)D^+(\lambda) - (\delta \tau^+, \vec{E}^+) + \hat{D}^+ \delta D^+(\lambda),$$

$$\hat{\chi}^- \delta \chi^-(x, \lambda)|_{x=\mp\infty} = (\delta \tau^-, \vec{E}^-) - \hat{D}^-(\delta \rho^-, \vec{E}^+)D^-(\lambda) + \hat{D}^- \delta D^-(\lambda),$$

and

$$\hat{\chi}'^+ \delta \chi'^+(x, \lambda)|_{x=\mp\infty} = (\delta \rho^+, \vec{E}^-)(\lambda) - D^+(\delta \tau^+, \vec{E}^+)\hat{D}^+(\lambda) + \hat{D}^+ \delta D^+(\lambda),$$

$$\hat{\chi}'^- \delta \chi'^-(x, \lambda)|_{x=\mp\infty} = D^-(\delta \tau^-, \vec{E}^-)\hat{D}^-(\lambda) - (\delta \rho^-, \vec{E}^+)(\lambda) + \hat{D}^- \delta D^-(\lambda),$$

• and the corresponding “inversion formulas” (here $\beta \in \Delta_1^+$)

$$\delta \rho^+_\beta = -i\left[[\text{ad}_{\bar{J}}^{-1}\delta Q(x), \mathbf{e}'^+_{\beta}]\right], \quad \delta \rho^-_{\beta} = i\left[[\text{ad}_{\bar{J}}^{-1}\delta Q(x), \mathbf{e}'^-_{\beta}]\right],$$

$$\delta \tau^+_\beta = i\left[[\text{ad}_{\bar{J}}^{-1}\delta Q(x), \mathbf{e}^+_{\beta}]\right], \quad \delta \tau^-_{\beta} = -i\left[[\text{ad}_{\bar{J}}^{-1}\delta Q(x), \mathbf{e}^-_{\beta}]\right].$$
• Assuming that the variation of $Q(x)$ is due to its time evolution, and consider variations of the type:

$$
\delta Q(x, t) = Q_t \delta t + \mathcal{O}((\delta t)^2).
$$

Keeping only the first order terms with respect to $\delta t$ we find:

$$
\frac{d\rho^+_{\beta}}{dt} = -i \left[ [\text{ad}^{-1} J Q_t(x), e'_{\beta}^+] \right], \quad \frac{d\rho^-_{\beta}}{dt} = i \left[ [\text{ad}^{-1} J Q_t(x), e'^{-}_\beta] \right],
$$

$$
\frac{d\tau^+_{\beta}}{dt} = i \left[ [\text{ad}^{-1} J Q_t(x), e^+_\beta] \right], \quad \frac{d\tau^-_{\beta}}{dt} = -i \left[ [\text{ad}^{-1} J Q_t(x), e^-_\beta] \right],
$$
Completeness of the ‘squared solutions’

- Two sets of ‘squared solutions’

\[ \{ \Psi \} = \{ \Psi \}_c \cup \{ \Psi \}_d, \quad \{ \Phi \} = \{ \Phi \}_c \cup \{ \Phi \}_d, \]

\[ \{ \Psi \}_c \equiv \{ e^+_{-\alpha}(x, \lambda), \ e^-_{\alpha}(x, \lambda), \ \lambda \in \mathbb{R}, \ \alpha \in \Delta^+_1 \} , \]

\[ \{ \Psi \}_d \equiv \{ e^\pm_{\pm\alpha; j}(x), \ \dot{e}^\pm_{\pm\alpha; j}(x), \ \alpha \in \Delta^+_1 \} , \]

\[ \{ \Phi \}_c \equiv \{ e^+_{\alpha}(x, \lambda), \ e^-_{-\alpha}(x, \lambda), \ \lambda \in \mathbb{R}, \ \alpha \in \Delta^+_1 \} , \]

\[ \{ \Phi \}_d \equiv \{ e^\pm_{\pm\alpha; j}(x), \ \dot{e}^\pm_{\pm\alpha; j}(x), \ \alpha \in \Delta^+_1 \} , \]

where \( j = 1, \ldots, N \) and the subscripts ‘c’ and ‘d’ refer to the continuous and discrete spectrum of \( L \), the latter consisting of \( 2N \) discrete eigenvalues \( \lambda^\pm_j \in \mathbb{C} \).
Theorem: The sets \( \{\Psi\} \) and \( \{\Phi\} \) form complete sets of functions in \( \mathcal{M}_J \). The corresponding completeness relation has the form:

\[
\delta(x - y)\Pi_{0J} = \frac{1}{\pi} \int_{-\infty}^{\infty} d\lambda (G_1^+(x, y, \lambda) - G_1^-(x, y, \lambda))
- 2i \sum_{j=1}^{N} (G_1^+_{1,j}(x, y) + G_1^-_{1,j}(x, y)),
\]

where

\[
\Pi_{0J} = \sum_{\alpha \in \Delta_1^+} (E_\alpha \otimes E_{-\alpha} - E_{-\alpha} \otimes E_\alpha),
\]

\[
G_1^\pm(x, y, \lambda) = \sum_{\alpha \in \Delta_1^+} e_\pm^\alpha(x, \lambda) \otimes e_\mp^\alpha(y, \lambda),
\]
\[
G_{1,j}^{\pm}(x, y) = \sum_{\alpha \in \Delta_1^+} (\dot{e}^{\pm}_{\pm \alpha; j}(x) \otimes e^{\pm}_{\mp \alpha; j}(y) + e^{\pm}_{\pm \alpha; j}(x) \otimes \dot{e}^{\pm}_{\mp \alpha; j}(y)).
\]

Expansions of \(Q(x)\) and \(\text{ad}^{-1}_J \delta Q(x)\).

- One can expand any generic element \(F(x)\) of the phase space \(\mathcal{M}\) over each of the complete sets of ‘squared solutions’:

\[
F(x) = \sum_{\alpha \in \Delta_1^+} (F^+_{-\alpha}(x) E_{-\alpha} + F^-_{\alpha}(x) E_{\alpha}).
\]
\[
F(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\lambda \sum_{\alpha \in \Delta_1^+} \left( e_{\alpha}(x, \lambda) \gamma_{F;\alpha}(\lambda) - e_{-\alpha}(x, \lambda) \gamma_{F;\alpha}(\lambda) \right)
\]

\[
- 2i \sum_{j=1}^{N} \sum_{\alpha \in \Delta_1^+} \left( Z_{F;\alpha,j}^+(x) + Z_{F;\alpha,j}^-(x) \right),
\]

\[
F(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} d\lambda \sum_{\alpha \in \Delta_1^+} \left( e_{-\alpha}(x, \lambda) \tilde{\gamma}_{F;\alpha}(\lambda) - e_{\alpha}(x, \lambda) \tilde{\gamma}_{F;\alpha}(\lambda) \right)
\]

\[
+ 2i \sum_{j=1}^{N} \sum_{\alpha \in \Delta_1^+} \left( \tilde{Z}_{F;\alpha,j}^+(x) + \tilde{Z}_{F;\alpha,j}^-(x) \right),
\]

\[
\gamma_{F;\alpha}(\lambda) = \left[ [e_{\pm\alpha}(y, \lambda), F(y)] \right], \quad \tilde{\gamma}_{F;\alpha}(\lambda) = \left[ [e_{\pm\alpha}(y, \lambda), F(y)] \right],
\]

\[
Z_{F;j}^\pm(x) = \text{Res}_{\lambda = \lambda_j^\pm} e_{\pm\alpha}(x, \lambda) \gamma_{F;\alpha}^\pm(\lambda), \quad \tilde{Z}_{F;j}^\pm(x) = \text{Res}_{\lambda = \lambda_j^\pm} e_{\pm\alpha}(x, \lambda) \tilde{\gamma}_{F;\alpha}^\pm(\lambda),
\]

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• **Example 1** Take $F(x) \equiv Q(x)$:

\[
Q(x) = -\frac{i}{\pi} \int_{-\infty}^{\infty} d\lambda \sum_{\alpha \in \Delta_1^+} (\tau_{\alpha}^+ (\lambda) e_{\alpha}^+(x, \lambda) - \tau_{\alpha}^- (\lambda) e_{-\alpha}^- (x, \lambda)) \\
- 2 \sum_{j=1}^{N} \sum_{\alpha \in \Delta_1^+} \left( \text{Res}_{\lambda=\lambda_j^+} \tau_{\alpha}^+ e_{\alpha}^+(x, \lambda) + \text{Res}_{\lambda=\lambda_j^-} \tau_{\alpha}^- e_{-\alpha}^- (x, \lambda) \right),
\]

\[
Q(x) = \frac{i}{\pi} \int_{-\infty}^{\infty} d\lambda \sum_{\alpha \in \Delta_1^+} (\rho_{\alpha}^+ (\lambda) e_{-\alpha}^+(x, \lambda) - \rho_{\alpha}^- (\lambda) e_{\alpha}^- (x, \lambda)) \\
+ 2 \sum_{j=1}^{N} \sum_{\alpha \in \Delta_1^+} \left( \text{Res}_{\lambda=\lambda_j^+} \rho_{\alpha}^+ e_{\alpha}^+(x, \lambda) + \text{Res}_{\lambda=\lambda_j^-} \rho_{\alpha}^- e_{\alpha}^- (x, \lambda) \right),
\]
Example 2 Take $F(x) \equiv \text{ad}^{-1} J \delta Q(x)$:

$$\text{ad}^{-1} J \delta Q(x) = \frac{i}{\pi} \int_{-\infty}^{\infty} d\lambda \sum_{\alpha \in \Delta_1^+} \left( (\delta \tau_+^\alpha(\lambda) e_+^\alpha(x, \lambda) + \delta \tau_-^\alpha(\lambda) e_-^\alpha(x, \lambda) ) \right)$$

$$+ 2 \sum_{j=1}^{N} \sum_{\alpha \in \Delta_1^+} \left( \text{Res}_{\lambda=\lambda_j^+} \delta \tau_+^\alpha e_+^\alpha(x, \lambda) - \text{Res}_{\lambda=\lambda_j^-} \delta \tau_-^\alpha e_-^\alpha(x, \lambda) \right),$$

$$\text{ad}^{-1} J \delta Q(x) = \frac{i}{\pi} \int_{-\infty}^{\infty} d\lambda \sum_{\alpha \in \Delta_1^+} \left( (\delta \rho_+^\alpha(\lambda) e_+^{\prime, +}^\alpha(x, \lambda) + \delta \rho_-^\alpha(\lambda) e_-^{\prime, -}^\alpha(x, \lambda) ) \right)$$

$$- 2 \sum_{j=1}^{N} \sum_{\alpha \in \Delta_1^+} \left( \text{Res}_{\lambda=\lambda_j^+} \delta \rho_+^{\prime, +} e_+^{\prime, +}(x, \lambda) - \text{Res}_{\lambda=\lambda_j^-} \delta \rho_-^{\prime, -} e_-^{\prime, -}(x, \lambda) \right).$$
• **Example 3** Take $F(x) \equiv \text{ad}^{-1} \frac{dQ}{dt}$:

\[
\text{ad}^{-1} \frac{dQ}{dt} = \frac{i}{\pi} \int_{-\infty}^{\infty} d\lambda \sum_{\alpha \in \Delta_1^+} \left( \frac{d\tau_\alpha^+}{dt} e_{\alpha}^+(x, \lambda) + \frac{d\tau_\alpha^-}{dt} e_{-\alpha}^-(x, \lambda) \right)
\]
\[
+ 2 \sum_{j=1}^{N} \sum_{\alpha \in \Delta_1^+} \left( \text{Res}_{\lambda=\lambda_j^+} \frac{d\tau_\alpha^+}{dt} e_{\alpha}^+(x, \lambda) - \text{Res}_{\lambda=\lambda_j^-} \frac{d\tau_\alpha^-}{dt} e_{-\alpha}^-(x, \lambda) \right),
\]

\[
\text{ad}^{-1} \frac{dQ}{dt} = \frac{i}{\pi} \int_{-\infty}^{\infty} d\lambda \sum_{\alpha \in \Delta_1^+} \left( \frac{d\rho_\alpha^+}{dt} e_{\alpha}^+(x, \lambda) + \frac{d\rho_\alpha^-}{dt} e_{-\alpha}^-(x, \lambda) \right)
\]
\[
- 2 \sum_{j=1}^{N} \sum_{\alpha \in \Delta_1^+} \left( \text{Res}_{\lambda=\lambda_j^+} \frac{d\rho_\alpha^+}{dt} e_{-\alpha}^+(x, \lambda) - \text{Res}_{\lambda=\lambda_j^-} \frac{d\rho_\alpha^-}{dt} e_{\alpha}^-(x, \lambda) \right).
\]
5. Hamiltonian formulation

Integrals of motion:

- One can use any of the matrix elements of \( m_1^\pm(\lambda) \) and \( m_2^\pm(\lambda) \) as generating functional of integrals of motion of our model.

Generically such integrals would have non-local densities and will not be in involution.

The principal series of integrals is generated by \( m_1^\pm(\lambda) \):

\[
\pm \ln m_1^\pm = \sum_{k=1}^{\infty} I_k \lambda^{-k}.
\]
The integrals of motion as functionals of $Q(x)$:

$$I_s = \frac{1}{s} \int_{-\infty}^{\infty} dx \int_{x}^{\infty} dy \langle [J, Q(y)], \Lambda_s^\pm Q(x) \rangle.$$ 

Using the explicit form of $\Lambda^\pm$:

$$\Lambda^\pm Q = i \text{ad}_J^{-1} \frac{dQ}{dx} = i \frac{dQ^+}{dx} - i \frac{dQ^-}{dx},$$

$$\Lambda^2_\pm Q = -\frac{d^2 Q}{dx^2} + [Q^+ - Q^-, [Q^+, Q^-]],$$

$$\Lambda^3_\pm Q = -i \frac{d^3 Q^+}{dx^3} + i \frac{d^3 Q^-}{dx^3} + 3i [Q^+, [Q_x^+, Q^-]] + 3i [Q^-, [Q^+, Q_x^-]],$$

where

$$Q^+(x, t) = (\vec{q}(x, t) \cdot \vec{E}_1^+), \quad Q^-(x, t) = (\vec{p}(x, t) \cdot \vec{E}_1^-).$$
one can get explicit formulas for $I_s$:

\[ I_1 = -i \int_{-\infty}^{\infty} dx \langle Q^+(x), Q^-(x) \rangle, \]

\[ I_2 = \frac{1}{2} \int_{-\infty}^{\infty} dx \left( \langle Q^+_x(x), Q^-(-x) \rangle - \langle Q^+(x), Q^-(x) \rangle \right), \]

\[ I_3 = i \int_{-\infty}^{\infty} dx \left( -\langle Q^+_x(x), Q^-_x(x) \rangle + \frac{1}{2} \langle [Q^+(x), Q^-(x)], [Q^+(x), Q^-(x)] \rangle \right). \]

$iI_1$ can be interpreted as the density of the particles,

$I_2$ is the momentum,

$-iI_3$ is the Hamiltonian of the MNLS equations.
Indeed, the Hamiltonian equations of motion provided by $H(0) = -iI_3$ with the Poisson brackets

$$\{ q_k(y, t), p_j(x, t) \} = i\delta_{kj}\delta(x - y),$$

• The above Poisson brackets are dual to the canonical symplectic form:

$$\Omega_0 = i \int_{-\infty}^{\infty} dx \, \text{tr} \left( \delta \bar{p}(x) \wedge \delta \bar{q}(x) \right)$$

$$= \frac{1}{i} \int_{-\infty}^{\infty} dx \, \text{tr} \left( \text{ad}^{-1}_J \delta Q(x) \wedge [J, \text{ad}^{-1}_J \delta Q(x)] \right)$$

$$= \frac{1}{i} \left[ [\text{ad}_J^{-1} \delta Q(x) \wedge \text{ad}_J^{-1} \delta Q(x)] \right].$$
• The symplectic form through the scattering data:

\[
\Omega_0 = \frac{1}{\pi i} \int_{-\infty}^{\infty} d\lambda \left( \Omega_0^+(\lambda) - \Omega_0^-(\lambda) \right)
\]

\[
- 2 \sum_{j=1}^{N} \left( \text{Res}_{\lambda=\lambda_j^+} \Omega_0^+(\lambda) + \text{Res}_{\lambda=\lambda_j^-} \Omega_0^-(\lambda) \right),
\]

\[
\Omega_0^\pm(\lambda) = \sum_{\alpha,\gamma \in \Delta_1^+} \delta \tau^\pm(\lambda) D_{\alpha,\gamma}^\pm \wedge \delta \rho_\gamma^\pm, \quad D_{\alpha,\gamma}^\pm = \langle \hat{D}^\pm E_{\mp \gamma} D^\pm(\lambda) E_{\pm \alpha} \rangle,
\]

• The classical $R$-matrix approach [Faddeev; Takhtajan; 1986], [Fordy, Kulish; 1983] is an effective method to determine the generating functionals of local integrals of motion which are in involution.

• From it there follows that such integrals are generated by expanding $\ln m_k^\pm(\lambda)$ over the inverse powers of $\lambda$ [Gerdjikov; 1987].
Here $m_k^{\pm}(\lambda)$ are the principal minors of $T(\lambda)$; in our case

$$
m_1^{+}(\lambda) = a_{11}^{+}(\lambda), \quad m_2^{+}(\lambda) = \det a^{+}(\lambda),
$$

$$
m_1^{-}(\lambda) = a_{22}^{-}(\lambda), \quad m_2^{-}(\lambda) = \det a^{-}(\lambda).
$$

– If we consider

$$
\ln m_k^{+}(\lambda) = \sum_{s=1}^{\infty} \lambda^{-k} I_s^{(k)},
$$

then one can prove that the densities of $I_s^{(k)}$ are local in $Q(x, t)$.

– The fact that [Gerdjikov;1987]:

$$
\{m_k^{\pm}(\lambda), m_j^{\pm}(\mu)\} = 0, \quad \text{for } k, j = 1, 2,
$$

and for all $\lambda, \mu \in \mathbb{C}_\pm$ allow one to conclude that $\{I_s^{(k)}, I_p^{(j)}\} = 0$ for all $k, j = 1, 2$ and $s, p \geq 1$.  

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• In particular, the Hamiltonian of our model is proportional to $I_3^{(2)}$, i.e.

$$H = 8iI_3^{(2)}.$$
6. Conclusions

- A special version of the models describing $\mathcal{F} = 1$ and $\mathcal{F} = 2$ spinor Bose-Einstein condensates is integrable by the ISM. The corresponding Lax pair is on $\text{BD.I.} \simeq \text{SO}(2r + 1)/\text{SO}(2) \times \text{SO}(2r - 1)$ - symmetric space.

- For a generic hyperfine spin $F$, the dynamics within the mean field theory is described by the $2F + 1$ component Gross-Pitaevskii equation in one dimension.

- If all the spin dependent interactions vanish and only intensity interaction exists, the multi-component Gross-Pitaevskii equation in one dimension is equivalent to the vector nonlinear Schrödinger equation with $2F + 1$ components [S. V. Manakov; 1974].

- One can also treat generalized Zakharov–Shabat systems related to other symmetric spaces.
For all these systems of equations one can construct soliton solutions, prove completeness of ‘squared solutions’ etc.

Another interesting and still open problem is the analysis of the soliton interactions in spinor Bose-Einstein condensates.
Thank you!

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