Multi-component NLS and MKdV models on symmetric spaces and generalized Fourier transforms

Vladimir S. Gerdjikov and **Georgi G. Grahovski**

Institute for Nuclear Research and Nuclear Energy,
Bulgarian Academy of Sciences, Sofia

and

Dublin City University, Ireland

grah@inrne.bas.bg

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INRNE

Plan of the talk:

1. Introduction
2. NLS and MKdV over symmetric spaces: algebraic and analytic aspects
3. Direct and the inverse scattering problem for L
4. The Generalized Fourier Transforms for Non-regular J
5. Hamiltonian formulation
6. Conclusions

1. Introduction

- In the one-dimensional approximation the dynamics of spinor BEC (in the $F = 1$ hyperfine state) is described by the following three-component nonlinear Schrödinger (MNLS) system in (1D) x-space [Ieda,Miyakawa,Wadati;2004]:

$$
\begin{aligned}
&i\partial_t\Phi_1 + \partial_x^2\Phi_1 + 2(|\Phi_1|^2 + 2|\Phi_0|^2)\Phi_1 + 2\Phi_{-1}^*\Phi_0^2 = 0,\\
&i\partial_t\Phi_0 + \partial_x^2\Phi_0 + 2(|\Phi_{-1}|^2 + |\Phi_0|^2 + |\Phi_1|^2)\Phi_0 + 2\Phi_0^*\Phi_1\Phi_{-1} = 0,\\
&i\partial_t\Phi_{-1} + \partial_x^2\Phi_{-1} + 2(|\Phi_{-1}|^2 + 2|\Phi_0|^2)\Phi_{-1} + 2\Phi_1^*\Phi_0^2 = 0.
\end{aligned}
$$

- This model is integrable by means of inverse scattering transform method [Ieda,Miyakawa,Wadati;2004].
  - It also allows an exact description of the dynamics and interaction of bright solitons with spin degrees of freedom.

  – Matter-wave solitons are expected to be useful in atom laser, atom interferometry and coherent atom transport.

- Lax pairs and geometric interpretation of our 3-component MNLS type model are given in [Fordy,Kulish;1983].

- Darboux transformation for this special integrable model is developed in [Li,Li,Malomed,Mihalache,Liu;2005].

- We will show that our system is related to the symmetric space $\mathbf{BD.I} \simeq \mathrm{SO}(2r+1)/\mathrm{SO}(2)\times\mathrm{SO}(2r-1)$ (in the Cartan classification [Helgasson;2001]) with canonical $\mathbb{Z}_2$-reduction and has a natural Lie algebraic interpretation.

- The model allows also a special class of soliton solutions.

- MKdV over symmetric spaces [Athorne, Fordy]:

$$\frac{\partial Q}{\partial t} + \frac{\partial^3 Q}{\partial x^3} + 3\left(Q_x Q^2 + Q^2 Q_x\right) = 0.$$

2. NLS and MKdV over symmetric spaces: algebraic and analytic aspects

- Our model belongs to the class of multi-component NLS equations that can be solved by the inverse scattering method

  It is a particular case of the MNLS related to the $\mathbf{BD.I}$ type symmetric space $\mathrm{SO(2r+1)/SO(2)\times SO(2r-1)}$ [Fordy,Kulish;1983].

MNLS over symmetric spaces

- These MNLS systems allow Lax representation with the generalized Zakharov–Shabat system as the Lax operator:

$$L\psi(x,t,\lambda) \equiv \mathrm{i}\frac{\partial\psi}{\partial x} + (Q(x,t) - \lambda J)\psi(x,t,\lambda) = 0.$$

$$M\psi(x,t,\lambda) \equiv i\frac{\partial\psi}{\partial t} + (V_0(x,t) + \lambda V_1(x,t) - \lambda^2 J)\psi(x,t,\lambda) = 0,$$

$$V_1(x,t) = Q(x,t), \qquad V_0(x,t) = i\,\mathrm{ad}_J^{-1}\frac{dQ}{dx} + \frac{1}{2}\left[\mathrm{ad}_J^{-1}Q, Q(x,t)\right].$$

where

$$Q = \left(\begin{array}{ccc} 0 & \vec{q}^T & 0 \\ \vec{p} & 0 & s_0\vec{q} \\ 0 & \vec{p}^T s_0 & 0 \end{array}\right), \qquad J = \mathrm{diag}(1, 0, \ldots 0, -1).$$

$$\vec{q} = (q_2, \ldots, q_r, q_{r+1}, q_{r+2}, \ldots, q_{2r})^T, \qquad \vec{p} = (p_2, \ldots, p_r, p_{r+1}, p_{r+2}, \ldots, p_{2r})^T,$$

$$S_0 = \sum_{k=1}^{2r+1} (-1)^{k+1} E_{k,2r+2-k} = \left(\begin{array}{ccc} 0 & 0 & 1 \\ 0 & -s_0 & 0 \\ 1 & 0 & 0 \end{array}\right), \qquad (E_{kn})_{ij} = \delta_{ik}\delta_{nj}$$

$$\vec{E}_1^{\pm} = (E_{\pm(e_1-e_2)}, \dots, E_{\pm(e_1-e_r)}, E_{\pm e_1}, E_{\pm(e_1+e_r)}, \dots, E_{\pm(e_1+e_2)}),$$

$$(\vec{q} \cdot \vec{E}_1^{+}) = \sum_{k=2}^{r} (q_k(x,t) E_{e_1-e_k} + q_{2r-k+2}(x,t) E_{e_1+e_k}) + q_{r+1}(x,t) E_{e_1}.$$

- Then the generic form of the potentials $Q(x,t)$ related to these type of symmetric spaces is

$$Q(x,t) = (\vec{q}(x,t) \cdot \vec{E}_1^{+}) + (\vec{p}(x,t) \cdot \vec{E}_1^{-}),$$

E_α – Weyl generators;
Δ_1^{+} is the set of all positive roots of $so(2r+1)$ such that $(\alpha, e_1) = 1$:

$$\Delta_1^{+} = \{e_1, \quad e_1 \pm e_k, \quad k = 2, \dots, r\}.$$

- The generic MNLS type equations on **BD.I.** symmetric spaces:

$$
\begin{aligned}
i\vec{q}_t & +\vec{q}_{xx} + 2(\vec{q},\vec{p})\vec{q} - (\vec{q}, s_0\vec{q})s_0\vec{p} = 0, \\
i\vec{p}_t & -\vec{p}_{xx} - 2(\vec{q},\vec{p})\vec{p} - (\vec{p}, s_0\vec{p})s_0\vec{q} = 0,
\end{aligned}
$$

$r = 2 \to \mathcal{F} = 1$ spinor BEC;
$r = 3 \to \mathcal{F} = 2$ spinor BEC;
⋮
$r \to \mathcal{F} = r - 1$ spinor BEC.

Example: $\mathcal{F} = 2$ spinor BEC

Introduce the variables: $\Phi_2 = q_2$, $\Phi_1 = q_3$, $\Phi_0 = q_4$, $\Phi_{-1} = q_5$,$\Phi_{-2} = q_6$.

The assembly of atoms in the $F = 2$ hyperfine state can be described by a normalized spinor wave vector

$$
\mathbf{\Phi}(x,t) = (\Phi_2(x,t), \Phi_1(x,t), \Phi_0(x,t), \Phi_{-1}(x,t), \Phi_{-2}(x,t))^T,
$$

whose components are labelled by the values of $m_F = 2, 1, 0, -1, -2$.

- The model equations read:

$$i\vec{\mathbf{\Phi}}_t \quad +\vec{\mathbf{\Phi}}_{xx} = -2\epsilon(\vec{\mathbf{\Phi}}, \vec{\mathbf{\Phi}}^*)\vec{\mathbf{\Phi}} + \epsilon(\vec{\mathbf{\Phi}}, s_0\vec{\mathbf{\Phi}})s_0\vec{\mathbf{\Phi}}^*,$$

or in explicit form by components:

$$i\partial_t\Phi_{\pm 2} + \partial_{xx}\Phi_{\pm 2} = -2\epsilon(\vec{\mathbf{\Phi}}, \vec{\mathbf{\Phi}}^*)\Phi_{\pm 2} + \epsilon(2\Phi_2\Phi_{-2} - 2\Phi_1\Phi_{-1} + \Phi_0^2)\Phi^*_{\mp 2},$$
$$i\partial_t\Phi_{\pm 1} + \partial_{xx}\Phi_{\pm 1} = -2\epsilon(\vec{\mathbf{\Phi}}, \vec{\mathbf{\Phi}}^*)\Phi_{\pm 1} - \epsilon(2\Phi_2\Phi_{-2} - 2\Phi_1\Phi_{-1} + \Phi_0^2)\Phi^*_{\mp 1},$$
$$i\partial_t\Phi_0 + \partial_{xx}\Phi_0 = -2\epsilon(\vec{\mathbf{\Phi}}, \vec{\mathbf{\Phi}}^*)\Phi_{\pm 0} + \epsilon(2\Phi_2\Phi_{-2} - 2\Phi_1\Phi_{-1} + \Phi_0^2)\Phi^*_0.$$

MKdV over symmetric spaces

- Lax representation

$$L\psi \equiv \left(i\frac{d}{dx} + Q(x,t) - \lambda J \right) \psi(x,t,\lambda) = 0,$$

$$Q(x,t) = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix},$$

$$M\psi \equiv \left(i\frac{d}{dt} + V_0(x,t) + \lambda V_1(x,t) + \lambda^2 V_2(x,t) - 4\lambda^3 J \right) \psi(x,t,\lambda) = \psi(x,t,\lambda) C(\lambda),$$

$$V_2(x,t) = 4Q(x,t), \qquad V_1(x,t) = 2iJQ_x + 2JQ^2, \qquad V_0(x,t) = -Q_{xx} - 2Q^3,$$

J and $Q(x,t)$ – $2r \times 2r$ matrices, J – block diagonal;

$Q(x,t)$ – block-off-diagonal matrix.

- The MMKdV equations take the form

$$\frac{\partial Q}{\partial t} + \frac{\partial^3 Q}{\partial x^3} + 3\left(Q_x Q^2 + Q^2 Q_x\right) = 0.$$

3. Direct and the inverse scattering problem for L

- Jost solutions $\phi = (\phi^+, \phi^-)$ and $\psi = (\psi^-, \psi^+)$:

$$\lim_{x\to -\infty} \phi(x,t,\lambda) e^{i\lambda J x} = \mathbb{1}, \qquad \lim_{x\to \infty} \psi(x,t,\lambda) e^{i\lambda J x} = \mathbb{1}$$

  - These definitions are compatible with the class of smooth potentials $Q(x,t)$ vanishing sufficiently rapidly at $x \to \pm\infty$.
  - It can be shown that ϕ^+ and ψ^+ (resp. ϕ^- and ψ^-) composed by 4 rows and 2 columns are analytic in the upper (resp. lower) half plane of λ.

- The scattering matrix:

$$T(\lambda,t) = \left(\begin{array}{ccc} m_1^+ & -\vec{b}^{-T} & c_1^- \\ \vec{b}^+ & \mathbf{T}_{22} & -s_0\vec{B}^- \\ c_1^+ & \vec{B}^{+T}s_0 & m_1^- \end{array} \right),$$

$\vec{b}^{\pm}(\lambda,t)$ – $2r-1$-component vectors,
$\mathbf{T}_{22}(\lambda)$ – $2r-1 \times 2r-1$ block
$m_1^{\pm}(\lambda)$, $c_1^{\pm}(\lambda)$ –scalar functions satisfying

$$c_1^+ = \frac{(\vec{b}^+\cdot s_0\vec{b}^+)}{2m_1^+} = \frac{(\vec{B}^+\cdot s_0\vec{B}^+)}{2m_1^-}, \qquad c_1^- = \frac{(\vec{B}^-\cdot s_0\vec{B}^-)}{2m_1^-} = \frac{(\vec{b}^-\cdot s_0\vec{b}^-)}{2m_1^+}.$$

- The fundamental analytic solutions (FAS) $\chi^{\pm}(x,t,\lambda)$ of $L(\lambda)$ **are analytic**

functions of λ for $\operatorname{Im}\lambda \gtrless 0$ and are related to the Jost solutions by:

$$\chi^{\pm}(x,t,\lambda) = \phi(x,t,\lambda)S_J^{\pm}(t,\lambda) = \psi(x,t,\lambda)T_J^{\mp}(t,\lambda).$$

Here $S_J^{\pm}$, $T_J^{\pm}$ upper- and lower- block-triangular matrices:

$$S_J^{\pm}(t,\lambda) = \exp\left(\pm(\vec{\tau}^{\pm}(\lambda,t)\cdot\vec{E}_1^{\pm})\right), \quad T_J^{\pm}(t,\lambda) = \exp\left(\mp(\vec{\rho}^{\pm}(\lambda,t)\cdot\vec{E}_1^{\pm})\right),$$

$$D_J^{+} = \begin{pmatrix} m_1^+ & 0 & 0 \\ 0 & \mathbf{m}_2^+ & 0 \\ 0 & 0 & 1/m_1^+ \end{pmatrix}, \qquad D_J^{-} = \begin{pmatrix} 1/m_1^- & 0 & 0 \\ 0 & \mathbf{m}_2^- & 0 \\ 0 & 0 & m_1^- \end{pmatrix},$$

where

$$\vec{\tau}^{+}(\lambda,t) = \frac{\vec{b}^-}{m_1^+}, \quad \vec{\rho}^{+}(\lambda,t) = \frac{\vec{b}^+}{m_1^+}, \quad \vec{\tau}^{-}(\lambda,t) = \frac{\vec{B}^+}{m_1^-}, \quad \vec{\rho}^{-}(\lambda,t) = \frac{\vec{B}^-}{m_1^-},$$

and

$$\mathbf{m}_2^+ = \mathbf{T}_{22} + \frac{\vec{b}^+\vec{b}^{-T}}{2m_1^+}, \qquad \mathbf{m}_2^- = \mathbf{T}_{22} + \frac{s_0\vec{b}^-\vec{b}^{+T}s_0}{2m_1^-}.$$

$$T_J^{\pm}(t,\lambda)\hat{S}_J^{\pm}(t,\lambda) = T(t,\lambda)$$

$\rightarrow T_J^{\pm}(t,\lambda)$ and $S_J^{\pm}(t,\lambda)$ and can be viewed as the factors of a generalized Gauss decompositions of $T(t,\lambda)$ [Gerdjikov;1994].

- If $Q(x,t)$ evolves according to our MNLS model then $\vec{b}^{\pm}(\lambda)$, $m_1^{\pm}(t,\lambda)$ and $\mathbf{m}_2^{\pm}(t,\lambda)$ satisfy the following **linear evolution equations**:

$$i\frac{d\vec{b}^{\pm}}{dt} \pm \lambda^2\vec{b}^{\pm}(t,\lambda) = 0, \qquad i\frac{dm_1^{\pm}}{dt} = 0, \qquad i\frac{d\mathbf{m}_2^{\pm}}{dt} = 0,$$

so the block-matrices $D^{\pm}(\lambda)$ can be considered as generating functionals of the integrals of motion.

- The fact that all $(2r-1)^2$ matrix elements of $\mathbf{m}_2^+(\lambda)$ for $\lambda \in \mathbb{C}_+$ (resp. of $\mathbf{m}_2^-(\lambda)$ for $\lambda \in \mathbb{C}_-$) generate integrals of motion reflect the superintegrability of the model and are due to the degeneracy of the dispersion law of our model.

- The FAS for real λ are linearly related

$$\chi^+(x,t,\lambda) = \chi^-(x,t,\lambda) G_J(\lambda,t), \qquad G_{0,J}(\lambda,t) = S_J^-(\lambda,t) S_J^+(\lambda,t).$$

So, the sewing function $G_j(x,\lambda,t)$ is uniquely determined by the Gauss factors $S_J^{\pm}(\lambda,t)$.

4. The Generalized Fourier Transforms for Non-regular J

- Wronskian relations

$$\langle \left(\hat{\chi}^{\pm} J \chi^{\pm}(x,\lambda) - J\right) E_{\beta}\rangle \Big|_{x=-\infty}^{\infty} = i \int_{-\infty}^{\infty} dx\, \langle \Big([J, Q(x)] \mathbf{e}_{\beta}^{\pm}(x,\lambda) \Big) \rangle,$$

$$\langle \left(\hat{\chi}'^{,\pm} J \chi'^{,\pm}(x,\lambda) - J\right) E_{\beta}\rangle \Big|_{x=-\infty}^{\infty} = i \int_{-\infty}^{\infty} dx\, \langle \Big([J, Q(x)] \mathbf{e}_{\beta}'^{,\pm}(x,\lambda) \Big) \rangle,$$

- ‘squared solutions’:

$$e_{\beta}^{\pm}(x,\lambda) = \chi^{\pm} E_{\beta} \hat{\chi}^{\pm}(x,\lambda), \qquad \boldsymbol{e}_{\beta}^{\pm}(x,\lambda) = P_{0J}(\chi^{\pm} E_{\beta} \hat{\chi}^{\pm}(x,\lambda)),$$

$$e_{\beta}'^{,\pm}(x,\lambda) = \chi'^{,\pm} E_{\beta} \hat{\chi}'^{,\pm}(x,\lambda), \qquad \boldsymbol{e}_{\beta}'^{,\pm}(x,\lambda) = P_{0J}(\chi'^{,\pm} E_{\beta} \hat{\chi}'^{,\pm}(x,\lambda)),$$

- Skew-scalar product in the "spectral space":

$$[[X,Y]] = \int_{-\infty}^{\infty} dx \langle X(x), [J, Y(x)] \rangle,$$

$\langle X, Y \rangle$ – the Killing form;

We assume that the Cartan-Weyl generators satisfy

$$\langle E_\alpha, E_{-\beta} \rangle = \delta_{\alpha,\beta} \quad \langle H_j, H_k \rangle = \delta_{jk}.$$

$[[X,Y]]$ is non-degenerate on the space of allowed potentials $\mathcal{M}$.

$$\rho_\beta^+ = -i\left[\left[Q(x), \boldsymbol{e}_\beta^{\prime,+}\right]\right], \qquad \rho_\beta^- = -i\left[\left[Q(x), \boldsymbol{e}_{-\beta}^{\prime,-}\right]\right],$$
$$\tau_\beta^+ = -i\left[\left[Q(x), \boldsymbol{e}_{-\beta}^{+}\right]\right], \qquad \tau_\beta^- = -i\left[\left[Q(x), \boldsymbol{e}_{\beta}^{-}\right]\right],$$

Thus the mappings $\mathfrak{F} : Q(x,t) \rightarrow \mathfrak{T}_i$ can be viewed as generalized Fourier transform in which $\boldsymbol{e}_{\beta}^{\pm}(x,\lambda)$ and $\boldsymbol{e}_{\beta}^{\prime,\pm}(x,\lambda)$ can be viewed as generalizations of the standard exponentials.

- In order to work out the contributions from the discrete spectrum of L we will need the explicit form of the singularities that the ‘squared solutions’ can develop in the vicinity of the discrete eigenvalues $\lambda_j^{\pm}$.

Lemma: *If all principal minors $m_k^\pm(\lambda)$ of $T(\lambda)$ only $m_1^\pm(\lambda)$ have zeroes, i.e.:*

$$m_1^\pm(\lambda) = \dot{m}_{1,k}^\pm(\lambda - \lambda_k^\pm) + \frac{1}{2}\ddot{m}_{1,k}^\pm(\lambda - \lambda_k^\pm)^2 + \mathcal{O}(\lambda - \lambda_k^\pm)^3.$$

then the structure of the singularities of $e_\alpha^\pm(x,\lambda)$ with $\alpha \in \Delta_1^+ \cup \Delta_1^-$ simplifies to:

$$e_\alpha^+(x,\lambda) = e_{\alpha;j}^+(x) + \dot{e}_{\alpha;j}^+(x)(\lambda - \lambda_j^+) + \mathcal{O}((\lambda - \lambda_j^+)^2),$$

$$e_{-\alpha}^+(x,\lambda) = \frac{e_{-\alpha;j}^+(x)}{(\lambda - \lambda_j^+)^2} + \frac{\dot{e}_{-\alpha;j}^+(x)}{\lambda - \lambda_j^+} + \mathcal{O}(1),$$

$$e_\alpha^-(x,\lambda) = \frac{e_{-\alpha;j}^-(x)}{(\lambda - \lambda_j^-)^2} + \frac{\dot{e}_{\alpha;j}^-(x)}{\lambda - \lambda_j^-} + \mathcal{O}(1),$$

$$e_{-\alpha}^-(x,\lambda) = e_{-\alpha;j}^-(x) + \dot{e}_{-\alpha;j}^-(x)(\lambda - \lambda_j^-) + \mathcal{O}((\lambda - \lambda_j^-)^2).$$

- One more type of Wronskian relations (relating the potential $\delta Q(x)$ to the corresponding variations of the scattering data:

$$\hat{\chi}^{+}\delta\chi^{+}(x,\lambda)\Big|_{x=-\infty}^{\infty} = \hat{D}^{+}(\delta\vec{\rho}^{+},\vec{E}^{-})D^{+}(\lambda) - (\delta\vec{\tau}^{+},\vec{E}^{+}) + \hat{D}^{+}\delta D^{+}(\lambda),$$
$$\hat{\chi}^{-}\delta\chi^{-}(x,\lambda)\Big|_{x=-\infty}^{\infty} = (\delta\vec{\tau}^{-},\vec{E}^{-}) - \hat{D}^{-}(\delta\vec{\rho}^{-},\vec{E}^{+})D^{-}(\lambda) + \hat{D}^{-}\delta D^{-}(\lambda),$$

and

$$\hat{\chi}^{\prime,+}\delta\chi^{\prime,+}(x,\lambda)\Big|_{x=-\infty}^{\infty} = (\delta\vec{\rho}^{+},\vec{E}^{-})(\lambda) - D^{+}(\delta\vec{\tau}^{+},\vec{E}^{+})\hat{D}^{+}(\lambda) + \hat{D}^{+}\delta D^{+}(\lambda),$$
$$\hat{\chi}^{\prime,-}\delta\chi^{\prime,-}(x,\lambda)\Big|_{x=-\infty}^{\infty} = D^{-}(\delta\vec{\tau}^{-},\vec{E}^{-})\hat{D}^{-}(\lambda) - (\delta\vec{\rho}^{-},\vec{E}^{+})(\lambda) + \hat{D}^{-}\delta D^{-}(\lambda),$$

- and the corresponding “inversion formulas” (here $\beta \in \Delta_1^{+}$)

$$\delta\rho_\beta^{+} = -i\big[\big[\mathrm{ad}_J^{-1}\delta Q(x), \boldsymbol{e}_\beta^{\prime,+}\big]\big], \qquad \delta\rho_\beta^{-} = i\big[\big[\mathrm{ad}_J^{-1}\delta Q(x), \boldsymbol{e}_{-\beta}^{\prime,-}\big]\big],$$
$$\delta\tau_\beta^{+} = i\big[\big[\mathrm{ad}_J^{-1}\delta Q(x), \boldsymbol{e}_{-\beta}^{+}\big]\big], \qquad \delta\tau_\beta^{-} = -i\big[\big[\mathrm{ad}_J^{-1}\delta Q(x), \boldsymbol{e}_{\beta}^{-}\big]\big],$$

- Assuming that the variation of $Q(x)$ is due to its time evolution, and consider variations of the type:

$$\delta Q(x,t) = Q_t \delta t + \mathcal{O}((\delta t)^2).$$

Keeping only the first order terms with respect to δt we find:

$$\frac{d\rho_\beta^+}{dt} = -i\left[\left[\mathrm{ad}_J^{-1} Q_t(x), \boldsymbol{e}_\beta^{\prime,+}\right]\right], \qquad \frac{d\rho_\beta^-}{dt} = i\left[\left[\mathrm{ad}_J^{-1} Q_t(x), \boldsymbol{e}_{-\beta}^{\prime,-}\right]\right],$$

$$\frac{d\tau_\beta^+}{dt} = i\left[\left[\mathrm{ad}_J^{-1} Q_t(x), \boldsymbol{e}_{-\beta}^{+}\right]\right], \qquad \frac{d\tau_\beta^-}{dt} = -i\left[\left[\mathrm{ad}_J^{-1} Q_t(x), \boldsymbol{e}_{\beta}^{-}\right]\right],$$

Completeness of the 'squared solutions'

- Two sets of 'squared solutions'

$$\{\boldsymbol{\Psi}\} = \{\boldsymbol{\Psi}\}_{\rm c} \cup \{\boldsymbol{\Psi}\}_{\rm d}, \qquad \{\boldsymbol{\Phi}\} = \{\boldsymbol{\Phi}\}_{\rm c} \cup \{\boldsymbol{\Phi}\}_{\rm d},$$

$$\begin{aligned}
\{\boldsymbol{\Psi}\}_{\rm c} &\equiv \left\{ \boldsymbol{e}^{+}_{-\alpha}(x,\lambda), \quad \boldsymbol{e}^{-}_{\alpha}(x,\lambda), \quad \lambda \in \mathbb{R}, \quad \alpha \in \Delta_1^+ \right\}, \\
\{\boldsymbol{\Psi}\}_{\rm d} &\equiv \left\{ \boldsymbol{e}^{\pm}_{\mp\alpha;j}(x), \quad \dot{\boldsymbol{e}}^{\pm}_{\mp\alpha;j}(x), \quad \alpha \in \Delta_1^+, \right\}, \\
\{\boldsymbol{\Phi}\}_{\rm c} &\equiv \left\{ \boldsymbol{e}^{+}_{\alpha}(x,\lambda), \quad \boldsymbol{e}^{-}_{-\alpha}(x,\lambda), \quad \lambda \in \mathbb{R}, \quad \alpha \in \Delta_1^+ \right\}, \\
\{\boldsymbol{\Phi}\}_{\rm d} &\equiv \left\{ \boldsymbol{e}^{\pm}_{\pm\alpha;j}(x), \quad \dot{\boldsymbol{e}}^{\pm}_{\pm\alpha;j}(x), \quad \alpha \in \Delta_1^+, \right\},
\end{aligned}$$

where $j = 1, \dots, N$ and the subscripts 'c' and 'd' refer to the continuous and discrete spectrum of L, the latter consisting of $2N$ discrete eigenvalues $\lambda_j^{\pm} \in \mathbb{C}_{\pm}$.

Theorem: *The sets* $\{\boldsymbol{\Psi}\}$ *and* $\{\boldsymbol{\Phi}\}$ *form complete sets of functions in* $\mathcal{M}_J$*. The corresponding completeness relation has the form:*

$$\delta(x-y)\Pi_{0J} = \frac{1}{\pi}\int_{-\infty}^{\infty} d\lambda (G_1^+(x,y,\lambda) - G_1^-(x,y,\lambda)) - 2i\sum_{j=1}^{N}(G_{1,j}^+(x,y) + G_{1,j}^-(x,y)),$$

where

$$\Pi_{0J} = \sum_{\alpha\in\Delta_1^+} (E_\alpha \otimes E_{-\alpha} - E_{-\alpha} \otimes E_\alpha),$$

$$G_1^{\pm}(x,y,\lambda) = \sum_{\alpha\in\Delta_1^+} \boldsymbol{e}_{\pm\alpha}^{\pm}(x,\lambda) \otimes \boldsymbol{e}_{\mp\alpha}^{+}(y,\lambda),$$

$$G_{1,j}{}^{\pm}(x,y) = \sum_{\alpha\in\Delta_1^+} (\dot{\boldsymbol{e}}^{\pm}_{\pm\alpha;j}(x) \otimes \boldsymbol{e}^{\pm}_{\mp\alpha;j}(y) + \boldsymbol{e}^{\pm}_{\pm\alpha;j}(x) \otimes \dot{\boldsymbol{e}}^{\pm}_{\mp\alpha;j}(y)).$$

Expansions of $Q(x)$ and $\mathrm{ad}_J^{-1}\delta Q(x)$.

- One can expand any generic element $F(x)$ of the phase space $\mathcal{M}$ over each of the complete sets of 'squared solutions':

$$F(x) = \sum_{\alpha\in\Delta_1^+} (F^{+}_{-\alpha}(x)E_{-\alpha} + F^{-}_{\alpha}(x)E_{\alpha}).$$

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} d\lambda \sum_{\alpha\in\Delta_1^+} \left(\boldsymbol{e}_{\alpha}^{+}(x,\lambda)\gamma_{F;-\alpha}^{+}(\lambda) - \boldsymbol{e}_{-\alpha}^{-}(x,\lambda)\gamma_{F;\alpha}^{-}(\lambda)\right)$$

$$- 2i \sum_{j=1}^{N} \sum_{\alpha\in\Delta_1^+} (Z_{F;\alpha,j}^{+}(x) + Z_{F;\alpha,j}^{-}(x)),$$

$$F(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} d\lambda \sum_{\alpha\in\Delta_1^+} \left(\boldsymbol{e}_{-\alpha}^{+}(x,\lambda)\tilde{\gamma}_{F;\alpha}^{+}(\lambda) - \boldsymbol{e}_{\alpha}^{-}(x,\lambda)\tilde{\gamma}_{F;-\alpha}^{-}(\lambda)\right)$$

$$+ 2i \sum_{j=1}^{N} \sum_{\alpha\in\Delta_1^+} (\tilde{Z}_{F;\alpha,j}^{+}(x) + \tilde{Z}_{F;\alpha,j}^{-}(x)),$$

$$\gamma_{F;\alpha}^{\pm}(\lambda) = [\![\boldsymbol{e}_{\pm\alpha}^{\pm}(y,\lambda), F(y)]\!], \qquad \tilde{\gamma}_{F;\alpha}^{\pm}(\lambda) = [\![\boldsymbol{e}_{\mp\alpha}^{\pm}(y,\lambda), F(y)]\!],$$

$$Z_{F;j}^{\pm}(x) = \operatorname*{Res}_{\lambda=\lambda_j^{\pm}} \boldsymbol{e}_{\mp\alpha}^{\pm}(x,\lambda)\gamma_{F;\mp\alpha}^{\pm}(\lambda), \qquad \tilde{Z}_{F;j}^{\pm}(x) = \operatorname*{Res}_{\lambda=\lambda_j^{+}} \boldsymbol{e}_{\pm\alpha}^{\pm}(x,\lambda)\gamma_{F;\pm\alpha}^{+}(\lambda),$$

- **Example 1** Take $F(x) \equiv Q(x)$:

$$Q(x) = -\frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda \sum_{\alpha\in\Delta_1^+} \left(\tau_\alpha^+(\lambda)\boldsymbol{e}_\alpha^+(x,\lambda) - \tau_\alpha^-(\lambda)\boldsymbol{e}_{-\alpha}^-(x,\lambda)\right)$$

$$-2\sum_{j=1}^{N}\sum_{\alpha\in\Delta_1^+}\left(\mathop{\mathrm{Res}}_{\lambda=\lambda_j^+} \tau_\alpha^+\boldsymbol{e}_\alpha^+(x,\lambda) + \mathop{\mathrm{Res}}_{\lambda=\lambda_j^-} \tau_\alpha^-\boldsymbol{e}_{-\alpha}^-(x,\lambda)\right),$$

$$Q(x) = \frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda \sum_{\alpha\in\Delta_1^+} \left(\rho_\alpha^+(\lambda)\boldsymbol{e}_{-\alpha}^{\prime;+}(x,\lambda) - \rho_\alpha^-(\lambda)\boldsymbol{e}_\alpha^{\prime;-}(x,\lambda)\right)$$

$$+2\sum_{j=1}^{N}\sum_{\alpha\in\Delta_1^+}\left(\mathop{\mathrm{Res}}_{\lambda=\lambda_j^+} \rho_\alpha^+\boldsymbol{e}_\alpha^{\prime;+}(x,\lambda) + \mathop{\mathrm{Res}}_{\lambda=\lambda_j^-} \rho_\alpha^-\boldsymbol{e}_\alpha^{\prime;-}(x,\lambda)\right),$$

- **Example 2** Take $F(x) \equiv \operatorname{ad}_J^{-1}\delta Q(x)$:

$$\operatorname{ad}_J^{-1}\delta Q(x) = \frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda \sum_{\alpha\in\Delta_1^+} \left(\delta\tau_\alpha^+(\lambda)\boldsymbol{e}_\alpha^+(x,\lambda) + \delta\tau_\alpha^-(\lambda)\boldsymbol{e}_{-\alpha}^-(x,\lambda)\right)$$

$$+ 2\sum_{j=1}^{N}\sum_{\alpha\in\Delta_1^+}\left(\operatorname*{Res}_{\lambda=\lambda_j^+}\delta\tau_\alpha^+\boldsymbol{e}_\alpha^+(x,\lambda) - \operatorname*{Res}_{\lambda=\lambda_j^-}\delta\tau_\alpha^-\boldsymbol{e}_{-\alpha}^-(x,\lambda)\right),$$

$$\operatorname{ad}_J^{-1}\delta Q(x) = \frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda \sum_{\alpha\in\Delta_1^+} \left(\delta\rho_\alpha^+(\lambda)\boldsymbol{e}_{-\alpha}^{\prime,+}(x,\lambda) + \delta\rho_\alpha^-(\lambda)\boldsymbol{e}_\alpha^{\prime,-}(x,\lambda)\right)$$

$$- 2\sum_{j=1}^{N}\sum_{\alpha\in\Delta_1^+}\left(\operatorname*{Res}_{\lambda=\lambda_j^+}\delta\rho_\alpha^+\boldsymbol{e}_{-\alpha}^{\prime,+}(x,\lambda) - \operatorname*{Res}_{\lambda=\lambda_j^-}\delta\rho_\alpha^-\boldsymbol{e}_\alpha^{\prime,-}(x,\lambda)\right).$$

- **Example 3** Take $F(x) \equiv \operatorname{ad}_J^{-1} \frac{dQ}{dt}$:

$$\operatorname{ad}_J^{-1} \frac{dQ}{dt} = \frac{i}{\pi} \int_{-\infty}^{\infty} d\lambda \sum_{\alpha \in \Delta_1^+} \left(\frac{d\tau_\alpha^+}{dt} \boldsymbol{e}_\alpha^+(x,\lambda) + \frac{d\tau_\alpha^-}{dt} \boldsymbol{e}_{-\alpha}^-(x,\lambda) \right)$$
$$+ 2 \sum_{j=1}^{N} \sum_{\alpha \in \Delta_1^+} \left(\operatorname*{Res}_{\lambda=\lambda_j^+} \frac{d\tau_\alpha^+}{dt} \boldsymbol{e}_\alpha^+(x,\lambda) - \operatorname*{Res}_{\lambda=\lambda_j^-} \frac{d\tau_\alpha^-}{dt} \boldsymbol{e}_{-\alpha}^-(x,\lambda) \right),$$

$$\operatorname{ad}_J^{-1} \frac{dQ}{dt} = \frac{i}{\pi} \int_{-\infty}^{\infty} d\lambda \sum_{\alpha \in \Delta_1^+} \left(\frac{d\rho_\alpha^+}{dt} \boldsymbol{e}_{-\alpha}^{\prime,+}(x,\lambda) + \frac{d\rho_\alpha^-}{dt} \boldsymbol{e}_\alpha^{\prime,-}(x,\lambda) \right)$$
$$- 2 \sum_{j=1}^{N} \sum_{\alpha \in \Delta_1^+} \left(\operatorname*{Res}_{\lambda=\lambda_j^+} \frac{d\rho_\alpha^+}{dt} \boldsymbol{e}_{-\alpha}^{\prime,+}(x,\lambda) - \operatorname*{Res}_{\lambda=\lambda_j^-} \frac{d\rho_\alpha^-}{dt} \boldsymbol{e}_\alpha^{\prime,-}(x,\lambda) \right).$$

5. Hamiltonian formulation

Integrals of motion:

- One can use any of the matrix elements of $m_1^{\pm}(\lambda)$ and $\mathbf{m}_2^{\pm}(\lambda)$ as generating functional of integrals of motion of our model.

  Generically such integrals would have non-local densities and will not be in involution.

  he principal series of integrals is generated by $m_1^{\pm}(\lambda)$:

$$\pm \ln m_1^{\pm} = \sum_{k=1}^{\infty} I_k \lambda^{-k}.$$

- The integrals of motion as functionals of $Q(x)$:

$$I_s = \frac{1}{s}\int_{-\infty}^{\infty} dx \int_{\pm\infty}^{x} dy \, \left\langle [J, Q(y)], \Lambda_{\pm}^{s} Q(x) \right\rangle .$$

Using the explicit form of $\Lambda_{\pm}$:

$$\Lambda_{\pm} Q = i\,\mathrm{ad}_{J}^{-1}\frac{dQ}{dx} = i\frac{dQ^{+}}{dx} - i\frac{dQ^{-}}{dx},$$

$$\Lambda_{\pm}^{2} Q = -\frac{d^2 Q}{dx^2} + \left[Q^{+} - Q^{-}, [Q^{+}, Q^{-}] \right],$$

$$\Lambda_{\pm}^{3} Q = -i\frac{d^3 Q^{+}}{dx^3} + i\frac{d^3 Q^{-}}{dx^3} + 3i\left[Q^{+}, [Q^{+}_{x}, Q^{-}] \right] + 3i\left[Q^{-}, [Q^{+}, Q^{-}_{x}] \right],$$

where

$$Q^{+}(x,t) = (\vec{q}(x,t)\cdot\vec{E}_1^{+}), \qquad Q^{-}(x,t) = (\vec{p}(x,t)\cdot\vec{E}_1^{-}).$$

one can get explicit formulas for I_s:

$$\begin{aligned} I_1 &= -i\int_{-\infty}^{\infty} dx\, \langle Q^+(x), Q^-(x)\rangle, \\ I_2 &= \frac{1}{2}\int_{-\infty}^{\infty} dx\, \left(\langle Q_x^+(x), Q^-(x)\rangle - \langle Q^+(x), Q_x^-(x)\rangle\right), \\ I_3 &= i\int_{-\infty}^{\infty} dx\, \left(-\langle Q_x^+(x), Q_x^-(x)\rangle + \frac{1}{2}\langle [Q^+(x), Q^-(x)], [Q^+(x), Q^-(x)]\rangle\right). \end{aligned}$$

iI_1 can be interpreted as the density of the particles,

I_2 is the momentum,

$-iI_3$ is the Hamiltonian of the MNLS equations.

Indeed, the Hamiltonian equations of motion provided by $H_{(0)} = -iI_3$ with the Poissson brackets

$$\{q_k(y,t), p_j(x,t)\} = i\delta_{kj}\delta(x-y),$$

- The above Poisson brackets are dual to the canonical symplectic form:

$$\begin{aligned} \Omega_0 &= i\int_{-\infty}^{\infty} dx \,\mathrm{tr}\, \left(\delta\vec{p}(x)\wedge\delta\vec{q}(x)\right) \\ &= \frac{1}{i}\int_{-\infty}^{\infty} dx \,\mathrm{tr}\, \left(\mathrm{ad}\,_J^{-1}\delta Q(x)\wedge [J, \mathrm{ad}\,_J^{-1}\delta Q(x)\right) \\ &= \frac{1}{i}[[\mathrm{ad}\,_J^{-1}\delta Q(x) \underset{'}{\wedge} \mathrm{ad}\,_J^{-1}\delta Q(x)]], \end{aligned}$$

- The symplectic form through the scattering data:

$$\begin{aligned}\Omega_0 &= \frac{1}{\pi i}\int_{-\infty}^{\infty} d\lambda \, \left(\Omega_0^+(\lambda) - \Omega_0^-(\lambda)\right) \\ &- 2\sum_{j=1}^{N}\left(\operatorname*{Res}_{\lambda=\lambda_j^+}\Omega_0^+(\lambda) + \operatorname*{Res}_{\lambda=\lambda_j^-}\Omega_0^-(\lambda)\right), \\ \Omega_0^{\pm}(\lambda) &= \sum_{\alpha,\gamma\in\Delta_1^+}\delta\tau^{\pm}(\lambda)D^{\pm}_{\alpha,\gamma}\wedge\delta\rho^{\pm}_{\gamma}, \qquad D^{\pm}_{\alpha,\gamma} = \left\langle \hat{D}^{\pm}E_{\mp\gamma}D^{\pm}(\lambda)E_{\pm\alpha}\right\rangle,\end{aligned}$$

- The classical R-matrix approach [Faddeev;Takhtajan;1986], [Fordy,Kulish;1983] is an effective method to determine the generating functionals of local integrals of motion which are in involution.

- From it there follows that such integrals are generated by expanding $\ln m_k^{\pm}(\lambda)$ over the inverse powers of λ [Gerdjikov;1987].

Here $m_k^{\pm}(\lambda)$ are the principal minors of $T(\lambda)$; in our case

$$m_1^+(\lambda) = a_{11}^+(\lambda), \qquad m_2^+(\lambda) = \det a^+(\lambda),$$

$$m_1^-(\lambda) = a_{22}^-(\lambda), \qquad m_2^-(\lambda) = \det a^-(\lambda).$$

– If we consider

$$\ln m_k^+(\lambda) = \sum_{s=1}^{\infty} \lambda^{-k} I_s^{(k)},$$

then one can prove that the densities of $I_s^{(k)}$ are local in $Q(x,t)$.

– The fact that [Gerdjikov;1987]:

$$\{m_k^{\pm}(\lambda), m_j^{\pm}(\mu)\} = 0, \qquad \text{for } k,j = 1,2,$$

and for all $\lambda, \mu \in \mathbb{C}_{\pm}$ allow one to conclude that $\{I_s^{(k)}, I_p^{(j)}\} = 0$ for all $k,j = 1,2$ and $s,p \geq 1$.

- In particular, the Hamiltonian of our model is proportional to $I_3^{(2)}$, i.e.

$$H = 8iI_3^{(2)}.$$

6. Conclusions

- A special version of the models describing $\mathcal{F}=1$ and $\mathcal{F}=2$ spinor Bose-Einstein condensates is integrable by the ISM. The corresponding Lax pair is on $\mathbf{BD.I.} \simeq \mathrm{SO(2r+1)/SO(2)\times SO(2r-1)}$ - symmetric space.

- For a generic hyperfine spin F, the dynamics within the mean field theory is described by the $2F+1$ component Gross-Pitaevskii equation in one dimension.

- If all the spin dependent interactions vanish and only intensity interaction exists, the multi-component Gross-Pitaevskii equation in one dimension is equivalent to the vector nonlinear Schrödinger equation with $2F+1$ components [S. V. Manakov;1974].

- One can also treat generalized Zakharov–Shabat systems related to other symmetric spaces.

- For all these systems of equations one can construct soliton solutions, prove completeness of 'squared solutions' etc.

- Another interesting and still open problem is the analysis of the soliton interactions in spinor Bose-Einstein condensates.

Thank you!

grah@inrne.bas.bg