Integrable discretizations of $Sp(m)$-invariant PDE's

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I plan to talk about integrable PDE's and their discrete analogues which are invariant under the action of the symplectic group $Sp(m)$ on the dependent variables. In contrast to the classes of $U(m)$-invariant systems (*e.g.* vector NLS) and $O(m)$-invariant systems (*e.g.* vector mKdV), only few studies have so far been made at $Sp(m)$-invariant systems. Typical examples in the space-discrete case include

$$\frac{\partial u_n^{(i)}}{\partial t} + \frac{u_{n+1}^{(i)} - u_n^{(i)}}{1 + \sum_{1\le j<k\le M} C_{jk}\bigl(u_{n+1}^{(j)}u_n^{(k)} - u_n^{(j)}u_{n+1}^{(k)}\bigr)} + \frac{u_n^{(i)} - u_{n-1}^{(i)}}{1 + \sum_{1\le j<k\le M} C_{jk}\bigl(u_n^{(j)}u_{n-1}^{(k)} - u_{n-1}^{(j)}u_n^{(k)}\bigr)} = 0, \quad i = 1,2,\dots,M,$$

$$\frac{\partial u_n^{(i)}}{\partial t} + \Delta_n \left[\frac{u_n^{(i)} + u_{n-1}^{(i)}}{1 - \sum_{1\le j<k\le M} C_{jk}\bigl(u_n^{(j)}u_{n-1}^{(k)} - u_{n-1}^{(j)}u_n^{(k)}\bigr)} \right] = 0, \quad i = 1,2,\dots,M,$$

$$\frac{\partial}{\partial \tau}(v_n^{(i)} - v_{n+1}^{(i)}) + v_n^{(i)} + v_{n+1}^{(i)} - 2\left[\sum_{1\le j<k\le M} C_{jk}(v_{n+1}^{(j)}v_n^{(k)} - v_n^{(j)}v_{n+1}^{(k)}) \right](v_n^{(i)} + v_{n+1}^{(i)}) = 0, \quad i = 1,2,\dots,M,$$

where Δ_n means the forward difference operator, $\Delta_n f_n(t) := f_{n+1}(t) - f_n(t)$. They are considered as integrable semi-discretizations of the following systems respectively:

$$\frac{\partial u_i}{\partial t} + \frac{\partial^3 u_i}{\partial x^3} + 3\left[\sum_{1\le j<k\le M} C_{jk}\Bigl(\frac{\partial u_j}{\partial x}u_k - u_j\frac{\partial u_k}{\partial x}\Bigr) \right]\frac{\partial u_i}{\partial x} = 0, \quad i = 1,2,\dots,M,$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial^3 u_i}{\partial x^3} + 3\frac{\partial}{\partial x}\left[\sum_{1\le j<k\le M} C_{jk}\Bigl(\frac{\partial u_j}{\partial x}u_k - u_j\frac{\partial u_k}{\partial x}\Bigr)u_i \right] = 0, \quad i = 1,2,\dots,M,$$

$$\frac{\partial^2 v_i}{\partial \tau \partial x} + v_i - \left[\sum_{1\le j<k\le M} C_{jk}\Bigl(\frac{\partial v_j}{\partial x}v_k - v_j\frac{\partial v_k}{\partial x}\Bigr) \right] v_i = 0, \quad i = 1,2,\dots,M.$$

In the canonical case of coupling constants, i.e. $C_{2j-1\,2k} = -C_{2k\,2j-1} = \delta_{jk}$, $C_{2j-1\,2k-1} = C_{2j\,2k} = 0$, $M = 2m$, all these systems are indeed invariant under the action of the symplectic group $Sp(m)$ in standard representation.