Approximate Lie symmetries of the Navier-Stokes equations

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Symmetries of differential equations are pivotal to a profound understanding of the physics of the underlying the problems under investigations. Symmetry group analysis of differential equations on the basis of Lie (Lie-Backlund) groups unify a wide variety of ad hoc methods to analyze and exactly solve differential equations. Detailed description of the methods and their applications to various problems of mathematical physics, fluid dynamics and others may be found e.g. in [1], [2]. In the context of Symmetry Group Methods an approach to derive certain turbulent scaling laws arising in the statistical theory of turbulence was given in [3]. In particular, it unifies a large set of scaling laws for the mean velocity of stationary parallel turbulent shear flows. The approach is derived from the Reynolds averaged Navier-Stokes equations, the fluctuations equations, and the velocity product equations, which are the dyad product of the velocity fluctuations with the equations for the velocity fluctuations. From the knowledge of the symmetries a broad variety of invariant solutions (scaling laws) were derived but these invariant solutions are fixed by using the symmetries of the Euler equations. From a physical point of view as the viscosity tends to zero the turbulence becomes highly intermittent, and the vorticity is concentrated on sets of a small measure. The use of symmetries of the Navier-Stokes equations do not enable us to introduce the Reynolds number dependence into scaling laws. The crucial point for understanding of Reynolds umber dependence is that viscosity is primarily significant for small scale turbulence at the order of the Kolmogorov length scale and, if wall bounded flows are considered, in the inner region (viscous sublayer) of a turbulent motion. The so-called outer region of this motion is mainly determined by the Euler equations. According to Kolmogorov's sub-range theory there is a region in correlation space obeying the limits where viscosity is negligible and large-scale influence are also asymptotically small. The eddies have a negligible amount of energy but provide the necessary dissipation for the energy balance equation. In contrast the energy containing large scale eddies determine the mean velocity, the Reynolds stress tensor and similar variables. It is this distinction and the corresponding difference in symmetries which is the basis for the understanding of the invariant solutions of turbulent flows. Barenblatt and Chorin in a series of papers [4], [5] investigate of the influence of the intermittency phenomenon on certain scaling laws presented by the von Karman-Prandtl universal logarithmic law of the wall (in the intermediate region of wall-bounded turbulence), and the Kolmogorov-Obukhov scaling for the local structure of turbulence. It was shown that when the viscosity is small the universal logarithmic law for the intermediate region of wall-bounded shear flow must be replaced by a power law. The concept of the so-called incomplete similarity and intermediate asymptotics was used to make a correction of the classical scaling laws when the Reynolds number is finite but large. The analysis extended the classical form of dependency between the velocity gradient and the spatial coordinate, the shear stress at the wall, the pipe diameter, the kinematic viscosity and density of a flow without using the Navier-Stokes equations directly.

Our aim is to find symmetries which corresponds to the Reynolds number dependence of a turbulent motion. As the first step we apply the theory of approximate symmetries developed by Fushchich and Shtelen [6], Euler et al. [7], Ibragimov, Bykov [8] and Gazizov [9] for studying differential equations with a small parameter and consider the Navier-Stokes equations as a perturbation of the Euler equations. We calculate the so-called approximate Lie symmetry tangent vector field to the manifold defined by the Navier-Stokes equations which is motivated by their application to the theory of turbulence. In particular, we show that the Lie symmetries of the Euler equations are inherited by the Navier-Stokes equations in the form of approximate symmetries. In the framework of the theory of approximate transformation groups proposed by Baikov, Gaziziv and Ibragimov, the first-order approximate symmetry operator is calculated for the Navier-Stokes equations. The symmetries of the coupled system obtained by expanding the dependent variables of the Navier-Stokes equations in the perturbation series with respect to a small parameter (viscosity) are used to derive approximate symmetries in the sense by Baikov et al.

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