Analytic functions in commutative algebras snd some problems of mathematical physics

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Yurii I. Samojlenko
(1932 – 2008)

The talk is devoted to the memory of Yurii Ivanovich Samojlenko.

Studying tidal deceleration models in systems of cosmic objects, Yu.I. Samojlenko (2006) formulated a boundary value problem the functions $u(x,y)$ and $v(x,y)$ satisfying the following system of equations:

$$\Delta^2 u(x,y) - \frac{\partial v(x,y)}{\partial x} = 0, \qquad \Delta v(x,y) + \frac{\partial u(x,y)}{\partial x} = 0,$$

where $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the two-dimensional Laplace operator. It follows from this system that the functions $u(x,y)$ and $v(x,y)$ satisfy the following equation:

$$\Delta^3 u(x,y) + \frac{\partial^2 u(x,y)}{\partial x^2} = 0.$$

An important achievement of mathematics is a description of plane potential fields by means analytic functions of complex variable:

- a potential $u(x,y)$ and a flow function $v(x,y)$ of plane stationary potential solenoid field satisfy the Cauchy–Riemann conditions

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y}, \qquad \frac{\partial u(x,y)}{\partial y} = -\frac{\partial v(x,y)}{\partial x},$$

  and they form the complex potential $F(x+iy) = u(x,y) + iv(x,y)$ being an analytic function of complex variable $x+iy$.

- In turn, every analytic function $F(x+iy)$ satisfies two-dimensional Laplace equation

$$\Delta_2 F := \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} \equiv F''(x+iy)\left(1^2 + i^2\right) = 0$$

  owing to the equality $1^2 + i^2 = 0$ for the unit 1 and the imaginary unit i of the algebra of complex numbers.

Analytic function methods in $\mathbb{C}$ for plane potential fields inspire searching analogous methods for spatial potential solenoid fields.

William Rowan Hamilton
(1806 – 1865)

Apparently, W. Hamilton (1843) made the first attempts to construct an algebra associated with the three-dimensional Laplace equation

$$\Delta_3 u(x,y,z) := \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u(x,y,z) = 0 \tag{1}$$

in that sense that components of hypercomlex functions satisfy Eq. (1). However, the Hamilton's quaternions form a noncommutative algebra.

Commutative harmonic algebras

Let $\mathbb{A}$ be a commutative associative Banach algebra of a rank n ($3 \le n \le \infty$) over the field $\mathbb{R}$ or the field $\mathbb{C}$.

$\{e_1, e_2, e_3\}$ — a part of the basis of $\mathbb{A}$.

Denote $\zeta := xe_1 + ye_2 + ze_3$, where $x, y, z \in \mathbb{R}$.

P.W. Ketchum (1928) discovered: if e_1, e_2, e_3 satisfy the condition

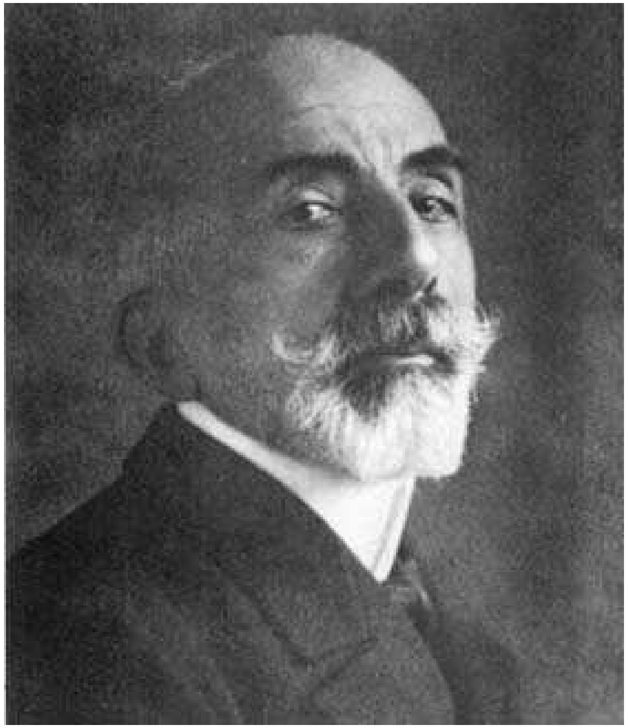
$$e_1^2 + e_2^2 + e_3^2 = 0\,, \tag{2}$$

then for every analytic function $\Phi(\zeta)$:

$$\Delta_3\Phi(\zeta) \equiv \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} \equiv \Phi''(\zeta)\,(e_1^2 + e_2^2 + e_3^2) = 0\,.$$

Definitions. *We say that an algebra $\mathbb{A}$ is harmonic if in $\mathbb{A}$ there exists a triad of linearly independent vectors $\{e_1, e_2, e_3\}$ satisfying the equality (2) provided that $e_k^2 \ne 0$ for $k = 1, 2, 3$. We say also that such a triad $\{e_1, e_2, e_3\}$ is harmonic.*

Segre algebra of quaternions



Corrado Segre
(1863 – 1924)

Apparently, the first harmonic algebra was constructed by C. Segre (1892). Indeed, in the Segre algebra of commutative quaternions the multiplication table for the basis $\{1, i, j, k\}$ is of the form:

$$i^2 = j^2 = -1\,, \quad k^2 = 1\,, \quad i\,j = k\,, \quad i\,k = -j\,, \quad j\,k = -i\,,$$

and there are harmonic triads, in particular: $e_1 = \sqrt{2}, e_2 = i, e_3 = j$ for which $e_1^2 + e_2^2 + e_3^2 = 0$.

- P.W. Ketchum (1928) considered the Segre algebra of quaternions in its relations with the three-dimensional Laplace equation: that means the satisfiability of the equalities

$$(3)\Delta_3\Phi(\zeta) \equiv \Phi''(\zeta)\,(e_1^2 + e_2^2 + e_3^2) = 0\,, \quad \zeta = xe_1 + ye_2 + ze_3\,,$$

owing to the equality $e_1^2 + e_2^2 + e_3^2 = 0$.

- I. Mel'nichenko (1975) noticed that doubly differentiable in the sense of Gateaux functions form the largest class of functions $\Phi(\zeta)$ satisfying the identity (3).

Differentiability in the sense of Gateaux

Denote $E_3 := \{\zeta := xe_1 + ye_2 + ze_3 \; : \; x, y, z \in \mathbb{R}\}$.

Definition. *A function $\Phi : Q \longrightarrow \mathbb{A}$ is differentiable in the sense of Gateaux in every point of domain $Q \subset E_3$ if for every $\zeta \in Q$ there exists an element $\Phi'(\zeta) \in \mathbb{A}$ such that*

$$\lim_{\delta \to 0+0} \left[\Phi(\zeta + \delta h) - \Phi(\zeta)\right] \delta^{-1} = \Phi'(\zeta) h \qquad \forall\, h \in E_3\,.$$

So, if the elements e_1, e_2, e_3 satisfy the condition $e_1^2 + e_2^2 + e_3^2 = 0$, then every doubly differentiable in the sense of Gateaux function $\Phi : Q \longrightarrow \mathbb{A}$ satisfies in Q the equality

$$\Delta_3 \Phi(\zeta) \equiv \Phi''(\zeta)\,(e_1^2 + e_2^2 + e_3^2) = 0\,, \qquad \zeta = xe_1 + ye_2 + ze_3\,.$$

In turn, if there exists a doubly differentiable in the sense of Gateaux function $\Phi : Q \longrightarrow \mathbb{A}$ satisfying the equation $\Delta_3\, \Phi = 0$ and the inequality $\Phi''(\zeta) \neq 0$ at least at one point $\zeta := xe_1 + ye_2 + ze_3 \in Q$, then in this case $e_1^2 + e_2^2 + e_3^2 = 0$.

Igor Mel'nichenko
(1938 – 2004)

I. Mel'nichenko (1975) developed an algebraic-analytic approach to equations of mathematical physics.

- An idea of such an approach means a finding of commutative associative Banach algebra such that differentiable in the sense of Gateaux functions with values in this algebra have components satisfying the given equations with partial derivatives.

Such algebras are constructed for the three-dimensional Laplace equation (I. Mel'nichenko (1975, 2003)) and the two-dimensional biharmonic equation (V. Kovalev and I. Mel'nichenko (1981), I. Mel'nichenko (1986) — $(e_1^2 + e_2^2)^2 = 0, \quad e_1^2 + e_2^2 \neq 0$) and some other equations.

- Inasmuch as differentiable in the sense of Gateaux functions taking values in a commutative associative Banach algebra form a functional algebra, note that a relation between these functions and solutions of equations with partial derivatives facilitates effective constructing of mentioned solutions.

It is quite natural that on such a way a quantity of fulfilled operations will be minimal in an algebra of minimal rank.

Three-dimensional harmonic algebras

Consider the problem on finding a three-dimensional harmonic algebra $\mathbb{A}$ with the unit 1, in which there exists a harmonic basis $\{e_1, e_2, e_3\}$ satisfying the following conditions

$$e_1^2 + e_2^2 + e_3^2 = 0\,, \qquad e_k^2 \neq 0 \ \text{ for } \ k = 1, 2, 3.$$

This problem was solved completely by I. Mel'nichenko (1975, 2003).

Theorem 1 (I. Mel'nichenko, 1975). *There does not exist a harmonic algebra of third rank with the unit over the field* $\mathbb{R}$.

I. Mel'nichenko (1975) constructed a three-dimensional harmonic algebra over the field $\mathbb{C}$.

Three-dimensional harmonic algebras

I. Mel'nichenko (2003) found all three-dimensional harmonic algebras (there are three exactly):

1) an algebra with 3 maximal ideals;
2) an algebra with 2 maximal ideals;
3) an algebra with 1 maximal ideals.

I. Mel'nichenko developed a method for finding all harmonic bases in these algebras.

Harmonic algebra $\mathbb{A}_3$

Let $\mathbb{A}_3$ be a three-dimensional commutative associative Banach algebra over the field $\mathbb{C}$ with the basis $\{1, \rho_1, \rho_2\}$ and the multiplication table: $\rho_1\rho_2 = \rho_2^2 = 0, \quad \rho_1^2 = \rho_2.$

$\mathbb{A}_3$ have the unique maximal ideal $I := \{\lambda_1\rho_1 + \lambda_2\rho_2 : \lambda_1, \lambda_2 \in \mathbb{C}\}$.

In $\mathbb{A}_3$ there exists a harmonic basis $\{e_1, e_2, e_3\}$:

$$e_1 = 1, \quad e_2 = i + 2i\rho_2, \quad e_3 = 2\rho_1$$

satisfying the conditions $e_1^2 + e_2^2 + e_3^2 = 0, \quad e_k^2 \neq 0$ for $k = 1, 2, 3$.
Doing calculations, one can easy change the harmonic basis $\{e_1, e_2, e_3\}$ into the basis $\{1, \rho_1, \rho_2\}$ and back:

$$1 = e_1, \quad \rho_1 = \frac{1}{2}e_3, \quad \rho_2 = -\frac{1}{2}e_1 - \frac{1}{2}ie_2.$$

Harmonic algebra $\mathbb{A}_3$

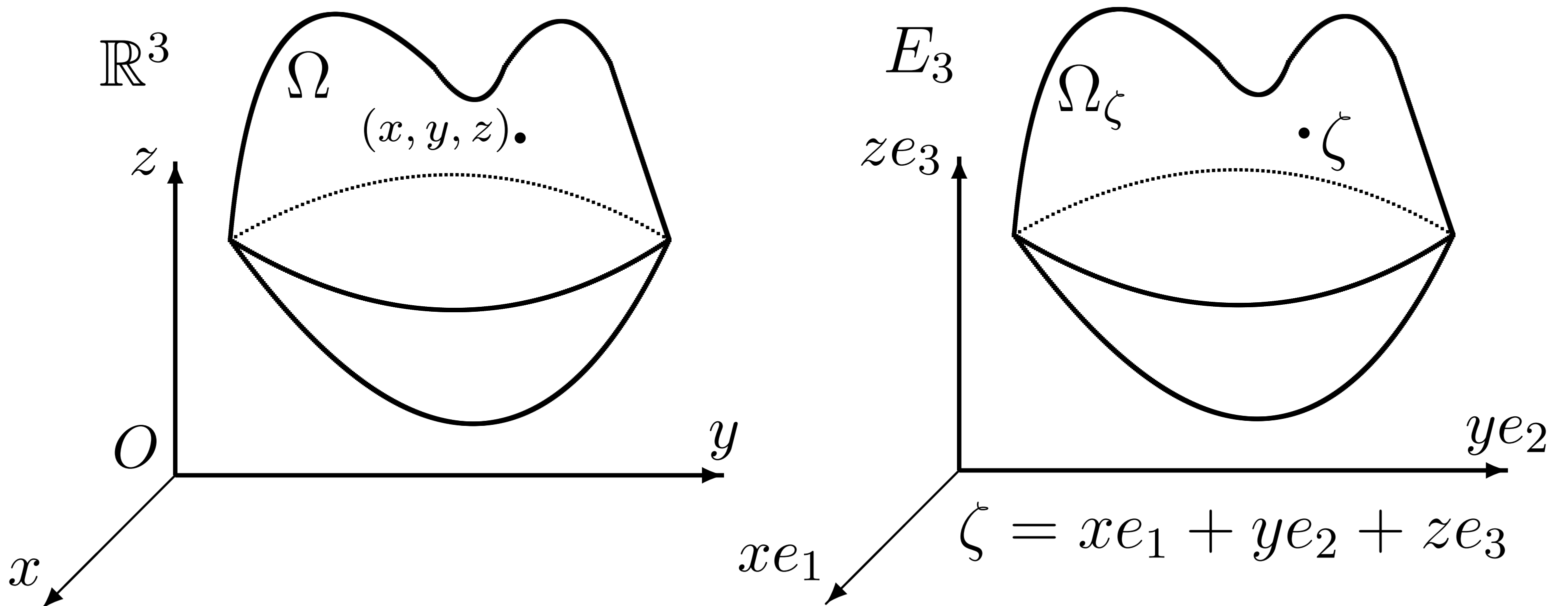
$\{e_1, e_2, e_3\}$ — a harmonic basis in the algebra $\mathbb{A}_3$:

$$e_1 = 1, \quad e_2 = i + 2i\rho_2, \quad e_3 = 2\rho_1\,.$$

$E_3 := \{\zeta = xe_1 + ye_2 + ze_3 \;:\; x, y, z \in \mathbb{R}\}$ — the linear space generated by the vectors e_1, e_2, e_3.

For a domain $\Omega \subset \mathbb{R}^3$ consider the domain

$\Omega_\zeta := \{\zeta = xe_1 + ye_2 + ze_3 : (x, y, z) \in \Omega\} \subset E_3$ congruent to Ω.

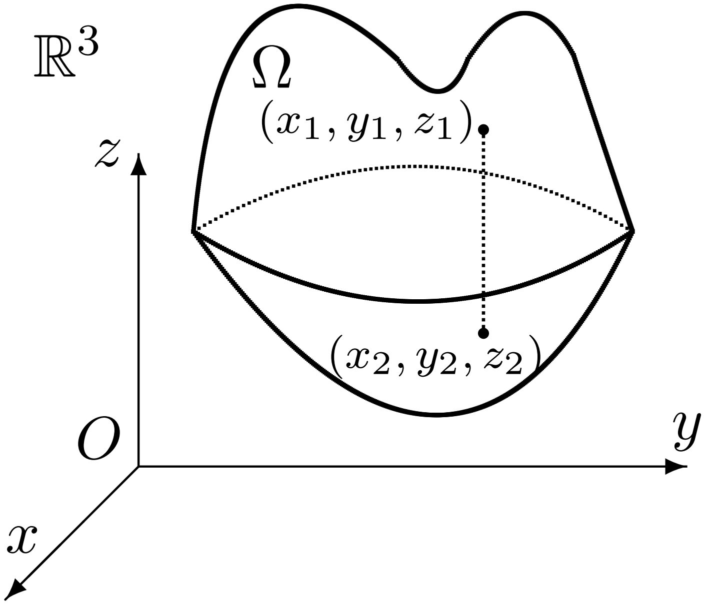


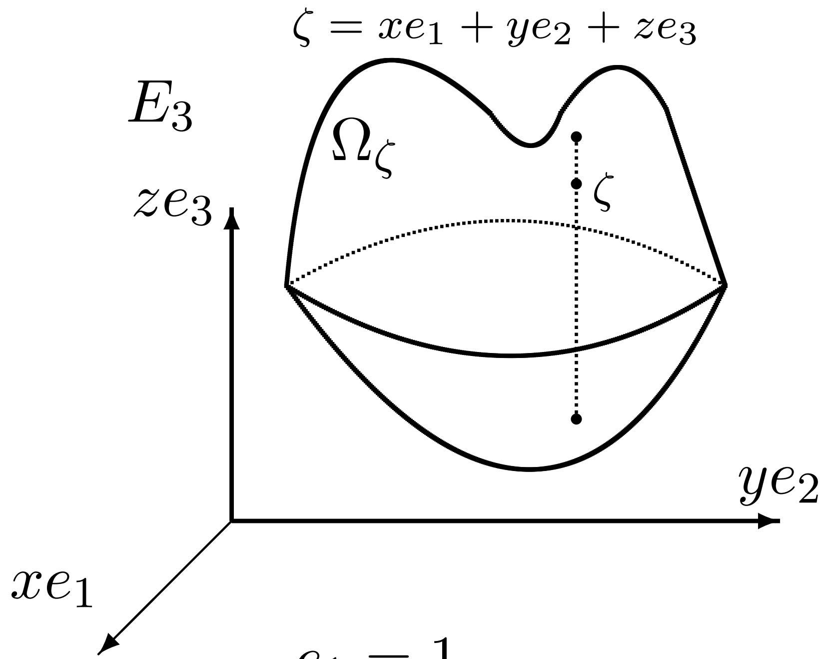
Monogenic functions

Definition. *We say that a locally bounded function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ is monogenic in Ω_ζ if Φ is differentiable in the sense of Gateaux in every point of Ω_ζ.*

Necessary and sufficient conditions (Cauchy – Riemann conditions) for monogenety of function $\Phi(\zeta)$ of the variable $\zeta = xe_1 + ye_2 + ze_3$ are of the form (I. Mel'nichenko, 2003):

$$\frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial x}\, e_2, \qquad \frac{\partial \Phi}{\partial z} = \frac{\partial \Phi}{\partial x}\, e_3.$$





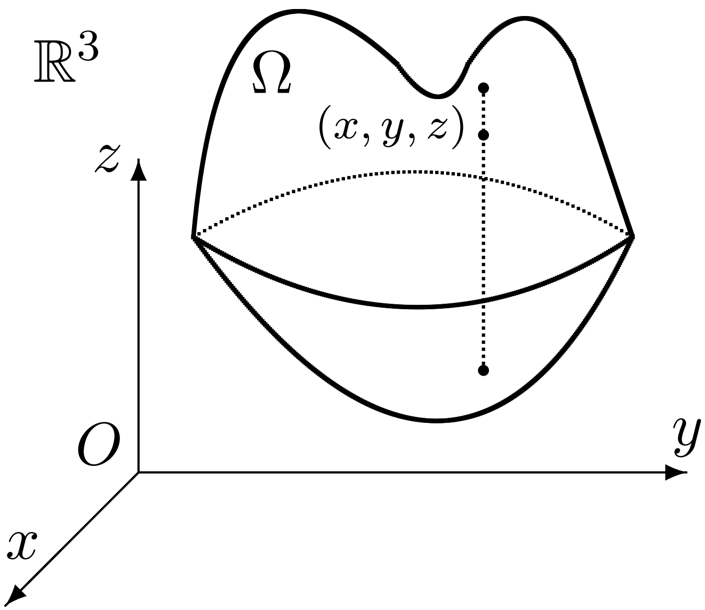
$$e_1 = 1$$
$$e_2 = i + 2i\rho_2$$
$$e_3 = 2\rho_1$$

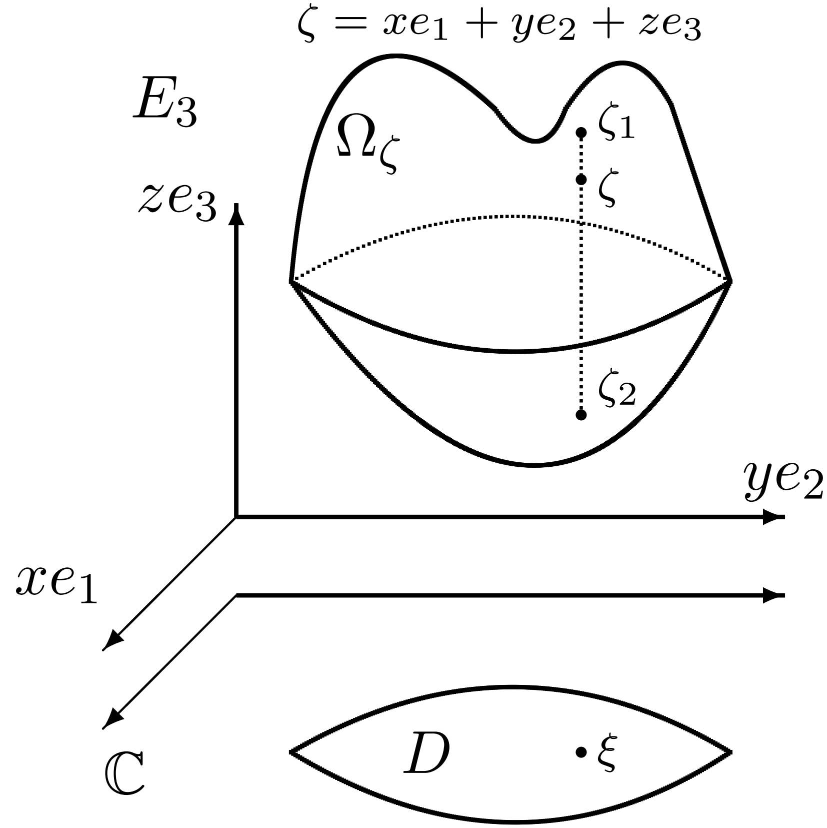
$\zeta = xe_1 + ye_2 + ze_3 \in E_3$ is noninvertible in $\mathbb{A}_3$ if and only if the point (x, y, z) is located on the axis Oz

$$I := \{\lambda_1\rho_1 + \lambda_2\rho_2 : \lambda_1, \lambda_2 \in \mathbb{C}\}$$

(the straight line $\{ze_3 : z \in \mathbb{R}\}$ is contained in the maximal ideal I).

- Let the domain $\Omega \subset \mathbb{R}^3$ be convex in the direction of the axis Oz (i.e. Ω contains every segment parallel to the axis Oz and connecting two points $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \Omega$).





$f \; : \; \mathbb{A}_3 \longrightarrow \mathbb{C}$ is the linear functional such that $f(e_1) = 1$, $f(e_2) = i$, $f(e_3) = 0$.

$\Phi : \Omega_\zeta \longrightarrow \mathbb{A}_3$ is monogenic.

$A : \Phi(\zeta) \longmapsto F(\xi) = f(\Phi(\zeta))$,

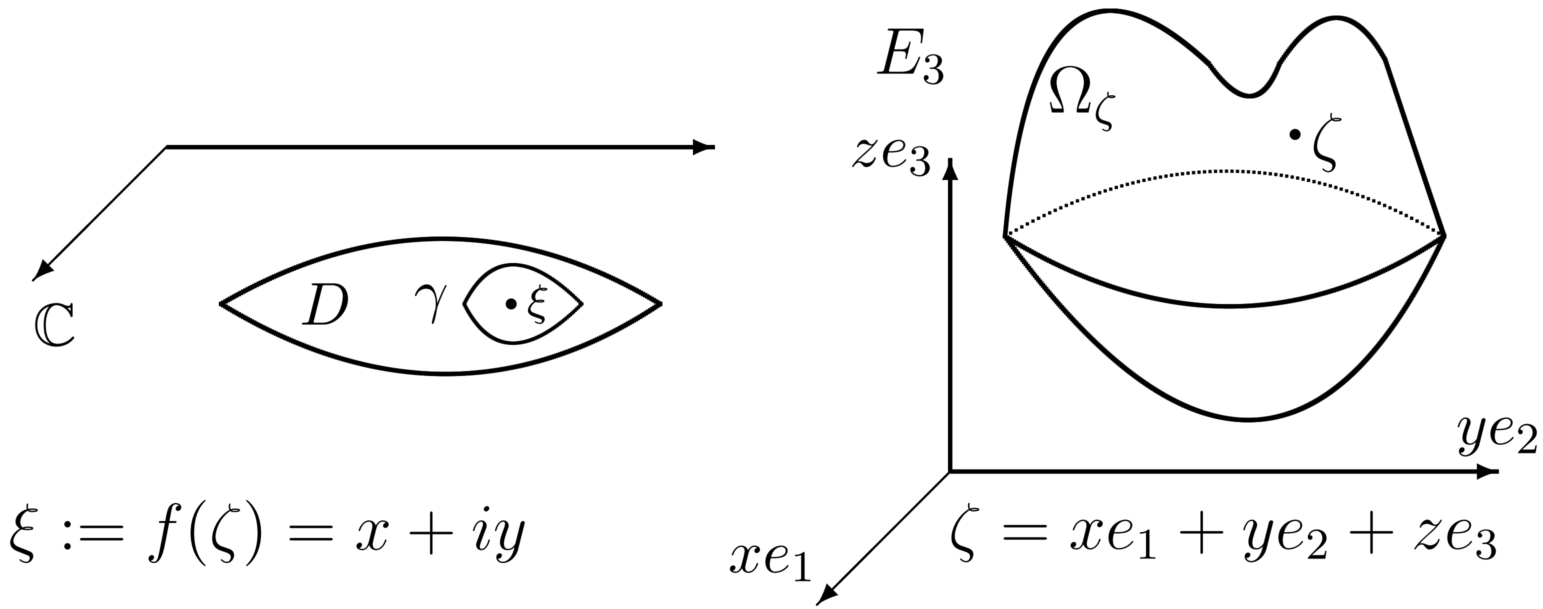
$$\xi := f(\zeta) = x + iy$$

$$\xi = f(\zeta) = f(\zeta_1) = f(\zeta_2)$$

where $\zeta = xe_1 + ye_2 + ze_3$ and $\xi := f(\zeta) = x + iy$ (we proved that the value $F(\xi)$ does not depend of a choice of a point ζ, for which $f(\zeta) = \xi$).

Φ is a monogenic function in $\Omega_\zeta \Longrightarrow F$ is an analytic function in the domain $D := f(\Omega_\zeta) \subset \mathbb{C}$.

Monogenic functions



Theorem 5 (S. Plaksa and V. Shpakivskyi, 2009). *Every monogenic function* Φ *in* Ω_ζ *can be expressed in the form*

$$\Phi(\zeta) = \frac{1}{2\pi i}\int\limits_{\gamma}(te_1 - \zeta)^{-1}F(t)dt + \Phi_0(\zeta), \qquad \zeta \in \Omega_\zeta, \;\; \Phi_0(\zeta) \in I.$$

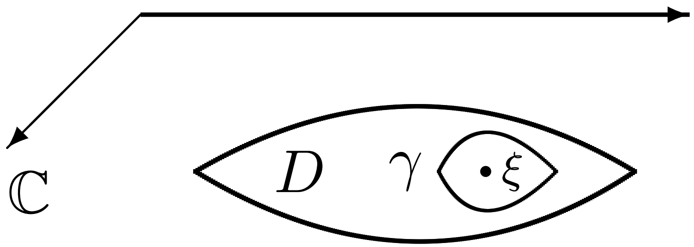
Here γ *is an arbitrary closed Jordan rectifiable curve in* D *that embraces the point* $f(\zeta)$ (the spectrum of ζ).

Monogenic functions

$$\Phi(\zeta) = \frac{1}{2\pi i} \int\limits_{\gamma} (te_1 - \zeta)^{-1} F(t) dt + \Phi_0(\zeta), \qquad \zeta \in \Omega_\zeta, \;\; \Phi_0(\zeta) \in I.$$

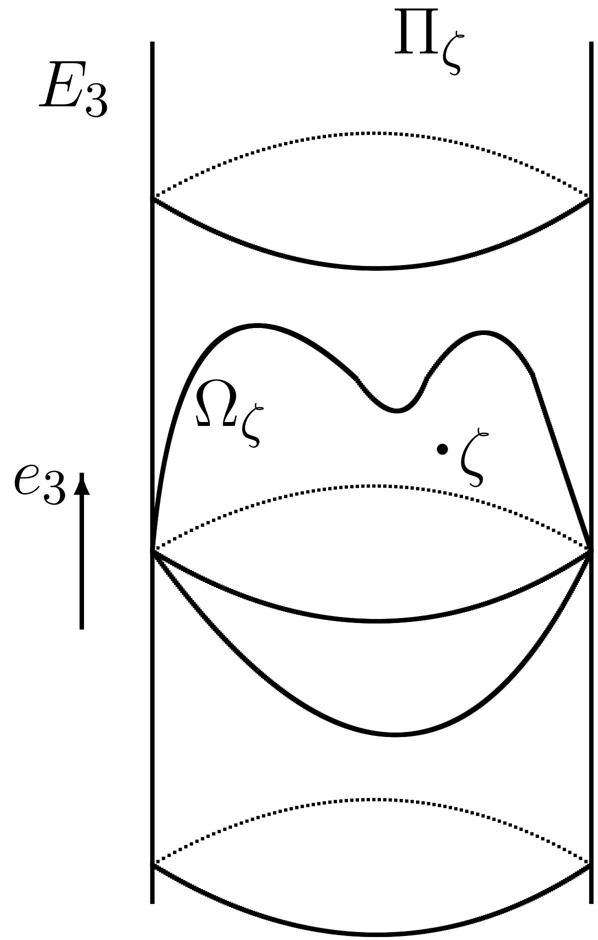
Thus, the algebra of monogenic functions is decomposed into the direct sum of the algebra of principal extensions of analytic functions and the algebra of monogenic functions taking values in the maximal ideal I.

The principal extension of analytic function $F : D \to \mathbb{C}$ into the infinite cylinder $\Pi_\zeta := \{\zeta \in E_3 : f(\zeta) \in D\}$ was constructed explicitly (I. Mel'nichenko and S. Plaksa, 2008).



$$\xi := f(\zeta) = x + iy$$

Π_ζ is an infinite cylinder.



$$\frac{1}{2\pi i}\int\limits_{\gamma} F(t)(t-\zeta)^{-1}dt = F(\xi) + 2zF'(\xi)\rho_1 + \Big(2iyF'(\xi) + 2z^2F''(\xi)\Big)\rho_2$$

$$\forall\,\zeta = xe_1 + ye_2 + ze_3 \in \Pi_\zeta, \quad \text{where} \quad \xi := x + iy\,.$$

Monogenic functions

$$\frac{1}{2\pi i}\int\limits_{\gamma} F(t)(t-\zeta)^{-1}dt = F(\xi) + 2zF'(\xi)\rho_1 + \Big(2iyF'(\xi) + 2z^2F''(\xi)\Big)\rho_2$$

$$\forall\,\zeta = xe_1 + ye_2 + ze_3 \in \Pi_\zeta, \ \ \text{where} \ \ \xi := x + iy\,.$$

We described all monogenic functions taking values in the maximal ideal I.

Theorem 6 (S. Plaksa and V. Shpakivskyi, 2009). *Every monogenic function* $\Phi_0 : \Omega_\zeta \to I$ *can be expressed in the form*

$$\Phi_0(\zeta) = F_1(\xi)\,\rho_1 + \Big(F_2(\xi) + 2zF_1'(\xi)\Big)\rho_2$$

$$\forall\zeta = xe_1 + ye_2 + ze_3 \in \Omega_\zeta\,,$$

where F_1, F_2 *are analytic in* D *and* $\xi = x + iy$.

Monogenic functions

Thus, we obtain the following representation of monogenic function $\Phi(\zeta)$ for all $\zeta = xe_1 + ye_2 + ze_3 \in \Omega_\zeta$:

$$\Phi(\zeta) = F(x+iy) + \Big(F_1(x+iy) + 2zF'(x+iy)\Big)\rho_1 +$$

$$+\Big(F_2(x+iy) + 2zF_1'(x+iy) + 2iyF'(x+iy) + 2z^2F''(x+iy)\Big)\rho_2.$$

Using this representation, one can construct all monogenic functions $\Phi : \Omega_\zeta \to \mathbb{A}_3$ by means arithmetic operations with arbitrary complex-valued analytic functions F, F_1, F_2 given in the domain $D \subset \mathbb{C}$.

We can write the following integral representation of monogenic function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ (Ω is convex in the direction of the axis Oz):

$$(5) \qquad \Phi(\zeta) = \frac{1}{2\pi i} \int\limits_{\gamma} \Big(F(t) + \rho_1 F_1(t) + \rho_2 F_2(t)\Big)(t-\zeta)^{-1} dt\,,$$

where the curve γ is the same as in the principal extension of F.

Using the integral representation (5), we obtain the following expression for the Gateaux n-th derivative $\Phi^{(n)}$:

$$\Phi^{(n)}(\zeta) = \frac{n!}{2\pi i} \int\limits_{\gamma} \Big(F(t) + \rho_1 F_1(t) + \rho_2 F_2(t)\Big)\Big((t-\zeta)^{-1}\Big)^{n+1} dt \ \ \forall \zeta \in \Omega_\zeta\,.$$

The following statement is true for monogenic functions in an arbitrary domain Ω_ζ.

Theorem 8 (S. Plaksa and V. Shpakivskyi, 2009). *For every monogenic function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ in an arbitrary domain Ω_ζ, the Gateaux n-th derivatives $\Phi^{(n)}$ are monogenic in Ω_ζ for any n. Every monogenic function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ is differentiable in the sense of Lorch.*

Differentiable functions

Comparison. *A function* $\Phi : \Omega_\zeta \to \mathbb{A}_3$ (*where* $\Omega_\zeta \subset E_3$) is:

- *differentiable in the sense of Lorch if for every* $\zeta \in \Omega_\zeta$ *there exists* $\Phi'(\zeta) \in \mathbb{A}_3$ *such that for every* $\varepsilon > 0$ *there exists* $\delta(\varepsilon) > 0$ *such that for all* w *with* $0 < \|w\| < \delta(\varepsilon)$:

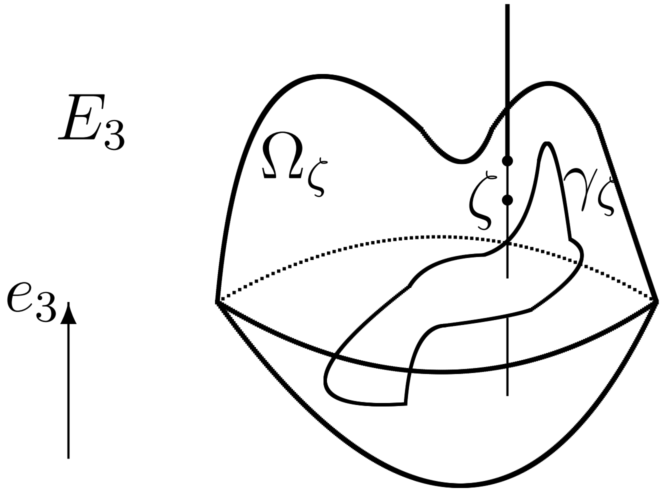
$$\left\| \frac{\Phi(\zeta + w) - \Phi(\zeta)}{\|w\|} - \frac{w}{\|w\|}\Phi'(\zeta) \right\| \le \varepsilon\,;$$

- *differentiable in the sense of Gateaux if for every* $\zeta \in \Omega_\zeta$ *there exists* $\Phi'(\zeta) \in \mathbb{A}_3$ *such that for every* $\varepsilon > 0$ *and for every* $h \in E_3$ *there exists* $\delta(\varepsilon, h) > 0$ *such that for all* $\delta \in \Big(0, \delta(\varepsilon, h)\Big)$:

$$\left\| \frac{\Phi(\zeta + \delta h) - \Phi(\zeta)}{\delta} - h\Phi'(\zeta) \right\| \le \varepsilon\,.$$

Φ is differentiable in the sense of Lorch $\Longrightarrow$ Φ is differentiable in the sense of Gateaux.

Cauchy integral formula



Theorem 9 (S. Plaksa and V. Shpakivskyi, 2009). *Let Ω be a domain convex in the direction of the axis Oz, and a function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ is monogenic in Ω_ζ. Then for every interior point $\zeta \in \Omega_\zeta$ the following formula is true:*

$$\Phi(\zeta) = \frac{1}{2\pi i} \int\limits_{\gamma_\zeta} \Phi(\tau)\,(\tau - \zeta)^{-1}\, d\tau,$$

where γ_ζ is an arbitrary closed Jordan rectifiable curve in Ω_ζ, which once around the straight line $\{\eta = \zeta + z e_3 : z \in \mathbb{R}\}$.

Equivalent definitions of monogenic functions

For monogenic functions $\Phi(\zeta)$ taking values in the algebra $\mathbb{A}_3$, we proved analogs of Cauchy theorems for surface integral and curvilinear integral. An analog of the Cauchy formula yields the Taylor expansion of monogenic function in the usual way. An analog of Morera theorem is also established.

Thus, as in the complex plane, for a continuous function $\Phi(\zeta)$, $\zeta = xe_1 + ye_2 + ze_3 \in \Omega_\zeta \subset E_3$, the statements are equivalent:

- Φ is a monogenic (differentiable in the sense of Gateaux) function;
- Cauchy – Riemann conditions $\dfrac{\partial\Phi}{\partial y} = \dfrac{\partial\Phi}{\partial x} e_2, \quad \dfrac{\partial\Phi}{\partial z} = \dfrac{\partial\Phi}{\partial x} e_3$ are satisfied;
- Φ is an analytic function (expanded into a power series);
- $\oint \Phi(\zeta)\, d\zeta = 0$.

The End