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Finite Group Actions on Quasi-Hamiltonian Spaces

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Abstract. We organize fundamental properties of quasi-Hamiltonian spaces on which a finite group acts, and we apply them to the theory of moduli spaces of flat connections on an oriented compact surface with boundary.

Key words: quasi-Hamiltonian geometry; moduli space of flat connections

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1 Introduction

Moduli spaces of flat connections on an oriented compact surface Σ with boundary have attracted much interest from both mathematicians and physicists. Atiyah and Bott discovered that such a space carries a Poisson structure obtained via the infinite-dimensional Hamiltonian reduction for the action of the gauge transformation group on the infinite-dimensional affine space consisting of connections on Σ [3]. Since then, several methods for constructing the Atiyah–Bott Poisson structure by finite-dimensional methods (without using the infinite-dimensional Hamiltonian reduction) have been discovered [2, 9, 10, 11].

In [2], Alekseev, Malkin, and Meinrenken developed the theory of quasi-Hamiltonian spaces which are multiplicative analogues of usual Hamiltonian spaces. In their theory, a moment map takes values in a compact Lie group G itself, not in the dual space $\mathfrak{g}^*$ of the Lie algebra of G. Moreover, using the framework of Dirac geometry, the theory has been extended to the case where G is not necessarily compact [1, 6]. Boalch also developed quasi-Hamiltonian geometry for complex reductive groups whose Lie algebras are equipped with non-degenerate invariant bilinear forms [4].

Let $\beta \subset \partial\Sigma$ be a set of base points with exactly one element in each connected component of $\partial\Sigma$. Then there exists a well-known correspondence between flat G-bundles on Σ framed at β and G-valued representations of the fundamental groupoid $\Pi_1(\Sigma,\beta)$ with base points β, $\operatorname{Hom}(\Pi_1(\Sigma,\beta),G)$. They showed that the representation space $\operatorname{Hom}(\Pi_1(\Sigma,\beta),G)$ becomes a quasi-Hamiltonian G^β-space where the group-valued moment map is the collection of monodromies around the boundary cycles of Σ, and that the Poisson structure on the moduli space is obtained via the quasi-Hamiltonian reduction.

In [5, 15], the notion of a quasi-Hamiltonian space was generalized to a twisted quasi-Hamiltonian space, so that a moment map could take values in a bitorsor. It was shown that the space of G-valued twisted representations of the fundamental groupoid (see Section 5.6) $\operatorname{Hom}_\kappa(\Pi_1(\Sigma,\beta),G)$ becomes a twisted quasi-Hamiltonian G^β-space. Many examples of twisted representation spaces and their twisted quasi-Hamiltonian structures are discussed in [17, 18].

In this paper, we show that twisted representation spaces associated with Σ and a set of base points containing several points on each boundary component also admit twisted quasi-Hamiltonian structures. To establish this, we introduce the notion of Γ-compatible quasi-Hamiltonian spaces and discuss some of their basic properties.

Now, we briefly state the main results of this paper. In Sections 2 and 3, we define Γ-compatible quasi-Hamiltonian G-spaces and study their basic properties, such as the fusion product of such spaces. Fix a group homomorphism $\Gamma \to \mathrm{Aut}(G)$. A Γ-compatible quasi-Hamiltonian G-space is a quasi-Hamiltonian G-space (M, ω, μ) where Γ acts on M, the quasi-Hamiltonian 2-form ω is Γ-invariant, and the group-valued moment map μ is Γ-equivariant. The following theorem is the main result of Sections 2 and 3.

Theorem 1.1 (Theorem 3.1)**.** *Let (M, ω, μ) be a Γ-compatible quasi-Hamiltonian G-space. Let M^Γ be the set of Γ-fixed points and G^Γ be the Lie subgroup of Γ-fixed points. Also we denote the identity component of G^Γ by H. Then, each connected component $N \subset M^\Gamma$ together with the 2-form $\omega|_N$ and the map $\mu|_N$, which takes values in an H-bitorsor $\mathbb{H} \subset \mathbb{G}$ such that $\mu(N) \subset \mathbb{H}$, is a twisted quasi-Hamiltonian H-space.*

This theorem allows us to obtain a new quasi-Hamiltonian space from a given Γ-compatible quasi-Hamiltonian space. In Sections 4 and 5, we apply the above results to framed moduli spaces of flat connections. In addition to the action of the finite group Γ on the compact Lie group G, we consider a free Γ-action on Σ and choose base points $\beta \subset \partial\Sigma$ that are invariant under the Γ-action on Σ and meet each connected component of $\partial\Sigma$. The following theorem is the main result of Sections 4 and 5.

Theorem 1.2 (Theorem 5.7)**.** *The representation space $\mathrm{Hom}(\Pi_1(\Sigma, \beta), G)$ has a structure of a Γ-compatible quasi-Hamiltonian G^β-space. Moreover, by taking the set of Γ-fixed points of $\mathrm{Hom}(\Pi_1(\Sigma, \beta), G)$, we obtain a structure of a twisted quasi-Hamiltonian $G^{\beta/\Gamma}$-space on a twisted representation space $\mathrm{Hom}_{\mathrm{mon}_{\Sigma/\Gamma,I}}(\Pi_1(\Sigma/\Gamma, \beta/\Gamma), G)$.*

While preparing this paper, we were aware of the work of Meinrenken [16], who showed that the space $\mathrm{Hom}(\Pi_1(\Sigma, \beta), G)$ has a quasi-Hamiltonian G^β-structure.

The paper is organized as follows. We give the definition of a Γ-compatible quasi-Hamiltonian G-space in Section 2. We discuss some of its basic properties in Section 3. In Sections 4 and 5, we apply the preceding results to moduli spaces of flat connections.

2 Finite group actions on quasi-Hamiltonian spaces

In this section, we give the definition of a finite group Γ-compatible q-Hamiltonian G-space and present a basic example that is an analogue of the double.

2.1 Compatibilities of bitorsors

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$ and let Γ be a finite group. We fix an action $\Gamma \to \mathrm{Aut}(G)$ of Γ on G. Also we give a G-bitorsor $\mathbb{G}$ equipped with a left Γ-action where a bitorsor $\mathbb{G}$ is a manifold endowed with left and right G-actions that are both simply transitive and commute with each other.

Definition 2.1. A Γ-compatible G-bitorsor $\mathbb{G}$ is a bitorsor $\mathbb{G}$ equipped with a Γ-action such that $\varphi \cdot (gx) = (\varphi \cdot g)(\varphi \cdot x)$, $\varphi \cdot (xg) = (\varphi \cdot x)(\varphi \cdot g)$, $(\varphi \in \Gamma,\ g \in G,\ x \in \mathbb{G})$.

Example 2.2. We define the space $\mathbb{G} := \mathrm{Map}(\mathbb{Z}/m\mathbb{Z}, G)$. Fixing an action $\varphi \colon \mathbb{Z}/m\mathbb{Z} \to \mathrm{Aut}(G)$, we define a $\mathbb{Z}/m\mathbb{Z}$-action on $\mathbb{G}$ by $(l \cdot g)(k) := \varphi(l)g(k-l)$, $l, k \in \mathbb{Z}/m\mathbb{Z}$, $g \in \mathbb{G}$. Then the space $\mathbb{G}$ is a $\mathbb{Z}/m\mathbb{Z}$-compatible $\mathrm{Map}(\mathbb{Z}/m\mathbb{Z}, G)$-bitorsor.

Remark 2.3. Let $\mathbb{G}$ be a Γ-compatible G-bitorsor. Define left and right actions of $G \rtimes \Gamma$ on $\mathbb{G} \times \Gamma$ by $(g, \varphi)(x, \psi) := (g(\varphi \cdot x), \varphi\psi)$, $(x, \psi)(g, \varphi) := (x(\psi \cdot g), \psi\varphi)$, where $(g, \varphi) \in G \rtimes \Gamma$ and $(x, \psi) \in \mathbb{G} \times \Gamma$. Then these actions make $\mathbb{G} \times \Gamma$ into a $G \rtimes \Gamma$-bitorsor. We denote this bitorsor by $\mathbb{G} \rtimes \Gamma$.

We define the inverse $\mathbb{G}^{-1}$ of a G-bitorsor $\mathbb{G}$ by the space $\mathbb{G}^{-1} := \{x^{-1} \mid x \in \mathbb{G}\}$, which is a copy of $\mathbb{G}$, equipped with the following G-actions, $gx^{-1} := (xg^{-1})^{-1}$, $x^{-1}g := (g^{-1}x)^{-1}$ for $g \in G$ and $x \in \mathbb{G}$.

Moreover, we define the product $\mathbb{G}_1 \cdot \mathbb{G}_2$ of two G-bitorsors $\mathbb{G}_1$ and $\mathbb{G}_2$ as the quotient of $\mathbb{G}_1 \times \mathbb{G}_2$ by the following action, $g(x_1, x_2) := (x_1 g^{-1}, gx_2)$ for $g \in G$ and $(x_1, x_2) \in \mathbb{G}_1 \times \mathbb{G}_2$. We denote an element $[x_1, x_2] \in \mathbb{G}_1 \cdot \mathbb{G}_2$ by $x_1 \cdot x_2$.

Proposition 2.4. *Let $\mathbb{G}$, $\mathbb{G}_1$, $\mathbb{G}_2$ be Γ-compatible G-bitorsors. Suppose that the fixed point sets $\mathbb{G}^\Gamma$, $\mathbb{G}_1^\Gamma$, $\mathbb{G}_2^\Gamma$ are non-empty. Then we have the following properties:*

(1) *The fixed point set $\mathbb{G}^\Gamma$ becomes a G^Γ-bitorsor under the restriction of the action of G on $\mathbb{G}$.*

(2) *The inverse $\mathbb{G}^{-1}$ of the bitorsor $\mathbb{G}$ and the product $\mathbb{G}_1 \cdot \mathbb{G}_2$ of $\mathbb{G}_1$ and $\mathbb{G}_2$ become Γ-compatible G-bitorsors.*

(3) *There exists an isomorphism of G^Γ-bitorsor $\mathbb{G}_1^\Gamma \cdot \mathbb{G}_2^\Gamma \simeq (\mathbb{G}_1 \cdot \mathbb{G}_2)^\Gamma$.*

Proof. Statement (1) is easy to verify. We define actions of Γ on $\mathbb{G}_1^{-1}$ and $\mathbb{G}_1 \cdot \mathbb{G}_2$ by $\varphi x^{-1} := (\varphi x)^{-1}$, $\varphi(x_1 \cdot x_2) := (\varphi x_1) \cdot (\varphi x_2)$, for $\varphi \in \Gamma$ and $x_j \in \mathbb{G}_j$. These actions satisfy the definition of Γ-compatibility.

To show statement (3), we see that the canonical map $\Phi \colon \mathbb{G}_1^\Gamma \cdot \mathbb{G}_2^\Gamma \to (\mathbb{G}_1 \cdot \mathbb{G}_2)^\Gamma$, $x_1 \cdot x_2 \mapsto x_1 \cdot x_2$ is equivariant with respect to the left and right G^Γ-actions. For any $g \in G^\Gamma$, $x_1 \cdot x_2 \in \mathbb{G}_1^\Gamma \cdot \mathbb{G}_2^\Gamma$,

$$\Phi(g(x_1 \cdot x_2)) = \Phi(gx_1 \cdot x_2) = gx_1 \cdot x_2 = g(x_1 \cdot x_2) = g\Phi(x_1 \cdot x_2).$$

The equivariance of Φ with respect to the right G^Γ-action is verified in the same way as above. Hence, by the simple transitivity of the left and right actions on bitorsors, Φ is an isomorphism. ■

2.2 Compatibilities of q-Hamiltonian G-spaces

First, we recall the definition of twisted quasi-Hamiltonian G-spaces [5, 15]. Throughout this paper, we fix an Ad-invariant positive definite inner product $(\ ,\ )$ on $\mathfrak{g}$ and consider only bitorsors whose corresponding elements in $\mathrm{Out}(G)$ preserve this inner product. Also we assume that the Γ-action on $\mathfrak{g}$ induced by the action on G preserves the inner product on $\mathfrak{g}$.

We define the left Maurer–Cartan form θ and the right Maurer–Cartan form $\overline{\theta}$ on a G-bitorsor $\mathbb{G}$ by $\theta_x(X) := x^{-1}X$, $\overline{\theta}_x(X) := Xx^{-1}$, $x \in \mathbb{G}$, $X \in T_x\mathbb{G}$, where $x^{-1}X \in \mathfrak{g}$ denotes the derivative of the map $x^{-1} \colon \mathbb{G} \to G$, $y \mapsto x^{-1}y$ for X.

Definition 2.5. A (twisted) quasi-Hamiltonian G-space (M, ω, μ) is a G-manifold M together with an invariant 2-form ω and a G-equivariant map $\mu \colon M \to \mathbb{G}$ to a G-bitorsor $\mathbb{G}$ such that

(QH1) $\mathrm{d}\omega = \mu^* \frac{1}{12}(\theta, [\theta, \theta])$.

(QH2) $\iota(\xi^\#)\omega = \frac{1}{2}\mu^*(\theta + \overline{\theta}, \xi)$, $(\xi \in \mathfrak{g})$, $\xi^\#|_p = \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-t\xi} \cdot p)|_{t=0}$.

(QH3) $\ker \omega_p \cap \ker d\mu_p = \{0\}$, $(p \in M)$,

where θ (resp. $\overline{\theta}$) denotes the left (resp. right) Maurer–Cartan form on $\mathbb{G}$.

We note that the restriction of the inner product $(\ ,\ )$ to the Lie algebra $\mathfrak{g}^\Gamma$ of the Γ-fixed point Lie subgroup $G^\Gamma := \{g \in G \mid \varphi \cdot g = g \text{ for all } \varphi \in \Gamma\}$ is also positive definite and invariant. Now we give the definition of a Γ-compatible quasi-Hamiltonian G-space.

Definition 2.6. A Γ-compatible quasi-Hamiltonian G-space (M, ω, μ) is a twisted quasi-Hamiltonian G-space equipped with a Γ-action such that

(1) $\varphi \cdot (g \cdot p) = (\varphi \cdot g) \cdot (\varphi \cdot p)$, $(\varphi \in \Gamma,\, g \in G,\, p \in M)$.

(2) The q-Hamiltonian 2-form ω is Γ-invariant.

(3) The moment map $\mu\colon M \to \mathbb{G}$ is Γ-equivariant.

We say that any two Γ-compatible quasi-Hamiltonian G-spaces are isomorphic if there exists a Γ-equivariant isomorphism of twisted quasi-Hamiltonian G-spaces.

From a Γ-compatible quasi-Hamiltonian G-space, one can naturally construct a twisted quasi-Hamiltonian $G \rtimes \Gamma$-space.

Remark 2.7. Let (M, ω, μ) be a Γ-compatible quasi-Hamiltonian G-space whose moment map takes values in a G-bitorsor $\mathbb{G}$. The actions of G and Γ naturally induce an action of $G \rtimes \Gamma$ on M given by $(g, \varphi) \cdot p := g \cdot (\varphi \cdot p)$, $((g, \varphi) \in G \rtimes \Gamma,\, p \in M)$, and a moment map $\tilde{\mu} := M \to \mathbb{G} \rtimes \Gamma, p \mapsto (\mu(p), 1_\Gamma)$, where $\mathbb{G} \rtimes \Gamma$ is the bitorsor introduced in Remark 2.3. Then we obtain a twisted quasi-Hamiltonian $G \rtimes \Gamma$-space $(M, \omega, \tilde{\mu})$.

2.3 The double $\mathbf{D}(\mathbb{G}_1, \mathbb{G}_2)$

We give a basic example of a Γ-compatible quasi-Hamiltonian space. Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be Γ-compatible G-bitorsors. We define a $G \times G$-action on $D := \mathbb{G}_1 \times \mathbb{G}_2$ by

$$(g_1, g_2) \cdot (a, b) := \big(g_1 a g_2^{-1}, g_2 b g_1^{-1}\big), \qquad (g_1, g_2) \in G \times G, \qquad (a, b) \in D.$$

Also we define a 2-form ω and a map μ by

$$\omega := -\frac{1}{2}\big(a^*\theta, b^*\overline{\theta}\big) - \frac{1}{2}\big(a^*\overline{\theta}, b^*\theta\big),$$
$$\mu\colon\ D \to \mathbb{G}_1 \cdot \mathbb{G}_2 \times \mathbb{G}_1^{-1} \cdot \mathbb{G}_2^{-1}, \qquad (a, b) \mapsto \big(ab, a^{-1}b^{-1}\big).$$

The tuple $\mathbf{D}(\mathbb{G}_1, \mathbb{G}_2) := (D, \omega, \mu)$ is a twisted quasi-Hamiltonian $(G \times G)$-space. Moreover, under the diagonal action of Γ on D and $\mathbb{G}_1\mathbb{G}_2 \times \mathbb{G}_1^{-1}\mathbb{G}_2^{-1}$, the space $\mathrm{D}(\mathbb{G}_1, \mathbb{G}_2)$ carries a Γ-compatible structure.

Remark 2.8. Let $\mathbb{C}$ and $\mathbb{H}$ be Γ-compatible G-bitorsors. We give the double $\mathbf{D}(\mathbb{C}, \mathbb{H})$ with different coordinates. For any $(C, h) \in \mathbf{D}(\mathbb{C}, \mathbb{H})$ and $(g, k) \in G \times G$, we define an action of $G \times G$ on $\mathbf{D}(\mathbb{C}, \mathbb{H})$ by $(g, k) \cdot (C, h) := \big(kCg^{-1}, khk^{-1}\big)$. We also define a quasi-Hamiltonian 2-form by

$$\omega := \frac{1}{2}\big(C^*\overline{\theta}, \mathrm{Ad}_h\, C^*\overline{\theta}\big) + \frac{1}{2}\big(C^*\overline{\theta}, h^*\theta + h^*\overline{\theta}\big),$$

and a moment map by

$$\mu\colon\ \mathbf{D}(\mathbb{C}, \mathbb{H}) \to \mathbb{C}^{-1}\mathbb{H}\mathbb{C} \times \mathbb{H}^{-1}, \qquad (C, h) \mapsto \big(C^{-1}hC, h^{-1}\big).$$

Then the twisted quasi-Hamiltonian $(G \times G)$-space $\mathbf{D}(\mathbb{C}, \mathbb{H})$ together with the natural Γ-action is also a Γ-compatible quasi-Hamiltonian $(G \times G)$-space and is isomorphic to the double of $\mathbb{C}^{-1}$ and $\mathbb{H}\mathbb{C}$ with standard coordinates.

3 Basic properties

Twisted quasi-Hamiltonian spaces have been investigated by several authors [5, 12, 15]. Although a Γ-compatible quasi-Hamiltonian G-space naturally gives rise to a twisted quasi-Hamiltonian $(G \rtimes \Gamma)$-space (see Remark 2.7), we study some basic properties of Γ-compatible quasi-Hamiltonian G-spaces, since this is the natural framework for the applications to twisted moduli spaces.

3.1 The set of fixed points of (M, ω, μ)

We fix an action $\Gamma \to \mathrm{Aut}(G)$. Let (M, ω, μ) be a Γ-compatible q-Hamiltonian G-space with the moment map $\mu\colon M \to \mathbb{G}$. Let M^Γ be the fixed point set of M for the action of Γ and let $\mathbb{G}^\Gamma$ be the fixed point set of $\mathbb{G}$. We denote the identity component of G^Γ by H. Fix a connected component N of M^Γ, and we can then take the connected component $\mathbb{H}$ of $\mathbb{G}$ so that $\mu(N) \subset \mathbb{H}$. Then $\mathbb{H}$ becomes an H-bitorsor and the following theorem holds.

Theorem 3.1. *The tuple $(N, \omega|_N, \mu|_N)$ is a quasi-Hamiltonian H-space.*

Proof. Let θ^Γ (resp. $\overline{\theta}^\Gamma$) be the left (resp. right) Maurer–Cartan form on $\mathbb{H}$. We denote the inclusions by $\iota_N\colon N \to M$ and $\iota_{\mathbb{H}}\colon \mathbb{H} \to \mathbb{G}$. Let the Cartan 3-forms on $\mathbb{G}$ and $\mathbb{H}$ be $\chi := \frac{1}{12}(\theta, [\theta, \theta])_{\mathfrak{g}}$ and $\chi^\Gamma := \frac{1}{12}\big(\theta^\Gamma, \big[\theta^\Gamma, \theta^\Gamma\big]\big)_{\mathfrak{g}^\Gamma}$, respectively. Since $\chi^\Gamma = \chi|_{\mathbb{H}}$, we obtain the following $\mathrm{d}\omega|_N = \big(\mu^*\chi\big)|_N = (\mu|_N)^*\chi^\Gamma$.

Next, we will show the moment map condition. For any $\xi \in \mathfrak{g}^\Gamma$,

$$\iota(\xi^\#)\omega|_N = \big(\iota(\xi^\#)\omega\big)|_N = \left(\frac{1}{2}\mu^*\big(\theta + \overline{\theta}, \xi\big)_{\mathfrak{g}}\right)\bigg|_N = \frac{1}{2}\mu|_N{}^*\big(\theta^\Gamma + \overline{\theta}^\Gamma, \xi\big)_{\mathfrak{g}^\Gamma}.$$

The degeneracy condition of $\omega|_N$ is shown by using the averaging over Γ. For any $x \in N$ and $X \in \ker \omega|_{N_x} \cap \ker \mathrm{d}_x\mu|_N$, $\mathrm{d}\mu \circ \mathrm{d}\iota_N(X) = \mathrm{d}\iota_{\mathbb{H}} \circ \mathrm{d}\mu|_N(X) = 0$. This leads to $\mathrm{d}\iota_N(X) \in \ker \mathrm{d}_x\mu$. For any $Y \in T_xM$, the following identity holds from the Γ-invariance of ω

$$\omega(\mathrm{d}\iota_N(X), Y) = \frac{1}{|\Gamma|}\sum_{\varphi \in \Gamma} \varphi^*\omega(\mathrm{d}\iota_N(X), Y) = \omega\left(\mathrm{d}\iota_N(X), \frac{1}{|\Gamma|}\sum \varphi_* Y\right) = 0.$$

Therefore, we obtain $\mathrm{d}\iota_N(X) \in \ker \omega_x \cap \ker \mathrm{d}_x\mu = \{0\}$. So, we conclude

$$\ker \omega|_N \cap \ker \mathrm{d}\mu|_N = \{0\}. \qquad \blacksquare$$

3.2 Fusion products and Γ-compatibilities

Let G, H_1, and H_2 be compact Lie groups. Let $\mathbb{G}$, $\mathbb{H}_1$, and $\mathbb{H}_2$ be bitorsors for G, H_1, and H_2 respectively. We fix actions of a finite group Γ on G and H_j. Let (M_j, ω_j, μ_j) be a Γ-compatible quasi-Hamiltonian $(G \times H_j)$-space for $j = 1, 2$. We denote the moment map $\mu_j\colon M_j \to \mathbb{G}_j \times \mathbb{H}_j$ by $\mu_j = \big(\mu_j^G, \mu_j^H\big)$.

Proposition 3.2. *The fusion product $M_1 \circledast M_2$ carries a Γ-compatible quasi-Hamiltonian $G \times H_1 \times H_2$-structure for the diagonal action of Γ on $M_1 \circledast M_2$, $G_1 \times H_1 \times H_2$ and $\mathbb{G}_1\mathbb{G}_2 \times \mathbb{H}_1 \times \mathbb{H}_2$.*

Proof. We show the quasi-Hamiltonian 2-form on $M_1 \circledast M_2$, $\omega = \omega_1 + \omega_2 - \frac{1}{2}\big(\big(\mu_1^G\big)^*\theta, \big(\mu_2^G\big)^*\overline{\theta}\big)$ is Γ-invariant. Since $\mathbb{G}_j$ are Γ-compatible and the inner product $(\ ,\ )$ on $\mathfrak{g}$ is invariant under the action of Γ, for any $\varphi \in \Gamma$,

$$\begin{aligned}\varphi^*\omega &= \varphi^*\omega_1 + \varphi^*\omega_2 - \frac{1}{2}\varphi^*\big(\big(\mu_1^G\big)^*\theta, \big(\mu_2^G\big)^*\overline{\theta}\big) = \omega_1 + \omega_2 - \frac{1}{2}\big(\varphi \cdot \big(\mu_1^G\big)^*\theta, \varphi \cdot \big(\mu_2^G\big)^*\overline{\theta}\big)\\ &= \omega_1 + \omega_2 - \frac{1}{2}\big(\big(\mu_1^G\big)^*\theta, \big(\mu_2^G\big)^*\overline{\theta}\big) = \omega.\end{aligned}$$

The conditions (1) and (3) in Definition 2.6 are easy to verify in this case. $\blacksquare$

Moreover, suppose that M_j^Γ are non-empty connected sets and $\mathbb{G}_j^\Gamma$ and $\mathbb{H}_j^\Gamma$ are non-empty. One can easily prove the following proposition.

Proposition 3.3. *There exists a canonical twisted quasi-Hamiltonian isomorphism $M_1^\Gamma \circledast M_2^\Gamma \simeq (M_1 \circledast M_2)^\Gamma$.*

Remark 3.4. In Proposition 3.3, if M_j^Γ are not connected, we can obtain a similar result as above by taking connected components of M_1^Γ and M_2^Γ.

Example 3.5 (fused double $\mathbb{D}(\mathbb{G}_1,\mathbb{G}_2)$)**.** Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be Γ-compatible G-bitorsors. The internal fusion of the double $\mathbf{D}(\mathbb{G}_1,\mathbb{G}_2)$ (see Example 2.3) is denoted by $\mathbb{D}(\mathbb{G}_1,\mathbb{G}_2)$, and is called the fused double. The space $\mathbb{D}(\mathbb{G}_1,\mathbb{G}_2)$ carries a Γ-compatible structure. Moreover, $\mathbb{G}_1^\Gamma$ and $\mathbb{G}_2^\Gamma$ are submanifolds of $\mathbb{G}_1$ and $\mathbb{G}_2$, respectively. We obtain an isomorphism of twisted quasi-Hamiltonian G^Γ-space $\mathbb{D}(\mathbb{G}_1,\mathbb{G}_2)^\Gamma \simeq \mathbb{D}\big(\mathbb{G}_1^\Gamma,\mathbb{G}_2^\Gamma\big)$.

4 Cyclic group actions on loop groups

The results of this section provide the foundation for the constructions in Section 5. We introduce the notion of Hamiltonian LG-spaces compatible with an action of a cyclic group $\Gamma := \mathbb{Z}/m\mathbb{Z}$, examine their relationship with Hamiltonian $L^{(\kappa)}G$-spaces, and discuss the equivalence theorem in the Γ-compatible setting. The constructions and computations involving the canonical 2-form on LG follow the same arguments as those in the (twisted) loop group setting developed in [2, 13].

4.1 Twisted loop groups

Let G be a compact connected simply connected Lie group with Lie algebra $\mathfrak{g}$. We fix an Ad-invariant inner product $(\ ,\ )$ on $\mathfrak{g}$ and a Lie group automorphism $\kappa \in \operatorname{Aut}(G)$ such that there exists a positive integer m such that $\kappa^m = 1$. We suppose that κ preserves the inner product $(\ ,\ )$ on $\mathfrak{g}$. Let S_m^1 denote $\mathbb{R}/m\mathbb{Z}$. We define the loop group LG as a space of maps

$$LG := \operatorname{Map}\big(S_m^1, G\big)$$

of a fixed Sobolev class $\lambda > 1/2$. Also we define the κ-twisted loop group $L^{(\kappa)}G$ as a subgroup of LG

$$L^{(\kappa)}G := \big\{g \in LG \mid \kappa g(t) = g(t+1) \text{ for all } t \in S_m^1\big\}.$$

We define the spaces $L^{(\kappa)}\mathfrak{g}$ and $L^{(\kappa)}\mathfrak{g}^*$ as subspaces of the space of maps $L\mathfrak{g} := \operatorname{Map}\big(S_m^1, \mathfrak{g}\big)$ of Sobolev class λ and the space of $\mathfrak{g}$-valued 1-forms $L\mathfrak{g}^* := \Omega^1\big(S_m^1, \mathfrak{g}\big)$ of Sobolev class $\lambda - 1$,

$$L^{(\kappa)}\mathfrak{g} := \big\{\xi \in L\mathfrak{g} \mid \kappa\xi(t) = \xi(t+1) \text{ for all } t \in S_m^1\big\},$$
$$L^{(\kappa)}\mathfrak{g}^* := \big\{A \in L\mathfrak{g}^* \mid \kappa A_t = A_{t+1} \text{ for all } t \in S_m^1\big\}.$$

The gauge transformation on $L^{(\kappa)}\mathfrak{g}^*$ is defined by $g \cdot A := gAg^{-1} + \mathrm{d}gg^{-1}$, $g \in L^{(\kappa)}G$, $A \in L^{(\kappa)}\mathfrak{g}^*$. There exists a canonical pairing $L^{(\kappa)}\mathfrak{g}^* \times L^{(\kappa)}\mathfrak{g} \to \mathbb{R}$, $(A,\xi) \mapsto \int_0^1 (A,\xi)$. We define the holonomy $\operatorname{Hol}_t^b(A)$ of A starting at $b \in S_m^1$ by the unique solution of the following differential equation:

$$\operatorname{Hol}_t^b(A)^{-1}\frac{\partial \operatorname{Hol}_t^b(A)}{\partial t} = -A, \qquad \operatorname{Hol}_b^b(A) = 1_G.$$

Then, the holonomy and the gauge action satisfy the following equation:

$$\operatorname{Hol}_t^b(g\cdot A) = g(b)\operatorname{Hol}_t^b(A)g(t)^{-1}, \qquad g \in L^{(\kappa)}G, \qquad A \in L^{(\kappa)}\mathfrak{g}^*.$$

Therefore, we have the holonomy map $\operatorname{Hol}^{(\kappa)}\colon L^{(\kappa)}\mathfrak{g}^* \to G^{(\kappa)}$, $A \mapsto \operatorname{Hol}_1^0(A)$, where $G^{(\kappa)} = G$ is a G-bitorsor with the following left and right actions $g\cdot x := gx$, $x \cdot g := x(\kappa g)$, $\big(x \in G^{(\kappa)}$, $g \in G\big)$.

Let θ (resp. $\overline{\theta}$) be the left (resp. right) Maurer–Cartan form on G. We define the 2-form on $L^{(\kappa)}\mathfrak{g}^*$ by

$$\varpi^{(\kappa)} := \frac{1}{2}\int_0^1 \mathrm{d}s \left(\mathrm{Hol}_s^{0*}\overline{\theta}, \frac{\partial}{\partial s}\mathrm{Hol}_s^{0*}\overline{\theta} \right).$$

The following proposition is analogous to [2, Proposition 8.1].

Proposition 4.1. *We have*

$$\varpi^{(\kappa)} \text{ is } L^{(\kappa)}G\text{-invariant}, \qquad \mathrm{d}\varpi^{(\kappa)} = -\mathrm{Hol}^{(\kappa)*}\frac{1}{12}(\theta, [\theta,\theta]),$$
$$\iota(\xi^\#)\varpi^{(\kappa)} = -\mathrm{d}\int_0^1 (A,\xi) - \frac{1}{2}\mathrm{Hol}^{(\kappa)*}\big(\theta+\overline{\theta},\xi\big), \qquad \xi \in L^{(\kappa)}\mathfrak{g}.$$

Proof. This is shown in the same way as in Proposition 4.2. ■

4.2 Cyclic group actions

In this subsection, let Γ be a cyclic group $\mathbb{Z}/m\mathbb{Z}$, and let S^1 denote $\mathbb{R}/m\mathbb{Z}$. We consider a Γ-action on G, i.e., a group homomorphism $\Gamma \to \mathrm{Aut}(G)$. Also a natural Γ-action on S^1 is defined by $t \cdot l := t - l$ for $t \in S^1$ and $l \in \Gamma$. Then we can define a Γ-action on the loop group LG by

$$l \cdot g(t) := \kappa^l g(t\cdot l), \qquad g \in LG, \qquad t \in S^1, \qquad l \in \Gamma.$$

Γ-actions on $L\mathfrak{g}$ and $L\mathfrak{g}^*$ are defined in the same way. Let β be a finite subset $0\cdot\Gamma = \{0,1,\dots, m-1\} \subset S^1$, and we denote $G^\beta := \mathrm{Map}(\beta, G)$. Also we denote Maurer–Cartan forms on G^β by θ and $\overline{\theta}$. Similar to the twisted case described above, gauge transformations, holonomies and natural pairing are defined. Furthermore, we define the 2-form on $L\mathfrak{g}^*$ by

$$\varpi := \frac{1}{2}\sum_{i=1}^m \int_i^{i+1} \mathrm{d}s \left(\mathrm{Hol}_s^{i*}\overline{\theta}, \frac{\partial}{\partial s}\mathrm{Hol}_s^{i*}\overline{\theta} \right).$$

Also the holonomy map $\mathrm{Hol}\colon L\mathfrak{g}^* \to \mathbb{H} := G^\beta$ is defined by $\mathrm{Hol}(A) := \big(\mathrm{Hol}_{i+1}^i(A)\big)_{i=1}^m$, $\big(A \in L\mathfrak{g}^*\big)$, where $\mathbb{H}$ is a Γ-compatible G^β-bitorsor (Example 2.2).

In this case, the following proposition holds in the same way as in [2, Proposition 8.1].

Proposition 4.2. *We have*

$$\varpi \text{ is } LG\text{-invariant}, \qquad \mathrm{d}\varpi = -\,\mathrm{Hol}^*\,\frac{1}{12}(\theta,[\theta,\theta]),$$
$$\iota(\xi^\#)\varpi_A = -\mathrm{d}\oint (A,\xi) - \frac{1}{2}\,\mathrm{Hol}^*\big(\theta+\overline{\theta}, \xi|_\beta\big), \qquad \xi \in L\mathfrak{g}, \qquad A \in L\mathfrak{g}^*,$$

where $\xi|_\beta = (\xi(i))_{i\in\beta} \in \mathfrak{g}^\beta$.

To prove it, we use the following lemma.

Lemma 4.3. *We have*

$$\iota\big(\xi^\#\big)\mathrm{Hol}_s^{i*}\overline{\theta} = \mathrm{Ad}\big(\mathrm{Hol}_s^i\big)\xi(s) - \xi(i), \qquad \xi \in L\mathfrak{g}.$$
$$\iota(\eta)\mathrm{Hol}_s^{i*}\overline{\theta} = -\int_i^s \mathrm{d}u\,\mathrm{Ad}\big(\mathrm{Hol}_u^i\big)\eta, \qquad \eta \in T_A L\mathfrak{g}^* = L\mathfrak{g}^*.$$

4.3 Γ-compatibilities of Hamiltonian *LG*-spaces

In this subsection, we state the definition of a Γ-compatibility on Hamiltonian LG-spaces. This is the usual Hamiltonian space version of a Γ-compatibility.

First, we recall the definition of a Hamiltonian LG-space [2].

Definition 4.4. A Hamiltonian LG-space is a Banach-manifold N together with an LG-action, an invariant 2-form $\sigma \in \Omega^2(N)^{LG}$, and an equivariant map $\Phi \in C^\infty\big(N, L\mathfrak{g}^*\big)^{LG}$ such that

(1) The 2-form σ is closed.

(2) The map Φ is a moment map for the LG-action $\iota\big(\xi^\#\big)\sigma = -\mathrm{d}\oint_{S^1}(m, \xi)$.

(3) The 2-form σ is weakly non-degenerate, that is, the induced map $\sigma_x^\flat\colon T_xN \to T_x^*N$ is injective.

Definition 4.5. Let (N, σ, Φ) be a Hamiltonian LG-space and we fix a Γ-action on N. The tuple (N, σ, Φ) together with the Γ-action is called a Γ-compatible Hamiltonian LG-space if it satisfies the following properties:

(1) $l \cdot (g \cdot x) = (l \cdot g) \cdot (l \cdot x)$, $l \in \Gamma$, $g \in LG$, $x \in N$.

(2) σ is Γ-invariant.

(3) $\Phi\colon N \to L\mathfrak{g}^*$ is Γ-equivariant.

We say that two Γ-compatible Hamiltonian LG-spaces are isomorphic if there exists a Γ-equivariant isomorphism of Hamiltonian LG-spaces.

4.4 Fixed points of a Γ-compatible space

Let (N, σ, Φ) be a Γ-compatible Hamiltonian LG-space. Then, we obtain a Hamiltonian $L^{(\kappa)}G$-space from a Γ-compatible Hamiltonian LG-space by taking fixed point objects. Now Γ-fixed objects are defined by

- $N^\Gamma := \{x \in N \mid l \cdot x = x,\ l \in \Gamma\}$,
- $\sigma^\Gamma := \sigma|_{N^\Gamma}$,
- $\Phi^\Gamma\colon N^\Gamma \to L\mathfrak{g}^*$, $x \mapsto \Phi(x)$.

In this paper, we denote Γ-fixed objects as above. For example, the Γ-fixed subgroup of LG is denoted by LG^Γ. Each connected component of N^Γ is a Banach submanifold of N. By considering left transformation of LG^Γ, the subgroup LG^Γ is a Banach Lie subgroup of LG (see Appendix C).

We thus obtain the following proposition.

Proposition 4.6. *For any connected component L of N^Γ, the tuple $\big(L, \sigma^\Gamma|_L, \Phi^\Gamma|_L\big)$ is a Hamiltonian LG^Γ-space, i.e., the following properties hold:*

- *LG^Γ acts on L.*
- *$\sigma^\Gamma|_L$ is a LG^Γ-invariant symplectic form on L.*
- *$\Phi^\Gamma|_L$ is LG^Γ-equivariant and $\sigma^\Gamma|_L$, $\Phi^\Gamma|_L$ satisfy the moment map condition*

$$\iota\big(\xi^\#\big)\sigma^\Gamma = -\mathrm{d}\oint\big(\Phi^\Gamma, \xi\big), \qquad \xi \in L\mathfrak{g}^\Gamma.$$

Proof. Since G is simply connected, the space LG^Γ is connected. Therefore, LG^Γ acts on a connected component L. The moment map condition for $\Phi^\Gamma|_L$ is a straightforward calculation and the weak non-degeneracy of σ^Γ is proved by averaging over Γ. Let $\iota\colon N^\Gamma \to N$ be the inclusion map. For any $\xi \in L\mathfrak{g}^\Gamma$,

$$\iota(\xi^\#)\sigma^\Gamma|_L = \big(\iota(\xi^\#)\sigma\big)|_L = -\mathrm{d}\oint(\Phi,\xi)\bigg|_L = -\oint\big(\Phi^\Gamma|_L,\xi\big).$$

Next, we will show the non-degeneracy of σ^Γ. Fix an element $x \in N^\Gamma$ and $X \in \ker\sigma^\Gamma$. Then for any $Y \in T_xN$,

$$\iota(Y)\iota(X)\sigma = \sigma(X,Y) = \frac{1}{|\Gamma|}\sum_{l\in\Gamma} l^*\sigma(X,Y) = \sigma\left(X,\ \frac{1}{|\Gamma|}\sum_{l\in\Gamma} l_*Y\right) = 0.$$

Thus we have proved the non-degeneracy condition. ■

Remark 4.7. Suppose that Γ is a cyclic group $\mathbb{Z}/m\mathbb{Z}$ and that the action of Γ on LG is the one defined in Section 4.1. Then we can identify LG^Γ with the twisted loop group $L^{(\kappa)}G$,

$$LG^\Gamma = \{g \in LG \mid l\cdot g = g,\ l\in\Gamma\} = \big\{g \in LG \mid \kappa g(t) = g(t+1) \text{ for all } t \in S^1_m\big\} = L^{(\kappa)}G.$$

For any $\xi \in L^{(\kappa)}\mathfrak{g} = L\mathfrak{g}^\Gamma$,

$$\iota(\xi^\#)\sigma^\Gamma = -\mathrm{d}\oint\big(\Phi^\Gamma,\xi\big) = -\mathrm{d}\sum_{i=1}^m\int_i^{i+1}\big(\Phi^\Gamma,\xi\big) = -m\mathrm{d}\int_0^1(\Phi,\xi).$$

Therefore, $\big(L, \frac{1}{m}\sigma^\Gamma, \Phi^{(\kappa)}\big)$ becomes a Hamiltonian $L^{(\kappa)}G$-space.

4.5 Equivalence theorem

It is known that there exists a natural correspondence between (twisted) Hamiltonian LG-spaces and (twisted) quasi-Hamiltonian G-spaces which is called the equivalence theorem [2, 13, 15]. Taking base points β with $\beta = 0\cdot\Gamma \subset S^1_m$, the equivalence theorem between Γ-compatible Hamiltonian LG-spaces and Γ-compatible quasi-Hamiltonian G^β-spaces can be shown. Let $\Omega^\beta G$ be a based loop group for β, i.e., $\Omega^\beta G = \{g \in LG \mid g|_\beta = 1_G\}$.

Theorem 4.8. *Taking the holonomy manifold for $\Omega^\beta G$ yields a bijective correspondence between*

- *the isomorphism class of Γ-compatible Hamiltonian LG-spaces (N,σ,Φ) with a proper moment map,*
- *the isomorphism class of Γ-compatible q-Hamiltonian G^β-spaces (M,ω,μ).*

5 Twisted connections and finite group actions

5.1 Preliminaries

Let Σ be a compact connected 2-manifold with boundary components $V_0, V_1, \dots, V_r$ and $P := \Sigma\times G$ a trivial bundle on Σ. Fix $\lambda > 1$ and let $\Omega^j(\Sigma,\mathfrak{g})$ be the space of $\mathfrak{g}$-valued j-forms of Sobolev class $\lambda - j$. Then the gauge $\mathcal{G}(\Sigma) := \mathrm{Map}(\Sigma, G)$ is a Banach Lie group modeled on $\mathrm{Lie}(\mathcal{G}(\Sigma)) = \Omega^0(\Sigma,\mathfrak{g})$. Let $\mathcal{A}(\Sigma) := \Omega^1(\Sigma,\mathfrak{g})$ be the space of connections on the trivial bundle on Σ. $\mathcal{A}(\Sigma)$ has a smooth $\mathcal{G}(\Sigma)$-action given by $g\cdot A = gAg^{-1} + g^*\overline{\theta}$ The space $\mathcal{A}(\Sigma)$ has a canonical symplectic form called Atiyah–Bott 2-form $\sigma = \frac{1}{2}\int_\Sigma(\delta A\wedge\delta A)$ for which $\mathcal{G}(\Sigma)$-action is Hamiltonian with moment map

$$\langle\Psi(A),\xi\rangle = \int_\Sigma(\operatorname{curv} A,\xi) - \int_{\partial\Sigma}(A,\xi), \qquad \xi\in\Omega^0(\Sigma,\mathfrak{g}),$$

where $\operatorname{curv}(A)$ is the curvature of A. The space of flat connections is denoted by $\mathcal{A}_{\text{flat}}(\Sigma)$.

5.2 Finite group actions

Let Γ be a finite group. Fix a free right Γ-action on Σ which preserves the orientation of Σ, and a left Γ-action on G which preserves the inner product on $\mathfrak{g}$. Then a right Γ-action on the trivial bundle is defined by

$$(x,a)\cdot\varphi = \bigl(x\cdot\varphi, \varphi^{-1}\cdot a\bigr), \qquad (x,a)\in P, \qquad \varphi\in\Gamma.$$

This action satisfies Γ-compatibility

$$(p\cdot g)\cdot\varphi = (p\cdot\varphi)\cdot(g\cdot\varphi), \qquad p\in P, \qquad g\in G, \qquad \varphi\in\Gamma,$$

where $g\cdot\varphi = \varphi^{-1}\cdot g$. We can also define left Γ-actions on $\Omega^j(P,\mathfrak{g})$ by

$$\varphi\cdot\alpha = \varphi\circ\alpha\circ d\varphi, \qquad \alpha\in\Omega^j(P,\mathfrak{g}), \qquad \varphi\in\Gamma,$$

and left Γ-actions on $\Omega^j(\Sigma,\mathfrak{g})$ can be defined in the same way.

Lemma 5.1. *The canonical 2-form σ is Γ-invariant and the curvature* $\operatorname{curv}\colon \mathcal{A}(\Sigma)\to\Omega^2(\Sigma,\mathfrak{g})$ *is Γ-equivariant. Moreover, the following Γ-compatibility holds:*

$$\varphi\cdot(g\cdot A) = (\varphi\cdot g)\cdot(\varphi\cdot A), \qquad \varphi\in\Gamma, \qquad g\in\mathcal{G}(\Sigma), \qquad A\in\mathcal{A}(\Sigma).$$

Proof. For any $\varphi\in\Gamma$ and $X,Y\in T_A\mathcal{A}(\Sigma)=\Omega^1(\Sigma,\mathfrak{g})$, the following holds:

$$\varphi^*\sigma(X,Y) = \int_\Sigma(\varphi X\circ \mathrm{d}\varphi\wedge\varphi Y\circ\mathrm{d}\varphi) = \int_\Sigma(X\wedge Y) = \sigma(X,Y).$$

Since the exterior derivative $\mathrm{d}\colon \Omega^j(\Sigma,\mathfrak{g})\to\Omega^{j+1}(\Sigma,\mathfrak{g})$ is Γ-equivariant, the Γ-equivariance of the curvature holds

$$\operatorname{curv}(\varphi\cdot A) = \mathrm{d}(\varphi\cdot A)+\frac{1}{2}[\varphi\cdot A\wedge\varphi\cdot A] = \varphi\cdot\mathrm{d}A+\frac{1}{2}\varphi\cdot[A\wedge A] = \varphi\cdot\operatorname{curv}A,$$

where $\varphi\in\Gamma$ and $A\in\mathcal{A}(\Sigma)$. Moreover, for any $g\in\mathcal{G}(\Sigma)$, $\varphi\in\Gamma$ and $A\in\mathcal{A}(\Sigma)$, the following holds $\varphi\circ\operatorname{Ad}(g)=\operatorname{Ad}(\varphi\cdot g)\circ\varphi$, $\varphi\cdot g^*\overline{\theta}=(\varphi\cdot g)^*\overline{\theta}$. Therefore, we obtain the Γ-compatibility of $\mathcal{G}(\Sigma)$ and $\mathcal{A}(\Sigma)$. ■

Γ-actions on $\mathcal{G}(\partial\Sigma)$ and $\Omega^j(\partial\Sigma,\mathfrak{g})$ are also defined in the same way.

5.3 The space $\mathcal{M}(\Sigma)$ and Γ-compatibilities

Since the restriction map $\mathcal{G}(\Sigma)\to\mathcal{G}(\partial\Sigma)$ is surjective, the following sequence is exact:

$$1\to\mathcal{G}_{\partial\Sigma}(\Sigma)\to\mathcal{G}(\Sigma)\to\mathcal{G}(\partial\Sigma)\to 1,$$

where $\mathcal{G}_{\partial\Sigma}(\Sigma) := \{g\in\mathcal{G}(\Sigma)\mid g|_{\partial\Sigma}=1_G\}$. Then $\mathcal{M}(\Sigma):=\mathcal{A}_{\mathrm{flat}}(\Sigma)/\mathcal{G}_{\partial\Sigma}(\Sigma)$ is a Banach manifold and a Hamiltonian $(\mathcal{G}(\partial\Sigma)\simeq LG^{r+1})$-space with proper moment map [2, 8]. The symplectic form σ on $\mathcal{M}(\Sigma)$ is induced from Atiyah–Bott 2-form on $\mathcal{A}(\Sigma)$ and the moment map is a restriction

$$\Phi\colon \mathcal{M}(\Sigma)\to\Omega^1(\partial\Sigma,\mathfrak{g}), \qquad A\mapsto A|_{\partial\Sigma}.$$

The Γ-compatibility of $\mathcal{M}(\Sigma)$ can be shown by using Lemma 5.1. A Γ-action on $\mathcal{M}(\Sigma)$ is defined by $\varphi\cdot[A]:=[\varphi\cdot A]$, $\varphi\in\Gamma$, $[A]\in\mathcal{M}(\Sigma)$. The Hamiltonian $\mathcal{G}(\partial\Sigma)$-space $(\mathcal{M}(\Sigma),\sigma,\Phi)$ together with the Γ-action is a Γ-compatible Hamiltonian $\mathcal{G}(\partial\Sigma)$-space.

Proposition 5.2. $(\mathcal{M}(\Sigma),\sigma,\Phi)$ *is a Γ-compatible Hamiltonian $\mathcal{G}(\partial\Sigma)$-space.*

The above proposition follows immediately from Lemma 5.1.

5.4 Fixed points of $\mathcal{M}(\Sigma)$

By taking a fixed point objects $\big(\mathcal{M}(\Sigma)^\Gamma, \omega^\Gamma, \Phi^\Gamma\big)$, we get Hamiltonian spaces for twisted loop groups actions.

We define a Γ-action on $I := \{0, 1, \dots, r\}$ by using a permutation of $\partial\Sigma = \coprod_{i=0}^r V_i$, $V_{i\cdot\varphi} = V_i \cdot \varphi$. Let $\Gamma(i)$ be a stabilizer of $i \in I$, i.e., $\Gamma(i) = \{\varphi \in \Gamma \mid i \cdot \varphi = i\}$.

Lemma 5.3. *For any $i \in I$, $\Gamma(i)$ is a cyclic group.*

Proof. Fix a point $b \in V_i$ and consider a $\Gamma(i)$-orbit $b \cdot \Gamma(i)$, we restrict the $\Gamma(i)$-action on $\partial\Sigma$ to $b\cdot\Gamma(i)$. Since this Γ-action on $\partial\Sigma$ is free, the restricted $\Gamma(i)$-action on $b\cdot\Gamma(i)$ is simply transitive. So by fixing a cyclic order of $b \cdot \Gamma(i) = \{b_0, b_1, \dots, b_{m_i-1}\}$, we get a unique element $b_1 = b_0 \cdot \kappa_i$. Then, the element κ_i is a generator of $\Gamma(i) = \langle\kappa_i\rangle$. ∎

We fix a generator κ_i of $\Gamma(i)$ for all $i \in I$. We set $m_i := |\Gamma(i)|$, and identify V_i with $S^1_{m_i} = \mathbb{R}/m_i\mathbb{Z}$ while keeping $\Gamma(i)$-action. So, the following isomorphism is obtained: $\mathcal{G}(V_i)^{\Gamma(i)} \simeq L^{(\kappa_i)}G$. So, by taking a complete system of representatives I' of I/Γ, and fixing κ_i for any $i \in I$, and taking $\tau_{ji} \in \Gamma$ such that $\kappa_j = \tau_{ji}\kappa_i\tau_{ji}^{-1}$ for $i, j \in i\cdot\Gamma$, we identify the fixed points subgroup $\mathcal{G}(\partial\Sigma)^\Gamma$ with a product of twisted loop groups,

$$\mathcal{G}(\partial\Sigma)^\Gamma \simeq \left\{ (g_i) \in \prod_{i\in I} L^{(\kappa_i)}G \;\middle|\; g_i = \tau_{ij}g_j \right\} \simeq \prod_{i\in I'} L^{(\kappa_i)}G.$$

In the same way, we obtain the following isomorphisms:

$$\Omega^1(\partial\Sigma, \mathfrak{g})^\Gamma \simeq \prod_{i\in I'} L^{(\kappa_i)}\mathfrak{g}^*; \qquad A \mapsto (A|_{V_i}), \qquad \Omega^0(\partial\Sigma, \mathfrak{g})^\Gamma \simeq \prod_{i\in I'} L^{(\kappa_i)}\mathfrak{g}; \qquad \xi \mapsto (\xi|_{V_i}).$$

We define Φ' and σ' by $\Phi'\colon \mathcal{M}(\Sigma)^\Gamma \to \prod_{i\in I'} L^{(\kappa_i)}\mathfrak{g}^*$; $A \mapsto (\Phi(A)|_{V_i})$, $\sigma' := \frac{1}{|\Gamma|}\sigma$. By considering the $\prod_{i\in I} L^{(\kappa_i)}G$-action on $\mathcal{M}(X)$, the following proposition holds.

Proposition 5.4. *Each connected component $\mathcal{N}$ of $\mathcal{M}(\Sigma)^\Gamma$ together with the 2-form $\sigma'|_{\mathcal{N}}$ and the moment map $\Phi'|_{\mathcal{N}}$ is a Hamiltonian $\Pi_{i\in I'} L^{(\kappa_i)}G$-space.*

Proof. It is trivial that σ' is symplectic and Φ' is equivariant for twisted loop group actions. So, we prove the moment map condition: for any $[A] \in \mathcal{M}(\Sigma)$ and $(\xi_i) = \xi \in \prod_{i\in I'} L^{(\kappa_i)}\mathfrak{g}$,

$$\begin{aligned}
\iota(\xi^\#)\sigma'_{[A]} &= \frac{1}{|\Gamma|}\iota(\xi^\#)\sigma_{[A]}|_\Gamma = -\frac{1}{|\Gamma|}\mathrm{d}\int_{\partial\Sigma}(\Phi(A), \xi) = -\frac{1}{|\Gamma|}\sum_{i\in I}\mathrm{d}\oint_{V_i}(A|_{V_i}, \xi|_{V_i}) \\
&= -\frac{1}{|\Gamma|}\sum_{i\in I'}|i\cdot\Gamma|\,\mathrm{d}\oint_{V_i}(A|_{V_i}, \xi_i) = -\frac{1}{|\Gamma|}\sum_{i\in I'}|i\cdot\Gamma|\,|\Gamma(i)|\,\mathrm{d}\int_0^1(A|_{V_i}, \xi_i) \\
&= -\sum_{i\in I'}\mathrm{d}\int_0^1\big(\Phi'(A)_i, \xi_i\big).
\end{aligned}$$
∎

5.5 The space $M(\Sigma)$ and Γ-compatibilities

Let $\beta \subset \partial\Sigma$ be a finite subset (base points) which is Γ-invariant and we fix a cyclic ordering of $\beta_i = \big(b^i_\lambda\big) := \beta \cap V_i$. We denote $\mathcal{G}_\beta(\Sigma) := \{g \in \mathcal{G}(\Sigma) \mid g|_\beta = 1\}$, $\mathcal{G}_\beta(\partial\Sigma) := \{g \in \mathcal{G}(\partial\Sigma) \mid g|_\beta = 1\}$. Set $M(\Sigma) := \mathcal{A}_{\mathrm{flat}}(\Sigma)/\mathcal{G}_\beta(\Sigma) \simeq \mathcal{M}(\Sigma)/\mathcal{G}_\beta(\partial\Sigma)$. An action of $\mathcal{G}(\partial\Sigma)/\mathcal{G}_\beta(\partial\Sigma) \simeq G^\beta$ on $M(\Sigma)$ is induced from the residual gauge action on $\mathcal{M}(\Sigma)$. The collection of holonomy map $\mathrm{Hol}^{b^i_\lambda}_{b^i_{\lambda+1}}(A)$ defines a G^β-equivariant map

$$\mu\colon\ M(\Sigma) \to \mathbb{H}, \qquad A \mapsto \mathrm{Hol}(A) := \big(\mathrm{Hol}^{b^i_\lambda}_{b^i_{\lambda+1}}(A)\big)_{b^i_\lambda\in\beta},$$

where $\mathbb{H} = G^\beta$ is a G^β-bitorsor under the following left and right actions, $(g\cdot h)\big(b^i_\lambda\big) = g\big(b^i_\lambda\big)h\big(b^i_\lambda\big)$, $(h\cdot g)\big(b^i_\lambda\big) = h\big(b^i_\lambda\big)g\big(b^i_{\lambda+1}\big)$, $g \in G^\beta$, $h \in \mathbb{H}$. An action of Γ on $M(\Sigma)$ is induced from the Γ-action on $\mathcal{M}(\Sigma)$. The following theorem, which is analogous to [2, Theorem 9.1], holds.

Theorem 5.5. *There is a natural G^β-invariant 2-form ω on $M(\Sigma)$ for which $(M(\Sigma), \omega, \mu)$ is a Γ-compatible q-Hamiltonian G^β-space.*

Proof. The proof is analogous to that of [2, Theorem 9.1]. In this proof, let $\pi\colon \mathcal{M}(\Sigma) \to M(\Sigma)$ be the quotient map. First, we will show the equivariance of $\mu\colon M(\Sigma) \to \mathbb{H}$. For any $g \in G^\beta$ and $\pi(A) \in M(\Sigma)$, $\mu(g\cdot\pi(A)) = \mu(\pi(\tilde{g}\cdot A)) = \mathrm{Hol}(\tilde{g}\cdot A) = \mathrm{Ad}_g\,\mu(\pi(A))$.

Let $R_j\colon \Omega^1(\partial\Sigma, \mathfrak{g}) \to \Omega^1(V_j, \mathfrak{g})$ be a restriction and let ϖ_j be the canonical 2-form on $\Omega^1(V_j, \mathfrak{g}) \simeq L\mathfrak{g}^*$. Denote $\varpi := \sum_{j=0}^r R_j^*\varpi_j$. We define a 2-form on $\mathcal{M}(\Sigma)$, $\tilde{\omega} := \sigma - \Phi^*\varpi$. Then the 2-form $\tilde{\omega}$ is basic for $\mathcal{G}_\beta(\partial\Sigma)$-action for $\mathcal{M}(\Sigma)$. Since σ and ϖ are $\mathcal{G}(\partial\Sigma)$-invariant, $\tilde{\omega}$ becomes a $\mathcal{G}(\partial\Sigma)$-invariant 2-form. Furthermore, for any $\xi \in \mathrm{Lie}(\mathcal{G}_\beta(\partial\Sigma))$,

$$\iota\big(\xi^\#\big)\tilde{\omega}_A = \iota\big(\xi^\#\big)\sigma_A - \iota\big(\xi^\#\big)\Phi^*\varpi_A = \iota\big(\xi^\#\big)\sigma_A - \sum_{j=0}^r {R_j}^*\big(\iota\big(\xi|_{V_j}{}^\#\big)\varpi_j\big)$$

$$= -\mathrm{d}\int_{\partial\Sigma}(A|_{\partial\Sigma}, \xi) - \sum_{j=0}^r {R_j}^*\left(-\mathrm{d}\oint(A|_{V_j}, \xi|_{V_j}) - \frac{1}{2}(\mathrm{Hol}_j)^*\big(\theta_j + \overline{\theta}_j, \xi|_{\beta_j}\big)\right) = 0,$$

where $\mathrm{Hol}_j\colon \Omega^1(V_j, \mathfrak{g}) \to \mathbb{H}^j := G^{\beta_j}$, $A_j \mapsto \big(\mathrm{Hol}^{b^j_\lambda}_{b^j_{\lambda}}(A_j)\big)$, and θ_j (resp. $\overline{\theta}$) is the left (resp. right) Maurer–Cartan form on $\mathbb{H}^j$. So, $\tilde{\omega}$ induces a unique G^β-invariant 2-form ω on $M(\Sigma)$ which satisfies $\pi^*\omega = \tilde{\omega}$.

Now we prove the tuple $(M(\Sigma), \omega, \mu)$ is a twisted quasi-Hamiltonian G^β-space. We denote the canonical 3-form on $\mathbb{H}$ by $\chi_\mathbb{H}$, i.e., $\chi_\mathbb{H} = \frac{1}{12}(\theta_\mathbb{H}, [\theta_\mathbb{H}, \theta_\mathbb{H}])$. Then, by $\mathrm{Hol}\circ\Phi = \mu\circ\pi$, $\pi^*\mathrm{d}\omega = \mathrm{d}\tilde{\omega} = \mathrm{d}(\sigma - \Phi^*\varpi) = \Phi^*\,\mathrm{Hol}^*\,\chi_\mathbb{H} = \pi^*\mu^*\chi_\mathbb{H}$. Therefore, we have proved $\mathrm{d}\omega = \mu^*\chi_\mathbb{H}$.

Next, we want to show the moment map condition. For any $\xi \in \mathfrak{g}^\beta$, there exists $\tilde{\xi} \in \Omega^0(\partial\Sigma, \mathfrak{g})$ such that $\tilde{\xi}|_\beta = \xi$. Then

$$\pi^*\big(\iota\big(\xi^\#\big)\omega\big) = \iota\big(\tilde{\xi}\big)\pi^*\omega = \iota\big(\tilde{\xi}^\#\big)\big(\sigma - \Phi^*\varpi\big) = \frac{1}{2}\sum_{j=0}^r {R_j}^*{\mathrm{Hol}_j}^*\big(\theta_j + \overline{\theta}_j, \tilde{\xi}|_{\beta_j}\big)$$

$$= \pi^*\left(\frac{1}{2}\mu^*\big(\theta_\mathbb{H} + \overline{\theta}_\mathbb{H}, \xi\big)\right).$$

So, the moment map condition holds.

Finally, we will prove the degeneracy of ω, i.e., we will show $\ker\omega_m \cap \ker d_m\mu = \{0\}$ for any $m \in M(\Sigma)$. Fix $v \in \ker\omega_m \cap \ker\mathrm{d}_m\mu$ and $A \in \mathcal{M}(\Sigma)$ such that $\pi(A) = m$. Since $\pi\colon \mathcal{M}(\Sigma) \to M(\Sigma)$ is a submersion, there exists a tangent vector $X \in T_A\mathcal{M}(\Sigma)$ such that $\mathrm{d}_A\pi(X) = v$.

For any $Y \in T_A\mathcal{M}(\Sigma)$, $\tilde{\omega}_A(X, Y) = \pi^*\omega_A(X, Y) = 0$. Therefore, $X \in \ker\tilde{\omega}_A$. By the definition of $\tilde{\omega}$, the following holds: $\iota(X)\sigma = \iota(X)\Phi^*\varpi$. Thus, $X \in (\ker\mathrm{d}_A\Phi)^\sigma$. Since the space $(\ker\mathrm{d}_A\Phi)^\sigma$ equals to the tangent space of the orbit $\mathcal{G}(\partial\Sigma)\cdot A$, i.e., there exists $\xi \in \Omega^0(\partial\Sigma, \mathfrak{g})$ that the generating vector $\xi^\#$ equals to X [2, Theorem 9.1].

Since $\xi|_\beta{}^\# \in \ker\mathrm{d}_m\mu$, we obtain $\xi|_\beta \in \ker(\mathrm{Ad}_{\mu(m)} - 1)$. Now we define $\eta_j \in \Omega^0(V_j, \mathfrak{g})$ by $\eta_j(s) := \mathrm{Ad}\big(\mathrm{Hol}^{b^j_0}_s(\Phi(A))^{-1}\big)\xi\big(b^j_0\big)$. Because $\mathrm{Ad}_{\mu(m)}\,\xi|_\beta = \xi|_\beta$,

$$\eta_j\big(b^j_{\lambda_j}\big) = \mathrm{Ad}\big(\mathrm{Hol}^{b^j_0}_{b^j_{\lambda_j}}(\Phi(A))\big)\xi\big(b^j_0\big) = \xi\big(b^j_{\lambda_j}\big).$$

Therefore, η_j is well defined. By collecting η_j for all $j = 0, \dots, r$, we define $\eta \in \Omega^0(\partial\Sigma, \mathfrak{g})$. Because $(\eta - \xi)|_\beta = 0$, $\eta^\# = (\eta - \xi)^\# + \xi^\#$ is in $\ker\big(\sigma - \Phi^*\varpi\big)$.

We now prove $\iota(\eta^{\#})\Phi^*\varpi = 0$. For any $\alpha \in T_A\mathcal{M}(\Sigma)$, by Proposition 4.2 and Lemma 4.3,

$$\begin{aligned}
\iota(\alpha)\iota(\eta^{\#})\Phi^*\varpi &= \sum_{j=0}^{r} \iota(\alpha|_{V_j})\iota(\eta|_{V_j}{}^{\#})\varpi_j \\
&= -\sum_{j=0}^{r} \iota(\alpha|_{V_j})\mathrm{d}\oint(A|_{V_j},\eta|_{V_j}) - \frac{1}{2}\iota(\alpha)\,\mathrm{Hol}^*\big(\theta_{\mathbb{H}} + \overline{\theta}_{\mathbb{H}}, \eta|_\beta\big) \\
&= -\sum_{j=0}^{r} \oint(\alpha|_{V_j},\eta|_{V_j}) - \iota(\alpha)\iota(\xi^{\#})\omega \\
&= -\sum_{j=0}^{r}\sum_{\lambda_j=1}^{m_j} \int_{b^j_{\lambda_j}}^{b^j_{\lambda_{j+1}}} \Big(\alpha|_{V_j}, \mathrm{Ad}\big(\mathrm{Hol}_s^{b^j_{\lambda_j}}(A|_{V_j})^{-1}\big)\xi\big(b^j_{\lambda_j}\big)\Big)\mathrm{d}s \\
&= \sum_{j=0}^{r}\sum_{\lambda_j=1}^{m_j} \int_{b^j_{\lambda_j}}^{b^j_{\lambda_{j+1}}} \Big(\mathrm{Ad}\big(\mathrm{Hol}_s^{b^j_{\lambda_j}}(A)\big)\alpha|_{V_j}, \xi\big(b^j_{\lambda_j}\big)\Big) \\
&= \sum_{j=0}^{r}\sum_{\lambda_j=1}^{m_j} \Big(\iota(\alpha|_{V_j})\big(\mathrm{Hol}_{b^j_{\lambda_{j+1}}}^{b^j_{\lambda_j}}\big)^*\overline{\theta}, \xi\big(b^j_{\lambda_j}\big)\Big) \\
&= \frac{1}{2}\sum_{j=0}^{r}\sum_{\lambda_j=0}^{m_j} \Big(\iota(\alpha|_{V_j})\big(\mathrm{Hol}_{b^j_{\lambda_{j+1}}}^{b^j_{\lambda_j}}\big)^*\overline{\theta}, \xi\big(b^j_{\lambda_j}\big)\Big) \\
&\quad + \frac{1}{2}\sum_{j=0}^{r}\sum_{\lambda_j=0}^{m_j} \Big(\iota(\alpha|_{V_j})\big(\mathrm{Hol}_{b^j_{\lambda_{j+1}}}^{b^j_{\lambda_j}}\big)^*\overline{\theta}, \mathrm{Ad}\big(\mathrm{Hol}_{b^j_{\lambda_{j+1}}}^{b^j_{\lambda_j}}\big)\xi\big(b^j_{\lambda_{j+1}}\big)\Big) \\
&= \iota(\alpha)\frac{1}{2}\,\mathrm{Hol}^*\big(\theta_{\mathbb{H}} + \overline{\theta}_{\mathbb{H}}, \xi|_\beta\big) = 0.
\end{aligned}$$

Considering $\iota(\eta^{\#})\Phi^*\varpi = 0$ and $\eta^{\#} \in \ker(\sigma - \Phi^*\varpi)$, we obtain $\iota(\eta^{\#})\sigma = \iota(\eta^{\#})\big((\sigma - \Phi^*\varpi) + \Phi^*\varpi\big) = 0$. Since σ is weak non-degenerate, $\eta^{\#}$ is zero. Therefore, we conclude $\xi|_\beta{}^{\#} = (\xi-\eta)|_\beta{}^{\#} + \eta|_\beta{}^{\#} = 0$. ∎

Remark 5.6.

(1) If the group Γ is trivial, we can choose any finite subset of $\partial\Sigma$ as the set of base points. Then the space $M(\Sigma)$ is a straightforward generalization of the space in [2, Theorem 9.1].

(2) Although Theorem 5.5 follows directly from Theorem 4.8, we include a proof because the almost same argument applies to the case where G is not simply connected.

5.6 Fixed points of $M(X)$

Taking fixed point objects of $(M(\Sigma),\omega,\mu)$, we obtain a twisted q-Hamiltonian $\big(G^\beta\big)^\Gamma$-space $\big(M(\Sigma)^\Gamma, \omega^\Gamma, \mu^\Gamma\big)$. We denote $\underline{\Sigma} := \Sigma/\Gamma$ and $\beta_{\underline{\Sigma}} := \pi(\beta)$, where $\pi\colon \Sigma \to \underline{\Sigma}$ is the covering map.

Taking a holonomy yields the following correspondence: $M(\Sigma) \simeq \mathrm{Hom}(\Pi_1(\Sigma,\beta),G)$. This correspondence is Γ-equivariant for an action on $\mathrm{Hom}(\Pi_1(\Sigma,\beta),G)$ $(\varphi\cdot\rho)(\gamma) = \varphi\rho(\gamma\cdot\varphi)$, $\varphi \in \Gamma$, $\rho \in \mathrm{Hom}(\Pi_1(\Sigma,\beta),G)$, $\gamma \in \Pi_1(\Sigma,\beta)$. Therefore, we obtain a bijection $M(\Sigma)^\Gamma \simeq \mathrm{Hom}(\Pi_1(\Sigma,\beta),G)^\Gamma$.

Now we want to consider the space $\mathrm{Hom}(\Pi_1(\Sigma,\beta),G)^\Gamma$ as the space on $\underline{\Sigma}$. Let $G_{\underline{\Sigma}}$ be a group bundle $(\Sigma\times G)/\Gamma$ on $\underline{\Sigma}$ with a fiber G. By fixing a complete system of representatives I of β for Γ-action, we obtain the monodromy representation of $G_{\underline{\Sigma}}$ denoted by $\mathrm{mon}_{\underline{\Sigma},I}\colon \Pi_1(\underline{\Sigma},\beta_{\underline{\Sigma}}) \to \Gamma$ (cf. Appendix B). We denote the space of representations twisted by the morphism $\mathrm{mon}_{\underline{\Sigma},I}$ by

$$\mathrm{Hom}_{\mathrm{mon}_{\underline{\Sigma},I}}(\Pi_1(\underline{\Sigma},\beta_{\underline{\Sigma}}),G) := \big\{\underline{\rho} \in \mathrm{Hom}(\Pi_1(\underline{\Sigma},\beta_{\underline{\Sigma}}),\Gamma\ltimes G) \mid \mathrm{pr}\circ\underline{\rho} = \mathrm{mon}_{\underline{\Sigma},I}\big\}.$$

Then, the covering map $\pi\colon \Sigma \to \underline{\Sigma}$ induces a bijection

$$\mathrm{Hom}_{\mathrm{mon}_{\underline{\Sigma},I}}\big(\Pi_1(\underline{\Sigma},\beta_{\underline{\Sigma}}),G\big) \simeq \mathrm{Hom}\big(\Pi_1(\Sigma,\beta),G\big)^{\Gamma}.$$

Therefore, we obtain a bijection $M(\Sigma)^{\Gamma} \simeq \mathrm{Hom}_{\mathrm{mon}_{\underline{\Sigma},I}}(\Pi_1(\underline{\Sigma},\beta_{\underline{\Sigma}}),G)$. By choosing a system of free generators of $\Pi_1(\underline{\Sigma},\beta_{\underline{\Sigma}})$, the space $\mathrm{Hom}_{\mathrm{mon}_{\underline{\Sigma},I}}(\Pi_1(\underline{\Sigma},\beta_{\underline{\Sigma}}),G)$ is diffeomorphic to a product of G-bitorsors. Since G is connected, this product of G-bitorsors is connected. Therefore, the tuple $\big(M(\Sigma)^{\Gamma},\omega^{\Gamma},\mu^{\Gamma}\big)$ is a twisted quasi-Hamiltonian $\big(G^{\beta}\big)^{\Gamma}$-space. Also $\mathrm{Hom}_{\mathrm{mon}_{\underline{\Sigma},I}}(\Pi_1(\underline{\Sigma},\beta_{\underline{\Sigma}}),G)$ carries a twisted quasi-Hamiltonian $G^{\beta_{\underline{\Sigma}}}$-structure. The above discussion is summarized in the following theorem.

Theorem 5.7. *The representation space* $\mathrm{Hom}(\Pi_1(\Sigma,\beta),G)$ *has a structure of a* Γ*-compatible quasi-Hamiltonian* G^{β}*-space. By taking the set of* Γ*-fixed points of* $\mathrm{Hom}(\Pi_1(\Sigma,\beta),G)$*, we obtain a structure of a twisted quasi-Hamiltonian* $G^{\beta_{\underline{\Sigma}}}$*-space on* $\mathrm{Hom}_{\mathrm{mon}_{\underline{\Sigma},I}}(\Pi_1(\underline{\Sigma},\beta_{\underline{\Sigma}}),G)$*.*

5.7 The 2-form on $M(\Sigma)$

In this subsection, suppose that the base points on each boundary component $\beta_i \subset V_i$ are in one orbit for Γ-action on β. Let $r_{\underline{\Sigma}}$ be a non-negative integer such that $r_{\underline{\Sigma}}+1$ is the number of boundary components of $\underline{\Sigma}$ and let $g_{\underline{\Sigma}}$ be the genus of $\underline{\Sigma}$. Then $\beta_{\underline{\Sigma}}$ has only one point on each boundary component of $\underline{\Sigma}$. We denote $\beta_{\underline{\Sigma}} = \big\{b^0,b^1,\dots,b^{r_{\underline{\Sigma}}}\big\}$. Let $Q_{\underline{\Sigma}}$ denote a quiver which generates free groupoid $\Pi_1(\underline{\Sigma},\beta_{\underline{\Sigma}})$, $Q_{\underline{\Sigma}} := \big\{a^k,b^k,\partial^j,\gamma^j \mid k=1,\dots,g_{\underline{\Sigma}},\ j=1,\dots,r_{\underline{\Sigma}}\big\}$, where a^k and b^k are a-cycles and b-cycles and a ∂^j is a loop along a boundary component based at b^j and a γ^j is a path from b^0 to b^j. Set $\beta_j := \pi^{-1}\big(b^j\big)$ and we label $\beta_0 = \big\{b^0_1,\dots,b^0_m\big\}$. Furthermore, let γ^j_λ be the unique lifting of γ^j starting at b^0_λ. By using γ^j_λ, we set $\beta^j_\lambda := t\big(\gamma^j_\lambda\big)$, where $t\big(\gamma^j_\lambda\big)$ is the target of γ^j_λ. Set a quiver, $Q_\Sigma := \big\{a^k_\lambda,b^k_\lambda,\partial^j_\lambda,\gamma^j_\lambda\big\}$, the fundamental groupoid $\Pi_1(\Sigma,\beta)$ is the free groupoid (cf. [14]) of Q_Σ.

We define four Γ-compatible G^m-bitorsors (see Example 2.2) by

$$\begin{aligned}
\mathbb{C}^j &:= \big\{C^{(j)} = \big(\rho\big(\gamma^j_\lambda\big)\big)_{\lambda=1}^m \mid \rho \in \mathrm{Hom}(\Pi_1(\Sigma,\beta),G)\big\},\\
\mathbb{H}^j &:= \big\{h^{(j)} = \big(\rho\big(\partial^j_\lambda\big)\big)_{\lambda=1}^m \mid \rho \in \mathrm{Hom}(\Pi_1(\Sigma,\beta),G)\big\},\\
\mathbb{A}^k &:= \big\{A^{(k)} = \big(\rho\big(a^k_\lambda\big)\big)_{\lambda=1}^m \mid \rho \in \mathrm{Hom}(\Pi_1(\Sigma,\beta),G)\big\},\\
\mathbb{B}^k &:= \big\{B^{(k)} = \big(\rho\big(b^k_\lambda\big)\big)_{\lambda=1}^m \mid \rho \in \mathrm{Hom}(\Pi_1(\Sigma,\beta),G)\big\}.
\end{aligned}$$

We define permutations t^j_∂, t^k_a, t^k_b of $\{1,\dots,m\}$ by

$$s\big(\partial^j_{t^j_\partial(\lambda)}\big) = t\big(\partial^j_\lambda\big), \qquad s\big(a^k_{t^k_a(\lambda)}\big) = t\big(a^k_\lambda\big), \qquad s\big(b^k_{t^k_b(\lambda)}\big) = t\big(b^k_\lambda\big).$$

Also we define left and right G^m-actions on each space by

$$\begin{aligned}
\big(g\cdot C^{(j)}\big)_\lambda &:= g_\lambda C^{(j)}_\lambda, & \big(C^{(j)}\cdot g\big)_\lambda &:= C^{(j)}_\lambda g_\lambda,\\
\big(g\cdot h^{(j)}\big)_\lambda &:= g_{t^j_\partial(\lambda)} h^{(j)}_\lambda, & \big(h^{(j)}\cdot g\big)_\lambda &:= h^{(j)}_\lambda g_\lambda,\\
\big(g\cdot A^{(k)}\big)_\lambda &\colon\ g_{t^k_a(\lambda)} A^{(k)}_\lambda, & \big(A^{(k)}\cdot g\big)_\lambda &\colon\ A^{(k)}_\lambda g_\lambda,\\
\big(g\cdot B^{(k)}\big)_\lambda &:= g_{t^k_b(\lambda)} B^{(k)}_\lambda, & \big(B^{(k)}\cdot g\big)_\lambda &:= B^{(k)}_\lambda g_\lambda.
\end{aligned}$$

So, we can construct the double $\mathbf{D}^{(j)} := \mathbf{D}\big(\mathbb{C}^j,\mathbb{H}^j\big)$, with coordinates in Remark 2.8, and the fused double $\mathbb{D}^{(k)} := \mathbb{D}\big(\mathbb{A}^k,\mathbb{B}^k\big)$. Fusing the $\mathbf{D}^{(j)}$'s with respect to the first component G^m

of the $G^m \times G^m$-action together with all $\mathbb{D}^{(k)}$, we obtain the Γ-compatible quasi-Hamiltonian G^β-space

$$\bigotimes_j \mathbf{D}^{(j)} \circledast \bigotimes_k \mathbb{D}^{(k)}.$$

Since $\Pi_1(\Sigma, \beta)$ is the free groupoid of Q_Σ, we have a structure of a Γ-compatible quasi-Hamiltonian G^β-space on $\operatorname{Hom}(\Pi_1(\Sigma, \beta), G)$, via the canonical bijection

$$\operatorname{Hom}(\Pi_1(\Sigma, \beta), G) = \bigotimes_j \mathbf{D}^{(j)} \circledast \bigotimes_k \mathbb{D}^{(k)}.$$

This structure of Γ-compatible quasi-Hamiltonian G^β-space coincides with the structure coming from $M(\Sigma)$. This is analogous to [2, Theorem 9.3].

Theorem 5.8. *$(M(\Sigma), \omega, \mu)$ is canonically isomorphic to the fusion product*

$$\bigotimes_j \boldsymbol{D}^{(j)} \circledast \bigotimes_k \mathbb{D}^{(k)}$$

as Γ-compatible quasi-Hamiltonian G^β-spaces.

Proof. The idea of this proof is based on [2, Theorem 9.3]. Let P denote the polyhedron obtained by cutting $\underline{\Sigma}$ along the paths γ^j, a^j, b^j. The boundary ∂P consists of $3r_{\underline{\Sigma}} + 4g_{\underline{\Sigma}} + 1$ segments $\partial P = (\gamma^1)^{-1}\partial^1(\gamma^1)\cdots(\gamma^{r_{\underline{\Sigma}}})^{-1}\partial^{r_{\underline{\Sigma}}}(\gamma^{r_{\underline{\Sigma}}})[a^1, b^1]\cdots[a^{g_{\underline{\Sigma}}}, b^{g_{\underline{\Sigma}}}]\partial^0$.

Let ∂P_λ be the lifting of ∂P starting at b^0_λ. Then the loop ∂P_λ is homotopic to a constant path. Let P_λ be a lifting of P such that its boundary is ∂P_λ. Then we write $\Sigma = \bigcup_{\lambda=1}^m P_\lambda$, where each P_λ intersects the others only along their boundary segments. We define a 2-form on $\mathcal{A}(\Sigma)$ by $\sigma_\lambda := \frac{1}{2}\int_{P_\lambda}(\delta A, \delta A)$. Then Atiyah–Bott 2-form can be written by $\sigma = \sum_{\lambda=1}^m \sigma_\lambda$. We obtain the following equation of 2-forms on $\mathcal{A}(\Sigma)$ using [2, Lemma 9.4] for each σ_λ

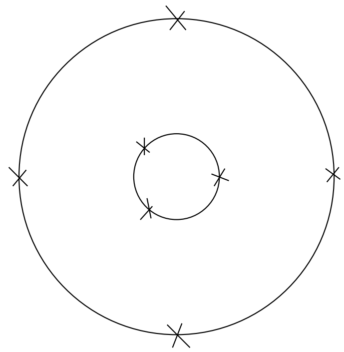
$$\begin{aligned}
\sigma - \sum_{i=0}^{r_{\underline{\Sigma}}} R_i^* \varpi_i &= \frac{1}{2}\sum_{j=1}^{r_{\underline{\Sigma}}}\sum_\lambda \big(C^{(j)}_{t^j_\partial(\lambda)}{}^*\overline{\theta}, \operatorname{Ad}_{h^{(j)}_\lambda} C^{(j)*}_\lambda\overline{\theta}\big) + \frac{1}{2}\sum_{j=1}^{r_{\underline{\Sigma}}}\sum_\lambda \big(C^{(j)}_{t^j_\partial(\lambda)}{}^*\overline{\theta}, h^{(j)}_{t^j_\partial(\lambda)}{}^*\theta + h^{(j)*}_\lambda\overline{\theta}\big) \\
&\quad - \frac{1}{2}\sum_{k=1}^{g_{\underline{\Sigma}}}\sum_\lambda \big(a^{(k)}_{t^k_b(\lambda)}{}^*\theta, b^{(k)*}_\lambda\overline{\theta}\big) - \frac{1}{2}\sum_{k=1}^{g_{\underline{\Sigma}}}\sum_\lambda \big(a^{(k)*}_\lambda\overline{\theta}, b^{(k)}_{t^{(k)}_a(\lambda)}{}^*\theta\big) \\
&\quad - \frac{1}{2}\sum_{k=1}^{g_{\underline{\Sigma}}}\sum_\lambda \big((a^{(k)}_{t^k_b(\lambda)} b^{(k)}_\lambda)^*\theta, (a^{(k)}_\lambda{}^{-1} b^{(k)}_{t^{(k)}_a(\lambda)}{}^{-1})^*\overline{\theta}\big) + \omega_{\mathrm{fus}},
\end{aligned}$$

where ω_{fus} is the pull back of extra terms coming from fusion. Moreover, the right-hand side is the quasi-Hamiltonian 2-form on $\operatorname{Hom}(\Pi_1(\Sigma, \beta), G)$. ■

A Generalizations of the twisted double

By considering the moduli space of an annulus Σ with base points $\beta = \beta_0 \cup \beta_\infty$ as above, we obtain the twisted quasi-Hamiltonian space which generalizes the twisted double, and we describe its quasi-Hamiltonian structure explicitly.

We fix a cyclic order $\beta_0 = \{b^0_1, \dots, b^0_{m_0}\}$ and $\beta_\infty = \{b^\infty_1, \dots, b^\infty_{m_\infty}\}$. We denote paths by $\gamma\colon b^0_1 \to b^\infty_1$, $\partial^\infty_i\colon b^\infty_i \to b^\infty_{i+1}$, $\partial^0_i\colon b^0_i \to b^0_{i+1}$. And for any $\rho \in \operatorname{Hom}(\Pi_1(\Sigma, \beta), G)$, we denote the corresponding values by $C := \rho(\gamma)$, $h_i := \rho(\partial^\infty_i)$, $h^0_i := \rho(\partial^0_i)$.



Then, the space $\mathrm{Hom}(\Pi_1(\Sigma,\beta),G)$ is identified with

$$\bigl\{\bigl(C,(h_i)_{i=1}^{m_\infty},\bigl(h_i^0\bigr)_{i=1}^{m_0}\bigr)\in G\times G^{m_\infty}\times G^{m_0}\mid C^{-1}h_{m_\infty}\cdots h_1Ch^0_{m_0}\cdots h^0_1=1\bigr\}.$$

Then the moment map μ and the 2-form ω are given as follows:

$$\mu\bigl(C,h,h^0\bigr)=\bigl(h^{-1},\bigl(h^0\bigr)^{-1}\bigr),$$

$$\omega=\frac{1}{2}\bigl(C^*\overline{\theta},\mathrm{Ad}_h C^*\overline{\theta}\bigr)+\frac{1}{2}\bigl(C^*\overline{\theta},h^*\overline{\theta}+h^*\theta\bigr)+\frac{1}{2}\sum_{i=1}^{m_\infty}\bigl(\bigl(k_i^\infty\bigr)^*\overline{\theta},h_i^*\theta\bigr)$$
$$+\frac{1}{2}\sum_{i=1}^{m_0}\bigl(\bigl(k_i^0\bigr)^*\overline{\theta},\bigl(h_i^0\bigr)^*\theta\bigr),$$

where k_i^∞ and k_i^0 are defined by $k_i^\infty:=h_{i-1}\cdots h_1$ and $k_i^0:=h^0_{i-1}\cdots h^0_1$, respectively.

We denote the q-Hamiltonian space $\mathrm{Hom}(\Pi_1(\Sigma,\beta),G)$ by ${}_{m_\infty}\mathbf{D}_{m_0}$. Then, if $m_\infty=m_0$, the space ${}_{m_\infty}\mathbf{D}_{m_0}$ is the double $\mathbf{D}\bigl(G^{m_0},G^{m_0}\bigr)$. It is easy to check that the space ${}_{m_\infty}\mathbf{D}_{m_0}$ is obtained by gluing ${}_{m_\infty}\mathbf{D}_1$ and ${}_1\mathbf{D}_{m_0}$.

B Twisted group bundles and monodromy

B.1 Setting

Let Σ be an oriented compact 2-manifold with boundary. Let Γ be a finite group. We consider a left Γ-action on G and a free right Γ-action on Σ. We define $\underline{\Sigma}:=\Sigma/\Gamma$, trivial group bundle $G_\Sigma:=\Sigma\times G$ on Σ, a group bundle $G_{\underline{\Sigma}}:=(\Sigma\times G)/\Gamma$ on $\underline{\Sigma}$.

Let $\beta_{\underline{\Sigma}}$ be a finite subset of $\underline{\Sigma}$, and let β_Σ be a pre-image of $\beta_{\underline{\Sigma}}$ for the quotient map $q\colon\Sigma\to\underline{\Sigma}$. We denote the fundamental groupoid by $\Pi_\Sigma:=\Pi_1(\Sigma,\beta_\Sigma)$ and $\Pi_{\underline{\Sigma}}:=\Pi_1(\underline{\Sigma},\beta_{\underline{\Sigma}})$.

B.2 The monodromy of $G_{\underline{\Sigma}}$

The trivial group bundle G_Σ has the trivial flat Ehresmann connection $\mathcal{H}_\Sigma$. We denote its monodromy by mon_Σ. Since G_Σ is trivial as a group bundle, under the natural identification $G_b\simeq G$ for all $b\in\beta_\Sigma$, we obtain the trivial monodromy representation $\mathrm{mon}_{\Sigma,\beta}\colon\Pi_\Sigma\to\mathrm{Aut}(G),\gamma\mapsto 1$.

Since $\mathcal{H}_\Sigma$ is Γ-invariant under the diagonal action of Γ on G_Σ, $\mathcal{H}_\Sigma$ induces a flat Ehresmann connection $\mathcal{H}_{\underline{\Sigma}}$ on $\underline{\Sigma}$. We denote its monodromy by $\mathrm{mon}_{\underline{\Sigma}}$. For any $\gamma\in\Pi_{\underline{\Sigma}}$, $\mathrm{mon}_{\underline{\Sigma}}(\gamma)\in\mathrm{Isom}(G_{s(\gamma)},G_{t(\gamma)})$.

Fix a complete system of representatives $I\subset\beta_\Sigma$ for the Γ-action. We construct a Γ-valued representation of $\mathrm{mon}_{\underline{\Sigma}}$. For any $\gamma\in\Pi_{\underline{\Sigma}}$, $i(s(\gamma))\in I$ denotes a representative of $q^{-1}(s(\gamma))$, and $i(t(\gamma))\in I$ denotes a representative of $q^{-1}(t(\gamma))$. Let $\tilde{\gamma}$ be a lift of $\gamma\in\Pi_{\underline{\Sigma}}$ starting at $i(s(\gamma))$, then there exists a unique element $\varphi_I(\gamma)\in\Gamma$ such that $t(\tilde{\gamma})=i(t(\gamma))\cdot\varphi_I(\gamma)$. Set $\mathrm{mon}_{\underline{\Sigma},I}(\gamma):=\varphi_I(\gamma)$, we obtain the monodromy representation $\mathrm{mon}_{\underline{\Sigma},I}\colon\Pi_{\underline{\Sigma}}\to\Gamma$.

B.3 The G-valued representation of $\Pi_{\underline{\Sigma}}$

We denote

$$\mathrm{Hom}_{\mathrm{mon}_{\underline{\Sigma},I}}(\Pi_{\underline{\Sigma}}, G) := \{\underline{\rho} \in \mathrm{Hom}(\Pi_{\underline{\Sigma}}, \Gamma \ltimes G) \mid \mathrm{pr}_\Gamma \circ \underline{\rho} = \mathrm{mon}_{\underline{\Sigma},I}\}.$$

In this appendix, we construct a bijection

$$\mathrm{Hom}(\Pi_\Sigma, G)^\Gamma \to \mathrm{Hom}_{\mathrm{mon}_{\underline{\Sigma},I}}(\Pi_{\underline{\Sigma}}, G), \qquad \rho \mapsto \underline{\rho}_I.$$

A Γ-action on $\mathrm{Hom}(\Pi_\Sigma, G)$ is defined by $(\varphi \cdot \rho)(\tilde{\gamma}) = \varphi\rho(\tilde{\gamma} \cdot \varphi)$, $\rho \in \mathrm{Hom}(\Pi_\Sigma, G)$, $\varphi \in \Gamma$, $\tilde{\gamma} \in \Pi_\Sigma$. For any $\rho \in \mathrm{Hom}(\Pi_\Sigma, G)^\Gamma$ and $\gamma \in \Pi_{\underline{\Sigma}}$, we denote the lifting of γ starting at $i(s(\gamma))$ by $\tilde{\gamma}$. We define $\underline{\rho}_I(\gamma) := (\mathrm{mon}_{\underline{\Sigma},I}(\gamma), \rho(\tilde{\gamma}))$.

Proposition B.1. *$\underline{\rho}_I \colon \Pi_{\underline{\Sigma}} \to \Gamma \ltimes G$ is a groupoid homomorphism.*

Proof. For any $\gamma_1, \gamma_2 \in \Pi_{\underline{\Sigma}}$ satisfying $s(\gamma_1) = t(\gamma_2)$, $\widetilde{(\gamma_1\gamma_2)} = (\tilde{\gamma}_1 \cdot \mathrm{mon}_{Y,I}(\gamma_2))\tilde{\gamma}_2$. Therefore,

$$\begin{aligned}\underline{\rho}_I(\gamma_1\gamma_2) &= (\mathrm{mon}_{Y,I}(\gamma_1\gamma_2), \rho(\widetilde{\gamma_1\gamma_2})) = (\mathrm{mon}_{Y,I}(\gamma_1)\,\mathrm{mon}_{Y,I}(\gamma_2), \rho(\tilde{\gamma}_1 \cdot \mathrm{mon}_{Y,I}(\gamma_2))\rho(\tilde{\gamma}_2)) \\ &= (\mathrm{mon}_{Y,I}(\gamma_1)\,\mathrm{mon}_{Y,I}(\gamma_2), (\rho(\tilde{\gamma}_1) \cdot \mathrm{mon}_{Y,I}(\gamma_2))\rho(\tilde{\gamma}_2)) = \underline{\rho}_I(\gamma_1)\underline{\rho}_I(\gamma_2).\end{aligned}$$

■

It is easy to see that the above correspondence is bijective.

C A C^∞ structure of fixed point sets

In this appendix, we show that each connected component of the fixed point set N^Γ for a C^∞-action of a finite group Γ on a Banach manifold N is a Banach submanifold of N.

Fix a point $x \in N^\Gamma$, a Γ-action on the tangent space T_xN is induced by the Γ-action on N. We denote the averaging by $\mathrm{ave}_x \colon T_xN \to (T_xN)^\Gamma$, $X \mapsto \frac{1}{|\Gamma|}\sum_{\varphi\in\Gamma} \varphi \cdot X$. Because the Γ-action on T_xN is top-linear, the averaging is also top-linear.

Lemma C.1. *$\ker(\mathrm{ave}_x)$ and $(T_xN)^\Gamma$ are closed subspaces of T_xN. Furthermore, a sequence*

$$0 \to \ker(\mathrm{ave}_x) \hookrightarrow T_xN \twoheadrightarrow (T_xN)^\Gamma \to 0$$

is split.

Proof. Since $(T_xN)^\Gamma$ is an intersection of closed subspaces $(T_xN)^\Gamma = \bigcap_{\varphi\in\Gamma} \ker(\varphi - 1)$, the space $(T_xN)^\Gamma$ is closed. Because ave_x is top-linear, the kernel $\ker(\mathrm{ave}_x)$ is a closed subspace. For any $X \in T_xN$,

$$(\mathrm{ave}_x)^2(X) = \mathrm{ave}_x\left(\frac{1}{|\Gamma|}\sum_{\varphi\in\Gamma} \varphi \cdot X\right) = \frac{1}{|\Gamma|}\sum_{\varphi\in\Gamma} \mathrm{ave}_x(\varphi \cdot X) = \frac{1}{|\Gamma|}\sum_{\varphi\in\Gamma} \varphi \cdot X = \mathrm{ave}_x(X).$$

Therefore, we have $T_xN = \ker(\mathrm{ave}_x) \oplus (T_xN)^\Gamma$. ■

Suppose that N^Γ is connected. Fix an element $x \in N^\Gamma$. We show that there exists a good chart around $x \in N$. By Bochner's linearization theorem for compact Lie group actions on Banach manifolds [7, Proposition 3.1.4], there exists a Γ-invariant open neighborhood U of $x \in N^\Gamma$ and a Γ-equivariant diffeomorphism $\chi \colon U \xrightarrow{\sim} V \subset T_xN$ which satisfies $\mathrm{d}_x\chi = 1_{T_xN}$. Since $T_xN = \ker(\mathrm{ave}_x) \oplus (T_xN)^\Gamma$, the above neighborhood U can be written as $U = \chi^{-1}\big(V' \times V^\Gamma\big)$ where V' $\big($resp. $V^\Gamma\big)$ is a sufficiently small open neighborhood of 0 in $\ker(\mathrm{ave}_x)$ $\big($resp. $(T_xN)^\Gamma\big)$. Because χ is Γ-equivariant, we have $N^\Gamma \cap U \simeq V^\Gamma = V \cap (T_xN)^\Gamma$. Therefore, each connected component of N^Γ is a Banach submanifold of N.

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