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Degenerate Addition Formulas of the KP Hierarchy and Applications

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Abstract. It is well known that tau functions of the KP hierarchy satisfy addition formulas. Among them we consider the formula which expresses a tau function with shifted arguments by $2n$ parameters in terms of the same tau function with shifted arguments by two parameters in the form of determinant. We then take the limits of it tending some of parameters to zero. As an important special case we obtain the formula which connects the shifted tau function to the Wronskian of functions obtained by substituting parameter values into the spectral variable of the wave function. As an application, we prove the equivalence of vertex operators and Darboux transformations. As another application, we derive a new addition formula for Riemann's theta functions of Riemann surfaces by considering theta function solutions of the KP hierarchy. It can be considered as a limit of Fay's famous formula (43) in the book [*Lecture Notes in Math.*, Vol. 352, Springer, Berlin, 1973]. But the formula we have derived is a different limit from the limits of (43) considered in that book.

Key words: KP hierarchy; tau function;addition formula; vertex operator; Darboux transformation;theta function

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1 Introduction

Motivated by the recent study by Kodama and others [1, 9, 10, 11] on shapes of soliton solutions of the KP equation and their relations to totally non-negative Grassmannians, cluster algebras, etc. the author [17] has studied the action of vertex operators to quasi-periodic solutions of the KP hierarchy, that is, solutions corresponding to smooth algebraic curves [12, 23]. The vertex operators of the KP hierarchy had been introduced by Date et al. [2] and act on the space of solutions of the KP hierarchy. In [17], it is shown that the solution obtained by applying a vertex operator to a solution corresponding to a smooth algebraic curve C becomes some solution corresponding to a singular algebraic curve whose normalization is the curve C. Moreover, the solution obtained in this way is shown to represent a soliton on the quasi-periodic background, where the background wave is the solution corresponding to C chosen at the beginning.

Darboux transformations of the KP hierarchy also carry solutions of the KP hierarchy to themselves [5, 7, 14, 20]. In [7], Kakei has studied in detail Darboux transformations of solutions corresponding to genus one algebraic curves motivated by the work of Li and Zhang [13] on elliptic solitons.

In [17], graphs of solutions obtained by applying vertex operators to genus one solutions are shown. Graphs of solutions obtained by applying Darboux transformations to solutions of genus one are shown in [7]. Surprisingly both graphs look quite similar. So it is natural to expect that two transformations are equivalent.

In this paper, we show this using the addition formula of the KP hierarchy. More precisely we first establish a certain addition formula (see Theorem 2.2) by taking a limit of the known general addition formula (2.5) proved in [24]. Using this, we prove that the vertex operator and Darboux transformation, if the parameters of the transformations are appropriately chosen, produce the same solution from the same initial solution which is not necessarily a solution corresponding to a genus one curve. We will explain the results in more detail below.

The addition formula for tau functions of the KP hierarchy was discovered by M. Sato and Y. Sato [22]. The simplest addition formula takes the form

$$\alpha_{12}\alpha_{34}\tau_{12}\tau_{34} - \alpha_{13}\alpha_{24}\tau_{13}\tau_{24} + \alpha_{14}\alpha_{23}\tau_{14}\tau_{23} = 0,$$

where $\alpha_{ij} = \alpha_i - \alpha_j$,

$$\tau_{ij} = \tau(t + [\alpha_i] + [\alpha_j]), \qquad t = {}^t(t_1, t_2, t_3, \dots), \qquad [\alpha] = {}^t\left(\alpha, \frac{\alpha^2}{2}, \frac{\alpha^3}{3}, \dots\right).$$

In [24, 25], this single equation is proved to be equivalent to the KP hierarchy itself. From this formula, the addition formula of the determinant form (2.5) had been derived in [24]. For mutually distinct parameters α_i, β_i, $1 \le i \le n$, it expresses the shifted tau function $\tau(t + \sum_{i=1}^n([\beta_i] - [\alpha_i]))$ in terms of $\tau(t + [\beta_i] - [\alpha_j])$, $1 \le i, j \le n$, as a certain determinant.

In order to prove the equivalence between vertex operators and Darboux transformations, we need some formula for $\tau\big(t - \sum_{i=1}^n [\alpha_i]\big)$. If we set $\beta_j = 0$, $j = 1, \dots, n$, in (2.5), we shall get a desired formula.

The problem here is that the expression (2.5) contains the difference product of $\{\beta_j\}$ in the denominator. It means that we need to differentiate the determinant with respect to β_j if we want to set $\beta_j = 0$. At a glance such a formula becomes very messy since if we rewrite the derivative with respect to β of the function like $\tau(t + [\beta])$ in terms of derivatives of t_j, $j \ge 1$, derivatives with respect to all t_j appear.

We have found that, if we use linear differential equations satisfied by the wave function (cf. (2.3)), the $\beta_1, \dots, \beta_n \to 0$ limit of (2.5) can be written entirely in terms of the wave function and its t_1-derivatives. Finally, we obtain a beautiful formula which connects $\tau\big(t - \sum_{i=1}^n [\alpha_i]\big)$ to the Wronskian of the functions obtained by substituting parameter values into the spectral variable of the wave function (see Theorem 2.2).

In a similar manner, it is possible to take the partial limits as $\beta_1, \dots, \beta_k \to 0$ or $\alpha_1, \dots, \alpha_k \to 0$ with $k \le n$ of the formula (2.5). The results are given by Theorems 2.5 and 2.6 which, together with Theorem 2.2, constitute main results of this paper.

As an application of Theorem 2.2, the equivalence of vertex operators and Darboux transformations is proved based on the expressions of solutions derived in [7, 16, 17].

As another application of main results, we prove identities of theta functions. Any identity for tau functions implies an identity for theta functions of Riemann surfaces if we consider solutions of the KP hierarchy expressed by theta functions. For example, the addition formula (2.5) corresponds to Fay's celebrated formula in [3, equation (43)] for Riemann's theta function $\theta(v)$ of a Riemann surface $\mathcal{C}$ of genus g. It is described as follows. Taking a symplectic basis of the homology group of C, let $\mathrm{d}v_i$, $i = 1, \dots, g$, be the normalized basis of holomorphic one forms, $\iota(p) = \int_{p_\infty}^p \mathrm{d}v$, $\mathrm{d}v = {}^t(\mathrm{d}v_1, \dots, \mathrm{d}v_g)$ the Abel–Jacobi map with a base point $p_\infty \in \mathcal{C}$ and $E(p, q)$ the prime form. Then for $n \ge 1$, mutually distinct points $p_i, q_i \in \mathcal{C}$, $i = 1, \dots, n$, and $c \in \mathbb{C}^g$ such that $\theta(c) \ne 0$ the following identity holds:

$$\begin{aligned} &\theta\left(\sum_{i=1}^n \iota(p_i) - \sum_{i=1}^n \iota(q_i) + c\right)\theta(c)^{n-1}\frac{\prod_{i<j} E(p_i, p_j)E(q_j, q_i)}{\prod_{i,j} E(p_i, q_j)} \\ &\quad = \det\left(\frac{\theta(\iota(p_i) - \iota(q_j) + c)}{E(p_i, q_j)}\right)_{1 \le i,j \le n}. \end{aligned} \tag{1.1}$$

In Fay's book [3], two kinds of limits of (1.1) are computed. The first one is the limit as $c \to c'$, where c' is such that $\theta(c') = 0$ and c' is not a singular point of the theta divisor $(\theta(v) = 0)$. The second one is the limit as all q_i approach the same point in the first limit. We derive the formula for theta functions corresponding to Theorem 2.2, which is given by Theorem 6.2. It corresponds to the limit of (1.1), where all q_j tend to the same point p_∞ while keeping $\theta(c) \neq 0$. Therefore, formulas in Theorem 6.2 are different from two limit formulas of Fay. In the case of genus one, the formulas of Theorem 6.2 had been considered and proved in [13] with the aid of some formulas of elliptic functions.

Finally, we would like to mention recent papers [26, 28] where Darboux transformations are studied from the view point of free fermions. They are closely related with the content of this paper. In particular, in [26], determinant formulas of some expectation values of free fermions are derived. Taking a generating function of them and using the boson-fermion correspondence, they correspond to the formulas in Theorem 2.6. They are derived using the determinantal structure of the Darboux transformations of [4]. While Theorem 2.6 is proved in another direct and simple way as explained above. Moreover, in our theorems it is important for applications that the formulas are written in the form of addition formulas. While the fermionic formulas in [26] do not apparently take the form of addition formulas.

The present paper is organized as follows. In Section 2 after a brief review of the KP hierarchy and the addition formula for tau functions, the main theorems are stated and proved. The vertex operators and related formulas are summarized in Section 3. Section 4 provides a brief review of the Darboux transformation and the necessary formulas. The equality of tau functions obtained by vertex operators and those obtained by Darboux transformations is proved in Section 5. In Section 6, new addition formula for Riemann's theta function of Riemann surfaces is given. In the case of genus one, the addition formulas rewritten in terms of Jacobi–Riemann theta functions and Weierstrass functions without prime form are also given.

2 KP hierarchy and addition formula for tau function

In this section, we first review the KP hierarchy [2, 21, 22] and addition formulas for tau functions [22, 24]. Then the main results of the paper are presented and proved.

Let $L(t) = \partial + \sum_{j=2}^{\infty} u_j \partial^{1-j}$, $t = {}^t(t_1, t_2, t_3, \dots)$, $x = t_1$, $\partial = \partial/\partial x$ be a monic microdifferential operator of order one without ∂^0 term. By defining, for $m \geq 1$ and a function f,

$$\partial^{-m} f = \sum_{j=0}^{\infty} \binom{-m}{j} f^{(j)} \partial^{-m-j}, \qquad f^{(j)} = \frac{\partial^j f}{\partial x^j},$$

the set of microdifferential operators $\left\{ \sum_{j \leq n} a_j \partial^j \text{ for some } n \right\}$, where a_j's belong to some ring of functions on which ∂ acts, becomes a ring. The differential operator part of $P = \sum_j a_j \partial^j$ is defined by $P_+ = \sum_{j \geq 0} a_j \partial^j$.

The KP hierarchy is the set of nonlinear differential equations for $\{u_j\}_{j=2}^{\infty}$ given by

$$\frac{\partial L(t)}{\partial t_m} = [B_m, L(t)], \qquad B_m = (L(t)^m)_+, \qquad m = 1, 2, 3, \dots, \tag{2.1}$$

where $[B_m, L(t)] = B_m L(t) - L(t) B_m$. In particular, $u = u_2$ satisfies the KP equation

$$3u_{t_2 t_2} + (-4u_{t_3} + 6uu_x + u_{xxx})_x = 0.$$

It describes shallow water waves [9], where (x, t_2) is the space variables and t_3 is the time variable.

For a solution $L(t)$ of the KP hierarchy, there exists a function $\Psi(t, z)$, called the wave function of $L(t)$, of the form

$$\Psi(t, z) = \left(1 + \sum_{j=1}^{\infty} w_j(t) z^j \right) \mathrm{e}^{\eta(t, z^{-1})}, \qquad \eta(t, z) = \sum_{j=1}^{\infty} t_j z^j,$$

which satisfies

$$\frac{\partial \Psi(t,z)}{\partial t_m} = B_m \Psi(t,z), \qquad L(t)\Psi(t,z) = z^{-1}\Psi(t,z). \tag{2.2}$$

So the variable z is the spectral parameter. A tau function of $L(t)$ can be introduced as a function $\tau(t)$ which satisfies

$$\Psi(t,z) = \frac{\tau(t-[z])}{\tau(t)} \mathrm{e}^{\eta(t,z^{-1})}, \qquad [z] = {}^t\left(z, \frac{z^2}{2}, \frac{z^3}{3}, \dots\right). \tag{2.3}$$

For an $L(t)$, a tau function exists and unique up to constant multiples [2].

The operator $L(t)$ can be recovered from a tau function through $\Psi(t,z)$ and the KP hierarchy can be expressed as a system of differential equations for $\tau(t)$ as

$$\operatorname*{Res}_{z=0} \tau(t-s-[z])\tau(t+s+[z]) \mathrm{e}^{-2\eta(s,z^{-1})} \frac{\mathrm{d}z}{z^2} = 0, \tag{2.4}$$

where $s = {}^t(s_1, s_2, \dots)$ is a set of parameters. Expanding (2.4) in $s_1, s_2, \dots$, we have a set of differential equations for $\tau(t)$ in the form of so called Hirota's bilinear equations.

The function $u_2(t)$ is recovered from $\tau(t)$ by $u_2(t) = 2\partial^2 \log \tau(t)$. It follows from (2.4) that, if $\tau(t)$ satisfies (2.4), so does $\mathrm{e}^{c_0 + \sum_{j=1}^{\infty} c_i t_i} \tau(t)$ for any constants c_i, $i = 0, 1, 2, \dots$. This transformation $\tau(t) \to \mathrm{e}^{c_0 + \sum_{j=1}^{\infty} c_i t_i} \tau(t)$ is called a gauge transformation.

It is discovered in [22] that a tau function of the KP hierarchy satisfies addition formulas and that the totality of addition formulas is equivalent to the KP hierarchy.

The following addition formula of the determinant form is proved in [24].

Theorem 2.1 ([24])**.** *Let $n \geq 1$ and $\alpha_1, \dots, \alpha_n$, $\beta_1, \dots, \beta_n$ be mutually distinct parameters. Then the following equation holds:*

$$\tau\left(t + \sum_{i=1}^{n} [\beta_i] - \sum_{i=1}^{n} [\alpha_i]\right)$$
$$= \tau(t)^{-n+1} \frac{\prod_{i,j=1}^{n} (\beta_i - \alpha_j)}{\prod_{i<j} (\alpha_{ij}\beta_{ji})} \det\left(\frac{\tau(t + [\beta_i] - [\alpha_j])}{\beta_i - \alpha_j}\right)_{1 \leq i,,j \leq n}, \tag{2.5}$$

where $\alpha_{ij} = \alpha_i - \alpha_j$, $\beta_{ij} = \beta_i - \beta_j$.

Our main aim is to get the formula for $\tau\big(t - \sum_{j=1}^{n} [\alpha_j]\big)$. To this end, we should set $\beta_j = 0$ for all j in (2.5). Notice here that the right-hand side of (2.5) contains a term $\prod_{i<j} \beta_{ji}$ in the denominator. So we consider the limit $\beta_1, \dots, \beta_n \to 0$. To take the limit, we need to take derivatives with respect to β_i. If we expand $\tau(t + [\beta_i] - [\alpha_j])$ in β_i, its coefficients contain derivatives of $\tau(t)$ in all t_j. However, we can prove that the final result contains derivatives with respect to only the variable x $(= t_1)$. The result is described in the following theorem which, together with Theorems 2.5 and 2.6, constitute main results of this paper.

Theorem 2.2. *Let $n \geq 1$ and $\alpha_1, \dots, \alpha_n$ be mutually distinct non-zero parameters. Then we have*

$$\tau\left(t - \sum_{i=1}^{n} [\alpha_i]\right) = \frac{1}{\prod_{i<j} (\alpha_j^{-1} - \alpha_i^{-1})} \operatorname{Wr}(\Psi(t,\alpha_1), \dots, \Psi(t,\alpha_n)) \tau(t) \mathrm{e}^{-\sum_{i=1}^{n} \eta(t,\alpha_i^{-1})},$$

where

$$\operatorname{Wr}(\Psi(t,\alpha_1), \dots, \Psi(t,\alpha_n)) = \det\big(\Psi^{(i-1)}(t,\alpha_j)\big)_{1 \leq i,j \leq n}, \qquad \Psi^{(i)}(t,\alpha) = \frac{\partial^i \Psi(t,\alpha)}{\partial x^i}$$

is the Wronskian of $\Psi(t,\alpha_1), \dots, \Psi(t,\alpha_n)$ with respect to the variable $x = t_1$.

Proof. Let us set $f(t,\alpha,\beta) = \frac{\tau(t+[\beta]-[\alpha])}{\beta-\alpha}$. Assuming $|\beta| < |\alpha|$, we consider $f(t,\alpha,\beta)$ as a series of β and we write it as

$$f(t,\alpha,\beta) = \sum_{j=0}^{\infty} f_j(t,\alpha)\beta^j. \tag{2.6}$$

Set

$$F = \frac{1}{\prod_{i<j}\beta_{ji}} D, \qquad D = \det\left(f(t,\alpha_j,\beta_i)\right)_{1\le i,j\le n} \tag{2.7}$$

and

$$\begin{aligned} &F_0 = F, \qquad F_k = F_{k-1}|_{\beta_k=0}, \qquad k \ge 1, \\ &\mathcal{B}_k = \prod_{k\le i<j\le n} \beta_{ji}, \qquad 1 \le k \le n-1, \qquad \mathcal{B}_n = \mathcal{B}_{n+1} = 1. \end{aligned} \tag{2.8}$$

Notice that each F_k is well defined due to (2.5).

Lemma 2.3. *For $1 \le k \le n$,*

$$F_k = \frac{D_k}{\mathcal{B}_{k+1}\prod_{j=k+1}^{n}\beta_j^k}, \tag{2.9}$$

where D_k is the following determinant:

$$D_k = \begin{vmatrix} f_0(t,\alpha_1) & \cdots & f_0(t,\alpha_n) \\ \vdots & & \vdots \\ f_{k-1}(t,\alpha_1) & \cdots & f_{k-1}(t,\alpha_n) \\ f(t,\alpha_1,\beta_{k+1}) & \cdots & f(t,\alpha_n,\beta_{k+1}) \\ \vdots & & \vdots \\ f(t,\alpha_1,\beta_n) & \cdots & f(t,\alpha_n,\beta_n) \end{vmatrix},$$

and the empty product $\prod_{j=k+1}^{n}\beta_j^k$ for $k = n$ is understood to be 1. In particular,

$$F_n = \det\left(f_{i-1}(t,\alpha_j)\right)_{1\le i,j\le n}. \tag{2.10}$$

Proof. Notice that $\mathcal{B}_k|_{\beta_k=0} = \mathcal{B}_{k+1}\prod_{j=k+1}^{n}\beta_j$. We prove (2.9) by induction on k. For $k = 1$, we have

$$\begin{aligned} F_1 = F|_{\beta_1=0} = \frac{1}{\mathcal{B}_2\prod_{j=2}^{n}\beta_j} D|_{\beta_1=0} = \frac{1}{\mathcal{B}_2\prod_{j=2}^{n}\beta_j} \begin{vmatrix} f_0(t,\alpha_1) & \cdots & f_0(t,\alpha_n) \\ f(t,\alpha_1,\beta_2) & \cdots & f(t,\alpha_n,\beta_2) \\ \vdots & \vdots & \vdots \\ f(t,\alpha_1,\beta_n) & \cdots & f(t,\alpha_n,\beta_n) \end{vmatrix} \\ = \frac{D_1}{\mathcal{B}_2\prod_{j=2}^{n}\beta_j}. \end{aligned}$$

Therefore, (2.9) is valid.

Suppose that (2.9) holds for k. Then

$$F_{k+1}=F_k|_{\beta_{k+1}=0}=\frac{D_k}{\mathcal{B}_{k+1}\prod_{j=k+1}^n\beta_j^k}\Big|_{\beta_{k+1}=0}=\frac{1}{\mathcal{B}_{k+1}|_{\beta_{k+1}=0}\prod_{j=k+2}^n\beta_j^k}\big(D_k\beta_{k+1}^{-k}\big)\Big|_{\beta_{k+1}=0}$$
$$=\frac{1}{\mathcal{B}_{k+2}\prod_{j=k+2}^n\beta_j^{k+1}}\big(D_k\beta_{k+1}^{-k}\big)\Big|_{\beta_{k+1}=0}.$$

Here, by expanding each element of $(k+1)$th row of D_k in β_{k+1} using (2.6), we get

$$D_k\beta_{k+1}^{-k}=\sum_{i=0}^{\infty}\begin{vmatrix} f_0(t,\alpha_1) & \cdots & f_0(t,\alpha_n)\\ \vdots & \vdots & \vdots\\ f_{k-1}(t,\alpha_1) & \cdots & f_{k-1}(t,\alpha_n)\\ f_i(t,\alpha_1) & \cdots & f_i(t,\alpha_n)\\ f(t,\alpha_1,\beta_{k+2}) & \cdots & f(t,\alpha_n,\beta_{k+2})\\ \vdots & \vdots & \vdots\\ f(t,\alpha_1,\beta_n) & \cdots & f(t,\alpha_n,\beta_n)\end{vmatrix}\beta_{k+1}^{i-k}.$$

Notice that the coefficient of β_{k+1}^{i-k} for $0\le i\le k-1$ becomes zero due to a property of determinants. Thus $\big(D_k\beta_{k+1}^{-k}\big)|_{\beta_{k+1}=0}=D_{k+1}$ and

$$F_{k+1}=\frac{D_{k+1}}{\mathcal{B}_{k+2}\prod_{j=k+2}^n\beta_j^{k+1}},$$

which shows (2.9) for $k+1$. Therefore, (2.9) is proved. ■

Next let us calculate the determinant of the right-hand side of (2.10). By (2.3),

$$\Psi(t+[\beta],z)=\frac{\tau(t+[\beta]-[z])}{\tau(t+[\beta])}\mathrm{e}^{\eta(t+[\beta],z^{-1})}. \tag{2.11}$$

Assuming $|\beta|<|z|$, we have

$$\mathrm{e}^{\eta(t+[\beta],z^{-1})}=\mathrm{e}^{\eta(t,z^{-1})+\eta([\beta],z^{-1})}=\mathrm{e}^{\eta(t,z^{-1})+\sum_{j=1}^\infty\frac{1}{j}(\frac{\beta}{z})^j}=\mathrm{e}^{\eta(t,z^{-1})-\log(1-\frac{\beta}{z})}=\frac{z\mathrm{e}^{\eta(t,z^{-1})}}{z-\beta}.$$

Putting it into (2.11) and replacing z by α, we obtain

$$\Psi(t+[\beta],\alpha)=-\alpha\frac{\tau(t+[\beta]-[\alpha])}{(\beta-\alpha)\tau(t+[\beta])}\mathrm{e}^{\eta(t,\alpha^{-1})}.$$

Therefore,

$$f(t,\alpha,\beta)=-\alpha^{-1}\Psi(t+[\beta],\alpha)\tau(t+[\beta])\mathrm{e}^{-\eta(t,\alpha^{-1})}. \tag{2.12}$$

Let us calculate $f_j(t,\alpha)$ by expanding the right-hand side of (2.12) in β. For this purpose, let

$$\mathrm{e}^{\eta(t,z)}=\sum_{k=0}^\infty p_k(t)z^k,\qquad \tilde\partial={}^t(\partial_1,\partial_2/2,\partial_3/3,\dots),\qquad \partial_k=\frac{\partial}{\partial t_k}. \tag{2.13}$$

Notice that the Schur function $p_k(t)$ is a homogeneous polynomial of degree k if we define $\deg t_k=k$. For example, $p_0=1$, $p_1=t_1$, $p_2=t_2+\frac{t_1^2}{2}$, $p_3=t_3+t_1t_2+\frac{t_1^3}{6}$. We have

$$\Psi(t+[\beta],\alpha)=\mathrm{e}^{\eta(\tilde\partial,\beta)}\Psi(t,\alpha)=\sum_{k=0}^\infty p_k\big(\tilde\partial\big)\Psi(t,\alpha)\beta^k,$$
$$\tau(t+[\beta])=\mathrm{e}^{\eta(\tilde\partial,\beta)}\tau(t)=\sum_{k=0}^\infty p_k\big(\tilde\partial\big)\tau(t)\beta^k.$$

Therefore,

$$\Psi(t+[\beta],\alpha)\tau(t+[\beta]) = \sum_{m=0}^{\infty}\sum_{k+l=m}\big(p_k\big(\tilde\partial\big)\Psi(t,\alpha)p_l\big(\tilde\partial\big)\tau(t)\big)\beta^m. \tag{2.14}$$

Lemma 2.4. *The equation*

$$p_k\big(\tilde\partial\big)\Psi(t,\alpha) = \big(\partial^k + (\textit{lower order})\big)\Psi(t,\alpha) \tag{2.15}$$

holds.

Proof. In the left-hand side of (2.15), we can replace ∂_j in $p_k\big(\tilde\partial\big)$ by ∂^j+(lower order) using (2.2) Replacing t_k by ∂^k/k in (2.13), we have

$$\sum_{k=0}^{\infty} p_k\big(\tilde\partial\big)|_{\partial_j=\partial^j} z^k = \mathrm{e}^{\sum_{k=1}^{\infty}\frac{(z\partial)^k}{k}} = \mathrm{e}^{-\log(1-z\partial)} = \frac{1}{1-z\partial} = \sum_{k=0}^{\infty}\partial^k z^k.$$

Therefore, $p_k\big(\tilde\partial\big)|_{\partial_j=\partial^j} = \partial^k$. Taking into account that $p_k(t)$ is a homogeneous polynomial, we have the assertion of the lemma. ∎

Using (2.12), (2.14) and (2.15), we have

$$f_l(t,\alpha) = C(t,\alpha)\sum_{k=0}^{l} A_{l,k}\partial^k\Psi(t,\alpha), \qquad C(t,\alpha) = (-\alpha)^{-1}\mathrm{e}^{-\eta(t,\alpha^{-1})}, \tag{2.16}$$

where $A_{l,k}$ is independent of α and $A_{l,l} = \tau(t)$. Namely,

$$f_0(t,\alpha) = C(t,\alpha)\tau(t)\Psi(t,\alpha), \qquad f_1(t,\alpha) = C(t,\alpha)(\tau(t)\partial + A_{1,0})\Psi(t,\alpha),$$
$$f_2(t,\alpha) = C(t,\alpha)\big(\tau(t)\partial^2 + A_{2,1}\partial + A_{2,0}\big)\Psi(t,\alpha), \qquad \dots.$$

Substituting them into the right-hand side of (2.10) and using the properties of determinants, we have

$$F_n = \det(f_{i-1}(t,\alpha_j)) = \tau(t)^n\left(\prod_{j=1}^{n} C(t,\alpha_j)\right)\det\big(\Psi^{(i-1)}(t,\alpha_j)\big). \tag{2.17}$$

By (2.5), (2.7), (2.8), we have

$$\tau\left(t - \sum_{i=1}^{n}[\alpha_i]\right) = \tau(t)^{-(n-1)}\frac{\prod_{i=1}^{n}(-\alpha_i)^n}{\prod_{i<j}\alpha_{ij}}F_n.$$

Substitute (2.17) into this equation and get

$$\tau\left(t - \sum_{i=1}^{n}[\alpha_i]\right) = \frac{\prod_{i=1}^{n}\alpha_i^{n-1}}{\prod_{i<j}\alpha_{ij}}\mathrm{Wr}(\Psi(t,\alpha_1),\dots,\Psi(t,\alpha_n))\tau(t)\mathrm{e}^{-\sum_{i=1}^{n}\eta(t,\alpha_i^{-1})}.$$

Finally, the proof of the theorem is completed if we notice

$$\frac{\prod_{i=1}^{n}\alpha_i^{n-1}}{\prod_{i<j}\alpha_{ij}} = \frac{1}{\prod_{i<j}\big(\alpha_j^{-1} - \alpha_i^{-1}\big)}.$$ ∎

In a similar way, it is possible to construct the formula for $\tau\big(t+\sum_{i=1}^{n}[\beta_i]\big)$ in terms of the Wronskian of the adjoint wave functions. Let

$$\Psi^*(t,z)=\frac{\tau(t+[z])}{\tau(t)}\mathrm{e}^{-\eta(t,z^{-1})}$$

be the adjoint wave function of $L(t)$. It satisfies

$$\frac{\partial\Psi^*(t,z)}{\partial t_m}=-B_m^*\Psi^*(t,z),$$

where B_m^* is the formal adjoint of B_m [2].

Theorem 2.5. *Let $n\geq 1$ and $\beta_1,\dots,\beta_n$ be mutually distinct non-zero parameters. Then we have*

$$\tau\left(t+\sum_{i=1}^{n}[\beta_i]\right)=\frac{1}{\prod_{i<j}\big(\beta_i^{-1}-\beta_j^{-1}\big)}\,\mathrm{Wr}(\Psi^*(t,\beta_1),\dots,\Psi^*(t,\beta_n))\tau(t)\mathrm{e}^{\sum_{i=1}^{n}\eta(t,\beta_i^{-1})}.$$

Proof. It is possible to prove the theorem by taking the limit as $\alpha_j\to 0$, $j=1,\dots,n$, of (2.5) in a similar manner to the proof of Theorem 2.2. Here we give another proof based on Theorem 2.2.

It is easily proved that if $\tau(t)$ is a solution of (2.4) so is $\tau(-t)$. We apply Theorem 2.2 to $\tilde{\tau}(t)=\tau(-t)$ and use

$$\tilde{\Psi}(t,z)=\Psi^*(-t,z), \tag{2.18}$$

where $\tilde{\Psi}(t,z)$ is the wave function of $\tilde{\tau}(t)$. Then we substitute t by $-t$ and obtain Theorem 2.5. ■

More generally, there are formulas which contain Theorems 2.2 and 2.5 as special cases. We simply tend $\beta_1,\dots,\beta_m\to 0$ or $\alpha_1,\dots,\alpha_m\to 0$ for $m\leq n$ in (2.5).

We set

$$\Psi(t,\alpha,\beta)=\frac{-\alpha}{\beta-\alpha}\frac{\tau(t+[\beta]-[\alpha])}{\tau(t)}\mathrm{e}^{\eta(t,\alpha^{-1})}, \tag{2.19}$$
$$\Psi^*(t,\alpha,\beta)=\frac{\beta}{\beta-\alpha}\frac{\tau(t+[\beta]-[\alpha])}{\tau(t)}\mathrm{e}^{-\eta(t,\beta^{-1})}.$$

Theorem 2.6. *Let m, n be nonnegative integers such that $n\geq m$.*

(i) *For mutually distinct non-zero parameters $\alpha_1,\dots,\alpha_n$, $\beta_1,\dots,\beta_{n-m}$, we have*

$$\tau\left(t+\sum_{i=1}^{n-m}[\beta_i]-\sum_{i=1}^{n}[\alpha_i]\right)$$
$$=C_{m,n}\mathrm{e}^{-\sum_{j=1}^{n}\eta(t,\alpha_j^{-1})}\tau(t)\begin{vmatrix}\Psi(t,\alpha_1) & \cdots & \Psi(t,\alpha_n)\\ \vdots & \cdots & \vdots\\ \Psi^{(m-1)}(t,\alpha_1) & \cdots & \Psi^{(m-1)}(t,\alpha_n)\\ \Psi(t,\alpha_1,\beta_1) & \cdots & \Psi(t,\alpha_n,\beta_1)\\ \vdots & \cdots & \vdots\\ \Psi(t,\alpha_1,\beta_{n-m}) & \cdots & \Psi(t,\alpha_n,\beta_{n-m})\end{vmatrix},$$

where

$$C_{m,n}=(-1)^{n(m-1)}\frac{\prod_{j=1}^{n}\alpha_j^{m-1}\prod_{i=1}^{n-m}\prod_{j=1}^{n}(\beta_i-\alpha_j)}{\prod_{i<j}^{n}\alpha_{ij}\prod_{i<j}^{n-m}\beta_{ji}\prod_{i=1}^{n-m}\beta_i^{m}}.$$

(*ii*) *For mutually distinct non-zero parameters* $\alpha_1,\dots,\alpha_{n-m}$, $\beta_1,\dots,\beta_n$, *we have*

$$\tau\left(t+\sum_{i=1}^{n}[\beta_i]-\sum_{i=1}^{n-m}[\alpha_i]\right)$$
$$= C^*_{m,n}\mathrm{e}^{\sum_{j=1}^{n}\eta(t,\beta_j^{-1})}\tau(t)\begin{vmatrix} \Psi^*(t,\beta_1) & \cdots & \Psi^*(t,\beta_n) \\ \vdots & \cdots & \vdots \\ \Psi^{*(m-1)}(t,\beta_1) & \cdots & \Psi^{*(m-1)}(t,\beta_n) \\ \Psi^*(t,\alpha_1,\beta_1) & \cdots & \Psi^*(t,\alpha_1,\beta_n) \\ \vdots & \cdots & \vdots \\ \Psi^*(t,\alpha_{n-m},\beta_1) & \cdots & \Psi^*(t,\alpha_{n-m},\beta_n) \end{vmatrix},$$

where

$$C^*_{m,n}=(-1)^{m(n-1)}\frac{\prod_{j=1}^{n}\beta_j^{m-1}\prod_{i=1}^{n}\prod_{j=1}^{n-m}(\beta_i-\alpha_j)}{\prod_{i<j}^{n-m}\alpha_{ij}\prod_{i<j}^{n}\beta_{ji}\prod_{i=1}^{n-m}\alpha_i^m}.$$

Proof. (i) Substitute (2.16) to (2.9) with $k=m$ and change the index of β_j as $(\beta_{m+1},\dots,\beta_n)$ to $(\beta_1,\dots,\beta_{n-m})$. Then we get (i) of Theorem 2.6 after a simple calculation.

(ii) As in the proof of Theorem 2.5, we should simply apply the formula (i) to $\tilde{\tau}(t)=\tau(-t)$. Let us denote the functions (2.19) corresponding to $\tilde{\tau}(t)$ by $\tilde{\Psi}(t,\beta,\alpha)$. Then

$$\tilde{\Psi}(t,\alpha,\beta)=\Psi^*(-t,\beta,\alpha). \tag{2.20}$$

Using (2.18) and (2.20) and substituting t by $-t$, we obtain (ii) of the theorem. ■

3 Vertex operator

The results of the previous section are used to prove the equivalence between solutions of the KP hierarchy generated by vertex operators and those generated by Darboux transformations. In this section, we recall the general formulas of solutions generated by vertex operators [17].

Let

$$X(p,q)=\mathrm{e}^{\eta(t,p)-\eta(t,q)}\mathrm{e}^{-\eta(\tilde{\partial},p^{-1})+\eta(\tilde{\partial},q^{-1})}$$

be the vertex operator of the KP hierarchy [2]. It acts on a function $f(t)$ as

$$X(p,q)f(t)=\mathrm{e}^{\eta(t,p)-\eta(t,q)}f\big(t-\big[p^{-1}\big]+\big[q^{-1}\big]\big). \tag{3.1}$$

The importance of the vertex operator is stated in the following theorem.

Theorem 3.1 ([2]). *If* $\tau(t)$ *is a solution of the KP-hierarchy, so is* $\mathrm{e}^{aX(p,q)}\tau(t)$ *for any* $a\in\mathbb{C}$.

We consider the composition of $\mathrm{e}^{aX(p,q)}$ with various values of a, p, q, and apply it to $\tau(t)$. The result can be written quite explicitly in a simple form.

The vertex operators satisfy the following commutation relations:

$$X(p_1,q_1)X(p_2,q_2)=X(p_2,q_2)X(p_1,q_1) \qquad \text{if } p_1\neq q_2 \text{ and } q_1\neq p_2,$$
$$X(p_1,q_1)X(p_2,q_2)=0 \qquad \text{if } p_1=p_2 \text{ or } q_1=q_2.$$

It follows that two vertex operators commute for general values of parameters and $X(p,q)^2=0$. Then

$$\mathrm{e}^{aX(p,q)}=1+aX(p,q). \tag{3.2}$$

Based on these facts, we consider the following general composition of vertex operators.

Let M, N be positive integers, $L=M+N$, $p_1,\dots,p_L$ mutually distinct non-zero complex parameters and $(a_{i,j})$ an $M\times N$ complex matrix. For these data, we set

$$G=\mathrm{e}^{\sum_{i=1}^M\sum_{j=1}^N a_{i,j}X(p_{N+i}^{-1},p_j^{-1})}. \tag{3.3}$$

Let $\tau(t)$ be a solution of the KP hierarchy. We apply G to $\tau(t)$. In order to have a formula similar to soliton solutions in [9], we first make a certain shift to $\tau(t)$, then apply G and make a gauge transformation. In this way, we define a new tau function $\tilde{\tau}(t)$ by

$$\tilde{\tau}(t)=\Delta\big(p_1^{-1},\dots,p_N^{-1}\big)\mathrm{e}^{\sum_{j=1}^N\eta(t,p_j^{-1})}G\tau\left(t-\sum_{j=1}^N[p_j]\right), \tag{3.4}$$

where $\Delta(p_1,\dots,p_n)=\prod_{1\le i<j\le n}(p_j-p_i)$. By (3.1) and (3.2), $\tilde{\tau}(t)$ is expressed as linear combination of shifts of $\tau(t)$. To give an explicit formula for it, let us define the $L\times N$ matrix $B=(b_{i,j})$ by

$$B=\begin{pmatrix}1&&\\&\ddots&\\&&1\\b_{N+1,1}&\cdots&b_{N+1,N}\\\vdots&&\vdots\\b_{N+M,1}&\cdots&b_{N+M,N}\end{pmatrix}, \tag{3.5}$$

where

$$b_{i,j}=\delta_{i,j},\qquad 1\le i,j\le N,$$

$$b_{N+i,j}=a_{i,j}\prod_{m\neq j}^N\frac{p_j^{-1}-p_m^{-1}}{p_{N+i}^{-1}-p_m^{-1}},\qquad 1\le i\le M,\qquad 1\le j\le N. \tag{3.6}$$

Notice that $a_{i,j}$ is recovered from $b_{N+i,j}$ uniquely by (3.6). Therefore, if a set of parameters $(p_1,\dots,p_L)$ is fixed, the set of $M\times N$ matrices $(a_{i,j})$ and the set of $L\times N$ matrices B of the form (3.5) correspond bijectively.

We use the following notation

$$[L]=\{1,\dots,L\},\qquad \binom{[L]}{N}=\{(i_1,\dots,i_N)\mid 1\le i_1<\dots<i_N\le L\},$$

and for $I=(i_1,\dots,i_N)\in\binom{[L]}{N}$

$$\Delta_I^-=\Delta\big(p_{i_1}^{-1},\dots,p_{i_N}^{-1}\big),\qquad \eta_I^-=\sum_{i\in I}\eta\big(t,p_i^{-1}\big),\qquad [p_I]=\sum_{i\in I}[p_i],$$

$$B_I=\det(b_{i_r,s})_{1\le r,s\le N}.$$

The formula for $\tilde{\tau}(t)$ is given in [16, 17] as described in the following proposition.

Proposition 3.2 ([16, 17])**.** *The function $\tilde{\tau}(t)$ defined by (3.4) is expressed as*

$$\tilde{\tau}(t)=\sum_{I\in\binom{[L]}{N}}B_I\Delta_I^-\mathrm{e}^{\eta_I^-}\tau(t-[p_I]). \tag{3.7}$$

This is the general formula for a solution of the KP hierarchy generated by vertex operators.

In [17], it is shown that, if $\tau(t)$ is a solution associated with a nonsingular algebraic curve (Riemann surface), $\tilde{\tau}(t)$ is a solution associated with a singular algebraic curve whose normalization is the algebraic curve corresponding to $\tau(t)$. In this case, it is also observed in [17] that $2\partial^2\log\tilde{\tau}(t)$ describes a soliton on a quasi-periodic background corresponding to $2\partial^2\log\tau(t)$.

4 Darboux transformation

Similarly to vertex operators, Darboux transformation also takes a solution of the KP hierarchy to another one. In this section, we review the definition and properties of Darboux transformations of the KP hierarchy following [5, 7]. Our aim here is to present the general formula for the tau function obtained by successive applications of elementary Darboux transformations.

We use the symbols M, N, L, p_i as in Section 3. Let $L(t)$ be a solution of the KP hierarchy.

Definition 4.1. A function $\varphi(t)$ is called an eigenfunction of $L(t)$ if it satisfies

$$\frac{\partial\varphi(t)}{\partial t_m} = B_m\varphi(t), \qquad B_m = (L(t)^m)_+.$$

For an eigenfunction $\varphi(t)$ of $L(t)$, we set

$$G_\varphi = \varphi\partial\varphi^{-1} = \partial - \partial(\log\varphi), \qquad L_\varphi(t) = G_\varphi L(t) G_\varphi^{-1}, \qquad \tau_\varphi(t) = \varphi(t)\tau(t).$$

Theorem 4.2 ([5, 6, 7, 20])**.** *Let $L(t)$ be a solution of the KP hierarchy (2.1) and $\varphi(t)$ be an eigenfunction of $L(t)$. Then*

(1) *The operator $L_\varphi(t)$ is a solution of the KP hierarchy and $\tau_\varphi(t)$ is a tau function of $L_\varphi(t)$.*

(2) *For any eigenfunction $\psi(t)$ of $L(t)$, $G_\varphi(\psi)$ is an eigenfunction of $L_\varphi(t)$.*

The transformation from $L(t)$ to $L_\varphi(t)$ or from $\tau(t)$ to $\tau_\varphi(t)$ is called an elementary Darboux transformation.

It is possible to repeat elementary Darboux transformations using a set of eigenfunctions of an initial solution $L(t)$ with the help of this theorem

Let $N \geq 2$ and $\varphi_1, \dots, \varphi_N$ be eigenfunctions of $L(t)$. We first consider the elementary Darboux transformation of $L(t)$ by φ_1. By Theorem 4.2 (2), the functions

$$G_{\varphi_1}(\varphi_j) =: \varphi_j^{[1]}, \qquad j = 2, \dots, N,$$

become eigenfunctions of $L_{\varphi_1}(t)$. Notice that $G_\varphi(\varphi) = 0$ for any φ. That is why we consider $j \geq 2$ here. Next we transform $L_{\varphi_1}(t)$ by $\varphi_2^{[1]}$. Then

$$G_{\varphi_2^{[1]}}\big(\varphi_j^{[1]}\big) = G_{\varphi_2^{[1]}} G_{\varphi_1}(\varphi_j) =: \varphi_j^{[21]}, \qquad j = 3, \dots, N,$$

are eigenfunctions of

$$(L_{\varphi_1})_{\varphi_2^{[1]}}(t) = (G_{\varphi_2^{[1]}} G_{\varphi_1}) L(t) (G_{\varphi_2^{[1]}} G_{\varphi_1})^{-1}.$$

We proceed in this way. In general, for $i \geq 2$ and $j \geq i+1$, we set

$$G_i = G_{\varphi_i^{[i-1,\dots,1]}} \cdots G_{\varphi_3^{[21]}} G_{\varphi_2^{[1]}} G_{\varphi_1}, \qquad \varphi_j^{[i,\dots,1]} = G_i(\varphi_j).$$

Then $\varphi_j^{[i,\dots,1]}$ is an eigenfunction of $G_i L(t) G_i^{-1}$ by Theorem 4.2 (2) and

$$\varphi_i^{[i-1,\dots,1]} \cdots \varphi_3^{[21]} \varphi_2^{[1]} \varphi_1 \tau(t)$$

is a tau function of $G_i L(t) G_i^{-1}$ by Theorem 4.2 (1). We set

$$\widehat{\tau}(t) = \varphi_N^{[N-1,\dots,1]} \cdots \varphi_3^{[21]} \varphi_2^{[1]} \varphi_1 \tau(t). \tag{4.1}$$

Example 4.3. For $N=2$, we have

$$\widehat{\tau}(t)=\varphi_2^{[1]}(t)\varphi_1(t)\tau(t)=G_{\varphi_1}(\varphi_2)\varphi_1\tau(t)=(\varphi_1\partial(\varphi_2)-\partial(\varphi_1)\varphi_2)\tau(t)=\mathrm{Wr}(\varphi_1,\varphi_2)\tau(t).$$

The appearance of the Wronskian is a general feature of the transformation.

Theorem 4.4 ([5, 6, 7, 20])**.** *The equality $\widehat{\tau}(t)=\mathrm{Wr}(\varphi_1,\dots,\varphi_N)\tau(t)$ holds.*

Let $C=(c_{i,j})$ be an $L\times N$ complex matrix of rank N. We take

$$\varphi_j(t)=\sum_{i=1}^{L}c_{i,j}\Psi(t,p_i),\qquad j=1,\dots,N, \tag{4.2}$$

as a set of eigenfunctions of $L(t)$. Then

$$\mathrm{Wr}(\varphi_1,\dots,\varphi_N)=\det\big(\big(\Psi^{(i-1)}(t,p_j)\big)_{1\le i\le N,1\le j\le L}C\big).$$

Using the Binet–Cauchy formula, we have

$$\mathrm{Wr}(\varphi_1,\dots,\varphi_N)=\sum_{I=(i_1,\dots,i_N)\in\binom{[L]}{N}}C_I\,\mathrm{Wr}(\Psi(t,p_{i_1}),\dots,\Psi(t,p_{i_N})),$$

where, for $I=(i_1,\dots,i_N)$, $C_I=\det(c_{i_r,s})_{1\le r,s\le N}$.

Thus we have the following proposition.

Proposition 4.5. *The tau function $\widehat{\tau}(t)$ defined by* (4.1) *with $\varphi_1,\dots,\varphi_N$ given by* (4.2) *is expressed as*

$$\widehat{\tau}(t)=\tau(t)\sum_{I=(i_1,\dots,i_N)\in\binom{[L]}{N}}C_I\,\mathrm{Wr}(\Psi(t,p_{i_1}),\dots,\Psi(t,p_{i_N})). \tag{4.3}$$

This is the general formula for a solution generated by Darboux transformations. In the next section, we compare this formula with the formula (3.7) in Proposition 3.2.

Remark 4.6. We denote by $\mathrm{Gr}(N,L)$ the Grassmannian which consists of N dimensional subspaces of $\mathbb{C}^L$. A point of $\mathrm{Gr}(N,L)$ is represented by an $L\times N$ matrix of rank N. Two $L\times N$ matrices Y_1 and Y_2 represent the same point of $\mathrm{Gr}(N,L)$ if and only if $Y_1=Y_2G$ for some invertible $N\times N$ matrix G.

If the matrix C in (4.2) is replaced by CG with an invertible constant matrix G, then $\widehat{\tau}(t)$ changes to $\det(G)\widehat{\tau}(t)$ and $u(t)=2\partial^2\log\widehat{\tau}(t)$ is unchanged. So if we are interested in solutions $u(t)$ of the KP equation, C can be considered as an element of $\mathrm{Gr}(N,L)$.

5 Connecting vertex operators to Darboux transformation

In this section, we prove, using Theorem 2.2, that two tau functions (3.7) and (4.3) obtained by vertex operators and Darboux transformations respectively coincide.

Consider the term corresponding to $I=(i_1,\dots,i_N)\in\binom{[L]}{N}$ in the formula (3.7). We take $n=N$, $(\alpha_1,\dots,\alpha_N)=(p_{i_1},\dots,p_{i_N})$ in Theorem 2.2. Then

$$\Delta_I^{-}\mathrm{e}^{\eta_I^{-}}\tau(t-[p_I])=\mathrm{Wr}\left(\Psi(t,p_{i_1}),\dots,\Psi(t,p_{i_N})\right)\tau(t).$$

Substituting this formula into (3.7), we have the following corollary.

Corollary 5.1. *If two matrices B of* (3.5) *and C given just before* (4.2) *coincide, the equation*

$$\tilde{\tau}(t) = \widehat{\tau}(t) \tag{5.1}$$

is valid.

It looks that the matrix B of (3.5) seems special compared with the matrix C which is an $L \times N$ matrix of rank N. But this is not the case.

Let S_L be the set of permutations of $\{1, \dots, L\}$. By the reason explained in Remark 4.6, we consider the matrix C as an element of $\mathrm{Gr}(N, L)$. Then we can prove the following.

Proposition 5.2. *For any $L \times N$ matrix C from $\mathrm{Gr}(N, L)$, there exists a matrix B and $\sigma \in S_L$ such that* (5.1) *holds up to constant multiples, where $\tilde{\tau}(t)$ corresponds to $\{B, (p_{\sigma(1)}, \dots, p_{\sigma(L)})\}$ and $\widehat{\tau}(t)$ corresponds to $\{C, (p_1, \dots, p_L)\}$.*

The strategy of the proof is as follows. We take C in the reduced column echelon form (cf. Example 5.4). We then change the order of rows of C appropriately so that the resulting matrix $\tilde{C}$ takes the form of (3.5) and define the components of B in such a way that $B = \tilde{C}$ holds.

Lemma 5.3. *Let $C = (c_{i,j})$ be an arbitrary $L \times N$ matrix and $p_1, \dots, p_L$ mutually distinct non-zero parameters. For an element σ of S_L, define the matrix $\tilde{C} = (\tilde{c}_{i,j})$ and parameter $\tilde{p}_1, \dots, \tilde{p}_L$ by $\tilde{c}_{i,j} = c_{\sigma(i),j}$, $\tilde{p}_i = p_{\sigma(i)}$, Then*

$$\sum_{I \in \binom{[L]}{N}} C_I \,\mathrm{Wr}(\Psi(p_I)) = \sum_{I \in \binom{[L]}{N}} \tilde{C}_I \,\mathrm{Wr}(\Psi(\tilde{p}_I)),$$

where, for $I = (i_1, \dots, i_N)$, $C_I = \det(c_{i_r,s})_{1 \le r,s \le N}$, $\mathrm{Wr}(\Psi(p_I)) = \mathrm{Wr}(\Psi(t, p_{i_1}), \dots, \Psi(t, p_{i_N}))$ and similarly for $\tilde{C}_I$, $\mathrm{Wr}(\Psi(\tilde{p}_I))$ in which $c_{i_r,s}$ and p_{i_j} are replaced by $\tilde{c}_{i_r,s}$ and $\tilde{p}_{i_j}$, respectively.

Proof. For $I = (i_1, \dots, i_N) \in \binom{[L]}{N}$ we have

$$\begin{aligned} \tilde{C}_I \,\mathrm{Wr}(\Psi(\tilde{p}_I)) &= \det(\tilde{c}_{i_r,s})_{1 \le r,s \le N} \det\big(\Psi^{(r-1)}(t, \tilde{p}_{i_s})\big)_{1 \le r,s \le N} \\ &= \det(c_{\sigma(i_r),s})_{1 \le r,s \le N} \det\big(\Psi^{(r-1)}(t, p_{\sigma(i_s)})\big)_{1 \le r,s \le N}. \end{aligned} \tag{5.2}$$

Let

$$\{\sigma(i_1), \dots, \sigma(i_N)\} = \{l_1, \dots, l_N\}, \qquad l_1 < \cdots < l_N,$$

and $\tilde{I} = (l_1, \dots, l_N)$. Due to the skew symmetry of the determinant, we have

$$\text{RHS of (5.2)} = \det(c_{l_r,s}) \det\big(\Psi^{(r-1)}(t, p_{l_s})\big) = C_{\tilde{I}} \,\mathrm{Wr}(\Psi(p_{\tilde{I}})),$$

and

$$\tilde{C}_I \,\mathrm{Wr}(\Psi(p_I)) = C_{\tilde{I}} \,\mathrm{Wr}(\Psi(p_{\tilde{I}})). \tag{5.3}$$

The lemma follows from (5.3) by summing up in I, since the map sending $(i_1, \dots, i_N)$ to $(l_1, \dots, l_N)$ is a bijection from $\binom{[L]}{N}$ to itself. ■

Proof of Proposition 5.2. Let $C \in \mathrm{Gr}(N, L)$ and B a matrix of the form (3.5) with its components $b_{N+i,j}$ to be determined. We can assume that C is in reduced column echelon form (see (5.4), for example). We suppose that pivots of C are in $(i_1, 1), \dots, (i_N, N)$ components with $1 \le i_1 < \cdots < i_N \le L$. Let $\{1, 2, \dots, L\} = \{i_1, \dots, i_N, j_1, \dots, j_{L-N})\}$ with $j_1 < \cdots < j_{L-N}$. Then we define $\sigma \in S_L$ by

$$\sigma(i_1, \dots, i_N, j_1, \dots, j_{L-N}) = (1, 2, \dots, L).$$

and the $L \times N$ matrix $\tilde{C} = (\tilde{c}_{i,j})$ by $\tilde{c}_{i,j} = c_{\sigma(i),j}$. Then $\tilde{C}$ becomes the matrix of the form (3.5). We define $b_{N+i,j}$ by $b_{N+i,j} = \tilde{c}_{N+i,j}$. Then the proposition follows from Lemma 5.3. ■

Example 5.4. The case of $\mathrm{Gr}(2,4)$. Consider the matrix C given by

$$C = \begin{pmatrix} 1 & 0 \\ a & 0 \\ 0 & 1 \\ b & c \end{pmatrix}. \tag{5.4}$$

and parameters (p_1, p_2, p_3, p_4). We shall show how to construct the matrix

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ b_{3,1} & b_{3,2} \\ b_{4,1} & b_{4,2} \end{pmatrix}$$

corresponding to C. The pivots of C are in $(1,1)$, $(3,2)$ components. So $i_1 = 1$, $i_2 = 3$, $j_1 = 2$, $j_4 = 4$ and $\sigma \in S_4$ is defined by $\sigma(1,3,2,4) = (1,2,3,4)$, that is, σ is the permutation of 2 and 3. The matrix $\tilde{C}$ is obtained by exchanging 2nd and 3rd rows of C,

$$\tilde{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a & 0 \\ b & c \end{pmatrix}.$$

We define $b_{3,1} = a$, $b_{3,2} = 0$, $b_{4,1} = b$, $b_{4,2} = c$. Then $B = \tilde{C}$. The associated parameters are $(p_{\sigma(1)}, \dots, p_{\sigma(4)}) = (p_1, p_3, p_2, p_4)$. With this B and parameters, $\tilde{\tau}(t)$ given by (3.7) and $\hat{\tau}(t)$ given by (4.3) coincide.

The vertex operator G corresponding to $(C, (p_1, \dots, p_4))$ is given by (3.3) with

$$a_{1,1} = a\frac{p_2^{-1} - p_3^{-1}}{p_1^{-1} - p_3^{-1}}, \qquad a_{1,2} = 0, \qquad a_{2,1} = b\frac{p_4^{-1} - p_3^{-1}}{p_1^{-1} - p_3^{-1}}, \qquad a_{2,2} = c\frac{p_4^{-1} - p_1^{-1}}{p_3^{-1} - p_1^{-1}}.$$

6 New addition formula for theta functions of Riemann surfaces

If we consider a solution of the KP hierarchy expressed by a theta function of a Riemann surface, the addition formula of Theorem 2.2 for tau functions becomes that for a theta function. In this section, we derive it.

Let $\mathcal{C}$ be a compact Riemann surface of genus g, $\{A_i, B_i\}$ a homology basis with the intersections $A_i \cdot A_j = 0$, $B_i \cdot B_j = 0$, $A_i \cdot B_j = \delta_{i,j}$, $\{\mathrm{d}v_i\}_{i=1}^g$ the normalized basis of Abelian differentials of the first kind, $\mathrm{d}v = {}^t(\mathrm{d}v_1, \dots, \mathrm{d}v_g)$, $\Omega_{i,j} = \int_{B_j} \mathrm{d}v_i$, $\Omega = (\Omega_{i,j})$ the normalized period matrix, p_∞ a point of $\mathcal{C}$, $\iota(p) = \int_{p_\infty}^p \mathrm{d}v$ the Abel map and z a local coordinate around p_∞ that satisfies $z(p_\infty) = 0$. We notice that, as a local coordinate around p_∞, we always use the same local coordinate z specified here unless otherwise stated.

Riemann's theta function with the characteristics $e = \binom{a}{b}$, $a, b \in \mathbb{R}^g$ is defined by

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (v, \Omega) = \sum_{m \in \mathbb{Z}^g} \exp\bigl(\pi \mathrm{i} {}^t(m+a)\Omega(m+a) + 2\pi \mathrm{i} {}^t(m+a)(v+b)\bigr),$$

where $v = {}^t(v_1, \dots, v_g)$. In the following, it is denoted by $\theta \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](v)$ by omitting the Ω dependence, since we do not consider changing Ω in the following. Moreover, the theta function with the characteristics $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ is denoted by $\theta(v)$.

We need the prime form $E(x,y)$, $x,y\in\mathcal{C}$ (see [3, 8, 15] for details). It is defined as follows. Let $\epsilon=\left[\begin{smallmatrix}\epsilon'\\ \epsilon''\end{smallmatrix}\right]$, $\epsilon',\epsilon''\in\frac{1}{2}\mathbb{Z}^g$ be a non-singular odd half period, which means that ϵ satisfies the following two conditions,

$$\theta[\epsilon](-v)=-\theta[\epsilon](v),\qquad \frac{\partial\theta[\epsilon]}{\partial v_i}(0)\neq 0\qquad \text{for some } i.$$

It can be proved that such ϵ exists but it is not necessarily unique [3, 15]. We take any one of them. Let $h_\epsilon(x)$ be such that

$$h_\epsilon(x)^2=\sum_{k=1}^{g}\frac{\partial\theta[\epsilon]}{\partial v_k}(0)\mathrm{d}v_k(x).$$

It can be considered as a section of a certain holomorphic line bundle on $\mathcal{C}$ and is unique up to sign [3]. We choose any sign and define $h_\epsilon(x)$. Then the prime form is defined by

$$E(x,y)=\frac{\theta[\epsilon](\iota(y)-\iota(x))}{h_\epsilon(x)h_\epsilon(y)}.$$

It is the section of a certain holomorphic line bundle on $\mathcal{C}\times\mathcal{C}$ independent of ϵ and the choice of the sign of $h_\epsilon(x)$. The prime form satisfies the following properties:

(i) $E(x,y)=-E(y,x)$,

(ii) $E(x,y)=0$ if and only if $x=y$,

(iii) for any point $p\in\mathcal{C}$, $x,y\in\mathcal{C}$ near p and a local coordinate u around p the expansion of $E(x,y)$ in $u(y)$ at $u(x)$ is of the form

$$E(x,y)\sqrt{\mathrm{d}u(x)}\sqrt{\mathrm{d}u(y)}=u(y)-u(x)+O\big((u(y)-u(x))^3\big).$$

For points x, y around p_∞, using the local coordinates $z=z(x)$, $w=z(y)$, define $E(z,w)$ and $E(p_\infty,y)$ by

$$E(x,y)=\frac{E(z,w)}{\sqrt{\mathrm{d}z}\sqrt{\mathrm{d}w}},\qquad E(p_\infty,y)=E(x,y)\sqrt{\mathrm{d}z}\Big|_{x=p_\infty}=\frac{E(0,w)}{\sqrt{\mathrm{d}w}}.$$

Notice that $E(p_\infty,y)$ depends on the choice of a local coordinate z.

We define $v_{i,j}$, $q_{i,j}$ as the expansion coefficients with respect to $z=z(x)$, $w=z(y)$

$$\mathrm{d}v_i=\sum_{j=1}^{\infty}v_{i,j}z^{j-1}\mathrm{d}z\qquad \text{near } p_\infty, \tag{6.1}$$

$$\mathrm{d}_z\mathrm{d}_w\log\frac{E(z,w)}{w-z}=\sum_{i,j=1}^{\infty}q_{i,j}z^{i-1}w^{j-1}\mathrm{d}z\mathrm{d}w\qquad \text{near } (0,0). \tag{6.2}$$

Since $E(z,w)$ is skew symmetric in z and w, we have $q_{i,j}=q_{j,i}$.

We set

$$\mathcal{V}=(v_{i,j})_{1\le i\le g,1\le j},\qquad \mathcal{V}_1={}^t(v_{1,1},v_{2,1},\dots,v_{g,1}), \tag{6.3}$$

$$q(t|s)=\sum_{i,j=1}^{\infty}q_{i,j}t_is_j,\qquad q(t)=q(t|t). \tag{6.4}$$

By Krichever's theory on Baker–Akhiezer function [12], solutions of the KP hierarchy expressed by Riemann's theta function of Riemann surfaces had been derived [8, 23].

Theorem 6.1 ([8, 23])**.** *For any $c = {}^t(c_1, \dots, c_g) \in \mathbb{C}^g$,*

$$\tau(t) = \mathrm{e}^{\frac{1}{2}q(t)}\theta(\mathcal{V}\cdot t + c) \tag{6.5}$$

is a solution of the KP hierarchy, where

$$\mathcal{V}\cdot t + c = {}^t\left(\sum_{i=1}^{\infty} v_{1,i}t_i + c_1, \dots, \sum_{i=1}^{\infty} v_{g,i}t_i + c_g\right).$$

We take (6.5) as $\tau(t)$ in Theorem 2.2 and derive the corresponding formula for $\theta(v)$. To this end we need to prepare some notation.

Let $\mathrm{d}r_k$, $k = 1, 2, \dots$, be the differentials of the second kind with a pole only at p_∞ normalized by the conditions

$$\int_{A_i} \mathrm{d}r_k = 0 \qquad \text{for any } i, \qquad \mathrm{d}r_k = \mathrm{d}\left(\frac{1}{z^k} + O(1)\right) \qquad \text{near } p_\infty.$$

It is well known that such $\mathrm{d}r_k$ exists and unique if a local coordinate around p_∞ is fixed. The differential $\mathrm{d}r_k$ can be described using the prime form. Let

$$\omega(x, y) = \mathrm{d}_x\mathrm{d}_y \log E(x, y) = \left(\frac{1}{(z-w)^2} + \sum_{i,j=1}^{\infty} q_{i,j}z^{i-1}w^{j-1}\right)\mathrm{d}z\mathrm{d}w$$

be the bilinear meromorphic differential on $\mathcal{C}\times\mathcal{C}$ with a double pole on the diagonal $\{x = y\}$. Due to the quasi-periodicity of the theta function, $\omega(x, y)$ satisfies

$$\int_{A_i(x)} \omega(x, y) = \int_{A_i(y)} \omega(x, y) = 0, \qquad 1 \le i \le g,$$

where $\int_{A_i(x)}$ signifies the integral over A_i in x variable and similarly for $\int_{A_i(y)}$.

We consider $w = z(y)$ as a parameter. Then we take the coefficient of $\mathrm{d}w$ of $\omega(x, y)$ and differentiate it several times in w. In this way, we get the differential in x variable which gives $\mathrm{d}r_k$. Explicitly,

$$\mathrm{d}r_k = -\frac{1}{(k-1)!}\frac{\partial^k}{\partial w^k}\mathrm{d}_z \log E(z, w)|_{w=0} = \mathrm{d}_z\left(\frac{1}{z^k} - \sum_{j=1}^{\infty} q_{j,k}\frac{z^j}{j}\right). \tag{6.6}$$

We denote $\int^x \mathrm{d}r_k$ the indefinite integral of $\mathrm{d}r_k$ such that the expansion near p_∞ in $z = z(x)$ has no constant term, that is,

$$\int^x \mathrm{d}r_k = \frac{1}{z^k} - \sum_{j=1}^{\infty} q_{j,k}\frac{z^j}{j}.$$

In particular,

$$\int^x \mathrm{d}r_1 = -\left.\frac{\partial}{\partial w}\log E(z, w)\right|_{w=0}.$$

Consider points $p_1, \dots, p_n$ of $\mathcal{C}$ and define, for $1 \le j \le n$,

$$D_j = \sum_{k=1}^{g} v_{k,1}\frac{\partial}{\partial v_k} + \int^{p_j} \mathrm{d}r_1. \tag{6.7}$$

In terms of $\mathrm{d}v_k$ and $E(z,w)$, it can be written as

$$D_j = \sum_{k=1}^{g} \frac{\mathrm{d}v_k}{\mathrm{d}z}\Big|_{z=0} \frac{\partial}{\partial v_k} - \frac{\partial}{\partial w} \log E(z(p_j), w)\Big|_{w=0}. \tag{6.8}$$

With these notation, we can rewrite the formula in Theorem 2.2 in terms of $\theta(v)$ which is the main theorem in this section.

Theorem 6.2. *Let $n \geq 1$, $p_1, \dots, p_n$ be mutually distinct points of $\mathcal{C}$ different from p_∞ and $c \in \mathbb{C}^g$ a vector satisfying $\theta(c) \neq 0$. Then we have the following assertions:*

(1)

$$\begin{aligned}&\frac{\theta\big(\mathcal{V}_1 x + c - \sum_{j=1}^{n} \iota(p_j)\big) \prod_{i<j} E(p_j, p_i)}{\theta(\mathcal{V}_1 x + c) \prod_{i=1}^{n} E(p_\infty, p_i)^{n-1}} \mathrm{e}^{x \sum_{j=1}^{n} \int^{p_j} \mathrm{d}r_1} \\ &\qquad = \det\left(\partial^{i-1}\left(\frac{\theta\left(\mathcal{V}_1 x + c - \iota(p_j)\right)}{\theta(\mathcal{V}_1 x + c)} \mathrm{e}^{x \int^{p_j} \mathrm{d}r_1}\right)\right)_{1\leq i,j\leq n}.\end{aligned}$$

(2)

$$\frac{\theta\big(c - \sum_{j=1}^{n} \iota(p_j)\big) \prod_{i<j} E(p_j, p_i)}{\theta(c) \prod_{i=1}^{n} E(p_\infty, p_i)^{n-1}} = \det\left(\phi_i(p_j)\right)_{1\leq i,j\leq n},$$

where

$$\phi_i(p_j) = D_j^{i-1} \frac{\theta\left(v - \iota(p_j)\right)}{\theta(v)}\Big|_{v=c}, \qquad i \geq 1.$$

Remark 6.3. The formula (1), (2) of this theorem can be considered as a limit of Fay's general addition formula (1.1). It is the limit as all $p_1, \dots, p_n$ approach the same point p_∞ while keeping c so that $\theta(c) \neq 0$. In [3], two limits of (1.1) are computed. The first one is the limit as c tends to c' where c' is a non-singular point of $(\theta(v) = 0)$. The second one is the limit, in the first limit, as all p_j tend to the same point. Therefore, the formulas (1), (2) are different from those in [3, Corollary 2.19]. Moreover, in [3] the notion of x derivative is not introduced. The x derivative structure in Theorem 6.2 (2) is reflected in the fact that the coefficient of $\partial/\partial v_k$, $1 \leq k \leq g$, in D_j consists of only the first term of the expansion of $\mathrm{d}v_k$ at p_∞ (see (6.1), (6.7), (6.8)).

Proof of Theorem 6.2. We use the following formulas:

$$\mathrm{e}^{q([z]|[w])} = \frac{zw}{w-z} \frac{E(z,w)}{E(0,z)E(0,w)} \quad \text{for } z \neq w, \qquad \mathrm{e}^{\frac{1}{2}q([z]|[z])} = \frac{z}{E(0,z)} \tag{6.9}$$

$$q(t|[z]) = \eta\big(t, z^{-1}\big) - \sum_{i=1}^{\infty} t_i \int^{p} \mathrm{d}r_i, \tag{6.10}$$

$$\mathcal{V}[z] = \iota(p) = \int_{p_\infty}^{p} \mathrm{d}v, \tag{6.11}$$

where $z = z(p)$ in (6.10) and (6.11). Those formulas can be verified using (6.2), (6.3), (6.4), (6.6).

Firstly, let us calculate the left-hand side of the formula of Theorem 2.2 divided by $\tau(t)$. Set $\alpha_i = z(p_i)$, $p_i \in \mathcal{C}$. By (6.11), we have

$$\tau\left(t - \sum_{i=1}^{n} [\alpha_i]\right) = \mathrm{e}^{\frac{1}{2} q(t - \sum_{i=1}^{n} [\alpha_i])} \theta\left(\mathcal{V} \cdot t - \sum_{i=1}^{n} \iota(p_i) + c\right).$$

Then, using (6.9), (6.10), we have

$$\frac{\tau\left(t-\sum_{i=1}^n[\alpha_i]\right)}{\tau(t)} = \frac{\prod_{i=1}^n \alpha_i}{\prod_{i<j}\left(\alpha_i^{-1}-\alpha_j^{-1}\right)} \frac{\prod_{i<j} E(\alpha_i,\alpha_j)}{\prod_{i=1}^n E(0,\alpha_i)^n} \mathrm{e}^{-\sum_{i=1}^n \eta(t,\alpha_i^{-1})} \\ \times \frac{\theta\left(\mathcal{V}\cdot t-\sum_{i=1}^n \iota(p_i)+c\right)}{\theta(\mathcal{V}\cdot t+c)} \mathrm{e}^{\sum_{i=1}^\infty t_i \sum_{j=1}^n \int^{p_j} \mathrm{d}r_i}. \tag{6.12}$$

Next, let us calculate the Wronskian part of the right-hand side of the formula in Theorem 2.2. The wave function at $z=\alpha_j$ becomes

$$\Psi(t,\alpha_j) = \frac{\alpha_j}{E(0,\alpha_j)} \frac{\theta(\mathcal{V}\cdot t+c-\iota(p_j))}{\theta(\mathcal{V}\cdot t+c)} \mathrm{e}^{\sum_{k=1}^\infty t_k \int^{p_j} \mathrm{d}r_k}.$$

Then

$$\mathrm{Wr}(\Psi(t,\alpha_1),\dots,\Psi(t,\alpha_n)) \\ = \prod_{i=1}^n \frac{\alpha_i}{E(0,\alpha_i)} \det\left(\partial^{i-1}\left(\frac{\theta(\mathcal{V}\cdot t+c-\iota(p_j))}{\theta(\mathcal{V}\cdot t+c)} \mathrm{e}^{x\int^{p_j}\mathrm{d}r_1}\right)\right) \mathrm{e}^{\sum_{j=1}^n\sum_{k=2}^\infty t_k \int^{p_j}\mathrm{d}r_k}. \tag{6.13}$$

By (6.12) and (6.13), the formula of Theorem 2.2 becomes

$$\frac{\theta\left(\mathcal{V}\cdot t-\sum_{i=1}^n \iota(p_i)+c\right)}{\theta(\mathcal{V}\cdot t+c)} \mathrm{e}^{x\sum_{j=1}^n \int^{p_j}\mathrm{d}r_1} \\ = \frac{\prod_{i=1}^n E(0,\alpha_i)^{n-1}}{\prod_{i<j} E(\alpha_j,\alpha_i)} \det\left(\partial^{i-1}\left(\frac{\theta(\mathcal{V}\cdot t+c-\iota(p_j))}{\theta(\mathcal{V}\cdot t+c)} \mathrm{e}^{x\int^{p_j}\mathrm{d}r_1}\right)\right). \tag{6.14}$$

The first product term of the right-hand side can be written as

$$\frac{\prod_{i=1}^n E(0,\alpha_i)^{n-1}}{\prod_{i<j} E(\alpha_j,\alpha_i)} = \frac{\prod_{i=1}^n \frac{E(0,z(p_i))^{n-1}}{\sqrt{\mathrm{d}z(p_i)}^{\,n-1}}}{\prod_{i<j} \frac{E(z(p_j),z(p_i))}{\sqrt{\mathrm{d}z(p_j)}\sqrt{\mathrm{d}z(p_i)}}} = \frac{\prod_{i=1}^n E(p_\infty,p_i)^{n-1}}{\prod_{i<j} E(p_j,p_i)}.$$

Substituting this into (6.14) and setting $t_1=x$, $t_j=0$ for $j\geq 2$, we get (1) of the theorem.

The formula (2) of the theorem follows from (1) as follows. If we set

$$f(v) = \frac{\theta(v-\iota(p_j))}{\theta(v)}, \qquad v={}^t(v_1,\dots,v_g),$$

then

$$\partial(f(\mathcal{V}_1 x+c)) = \sum_{k=1}^g v_{k,1} \left.\frac{\partial f(v)}{\partial v_k}\right|_{v=\mathcal{V}_1 x+c}.$$

Therefore,

$$\mathrm{e}^{-x\int^{p_j}\mathrm{d}r_1}\partial^{i-1}\left(\mathrm{e}^{x\int^{p_j}\mathrm{d}r_1} f(\mathcal{V}_1 x+c)\right) \\ = \left(\mathrm{e}^{-x\int^{p_j}\mathrm{d}r_1}\cdot\partial\cdot\mathrm{e}^{x\int^{p_j}\mathrm{d}r_1}\right)^{i-1} f(\mathcal{V}_1x+c) \\ = \left(\partial+\int^{p_j}\mathrm{d}r_1\right)^{i-1} f(\mathcal{V}_1x+c) \\ = \left.\left(\left(\sum_{k=1}^g v_{k,1}\frac{\partial}{\partial v_k}+\int^{p_j}\mathrm{d}r_1\right)^{i-1} f(v)\right)\right|_{v=\mathcal{V}_1x+c} = \left.\left(D_j^{i-1}f(v)\right)\right|_{v=\mathcal{V}_1x+c}. \tag{6.15}$$

Now, multiply the formula (1) of the theorem by $\mathrm{e}^{-x\sum_{j=1}^n\int^{p_j}\mathrm{d}r_1}$, use (6.15) and set $x=0$. Then we get (2). ■

The case of $g = 1$. In the $g = 1$ case, a non-singular odd half period used to define the prime form is explicitly known so that the formulas in Theorem 6.2 can be rewritten in terms of theta functions only without using the prime form. We shall give them here.

Let ω_1, ω_2 be complex numbers such tat $\Omega = \omega_2/\omega_1$ has positive imaginary part, $\sigma(u)$, $\zeta(u)$, $\wp(u)$ Weierstrass sigma, zeta, $\wp$-functions corresponding to $2\omega_1$, $2\omega_2$ respectively [27]. We denote by X the Weierstrass cubic $y^2 = 4x^3 - g_2x - g_3$, which is parametrized as $(x, y) = (\wp(u), \wp'(u))$. The segments $0(2\omega_1)$, $0(2\omega_2)$ define closed curves on X which we denote by A, B. Then $\{A, B\}$ is a basis of the homology group of X with the intersections $A \cdot A = B \cdot B = 0$, $A \cdot B = 1$. Let $\mathrm{d}u = \mathrm{d}x/y$. Then

$$\int_A \mathrm{d}u = 2\omega_1, \qquad \int_B \mathrm{d}u = 2\omega_2,$$

and the normalized differential of the first kind becomes $\mathrm{d}v = \frac{1}{2\omega_1}\mathrm{d}u$. The non-singular odd half period of X is $\Omega/2 + 1/2$ whose characteristics is $\epsilon = {}^t(1/2, 1/2)$. We set $\theta_{11}(v) = \theta[\epsilon](v, \Omega)$. We take $p_\infty = \infty$ as a base point and set $v(p) = \int_\infty^p \mathrm{d}v$, which we adopt as a local coordinate around ∞, $z = v(p)$. Then $v_{1,j} = \delta_{1,j}$ and $\mathcal{V}_1 = {}^t(1, 0, 0, \dots)$. We have $h_\epsilon^2(p) = \theta'_{11}(0)\mathrm{d}v(p)$, and the prime form is given by

$$E(p, q) = \frac{\theta_{11}\big(\int_p^q \mathrm{d}v\big)}{h_\epsilon(p)h_\epsilon(q)} = \frac{\theta'_{11}(0)^{-1}\theta_{11}(z(q) - z(p))}{\sqrt{\mathrm{d}z(p)}\sqrt{\mathrm{d}z(q)}},$$

where $\theta'_{11}(v)$ denotes the derivative of $\theta_{11}(v)$. Therefore, $E(z, w) = \theta'_{11}(0)^{-1}\theta_{11}(w - z)$. We also have

$$E(p_\infty, p) = \frac{E(0, z(p))}{\sqrt{\mathrm{d}z(p)}} = \frac{\theta'_{11}(0)^{-1}\theta_{11}(z(p))}{\sqrt{\mathrm{d}z(p)}}.$$

The normalized differential of the second kind $\mathrm{d}r_1$ is given by

$$\mathrm{d}r_1 = \frac{\mathrm{d}^2}{\mathrm{d}z^2}\log\theta_{11}(z)\mathrm{d}z.$$

Set $\widehat{\zeta}(z) = \frac{\mathrm{d}}{\mathrm{d}z}\log\theta_{11}(z)$. Then $D_j = \frac{\partial}{\partial v} + \widehat{\zeta}(\alpha_j)$, $\alpha_j = z(p_j)$.

Theorem 6.2 for $g = 1$: In terms of Jacobi–Riemann theta functions. Let $n \ge 1$, $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ be parameters which are mutually distinct and non-zero mod $\mathbb{Z} + \mathbb{Z}\Omega$ and $c \in \mathbb{C}$.

(1) The following equation holds as a function of x:

$$\begin{aligned}&\frac{\theta\big(x + c - \sum_{j=1}^n \alpha_j\big)\prod_{i<j}\theta_{11}(\alpha_i - \alpha_j)}{\theta(x + c)\prod_{i=1}^n\theta_{11}(\alpha_i)^{n-1}}\mathrm{e}^{x\sum_{j=1}^n\widehat{\zeta}(\alpha_j)}\\ &\quad = \theta'_{11}(0)^{-\frac{1}{2}n(n-1)}\det\left(\partial^{i-1}\left(\frac{\theta(x + c - \alpha_j)}{\theta(x + c)}\mathrm{e}^{x\widehat{\zeta}(\alpha_j)}\right)\right)_{1\le i,j\le n}.\end{aligned} \tag{6.16}$$

(2) If $\theta(c) \ne 0$, we have

$$\frac{\theta\big(c - \sum_{j=1}^n \alpha_j\big)\prod_{i<j}\theta_{11}(\alpha_i - \alpha_j)}{\theta(c)\prod_{i=1}^n\theta_{11}(\alpha_i)^{n-1}} = \theta'_{11}(0)^{-\frac{1}{2}n(n-1)}\det\left(\phi_i(\alpha_j)\right)_{1\le i,j\le n},$$

where

$$\phi_i(\alpha_j) = D_j^{i-1}\left(\frac{\theta(v - \alpha_j)}{\theta(v)}\right)\bigg|_{v=c}, \qquad i \ge 1.$$

Next, we rewrite these formulas in terms of Weierstrass functions $\sigma(u)$, $\zeta(u)$.

Firstly, replace c by $c+\Omega/2+1/2$ in (6.16). Then replace x, c, α_j by $x/2\omega_1$, $c/2\omega_1$, $\alpha_j/2\omega_1$, respectively. After that use

$$\theta_{11}\left(\frac{u}{2\omega_1}\right)=\frac{\theta_{11}'(0)}{2\omega_1}\mathrm{e}^{-\frac{\eta_1}{2\omega_1}u^2}\sigma(u), \qquad \eta_i=\zeta(\omega_i),$$
$$\theta\left(v+\frac{\Omega}{2}+\frac{1}{2}\right)=-\mathrm{i}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}\Omega-\pi\mathrm{i}v}\theta_{11}(v).$$

We express the final formula using the Lamé function [27] (or Baker–Akhiezer function of genus one [12]) defined by

$$\Phi(x,\alpha)=\frac{\sigma(x-\alpha)}{\sigma(x)\sigma(\alpha)}\mathrm{e}^{x\zeta(\alpha)}.$$

Set $\Phi^{(i)}(x,\alpha)=\partial^i\Phi(x,\alpha)$. Then we have the following formulas.

Theorem 6.2 for $g=1$: In terms of Weierstrass functions. Let $n\geq 1$ and $\alpha_1,\dots,\alpha_n$ be parameters which are mutually distinct and non-zero mod $\mathbb{Z}(2\omega_1)+\mathbb{Z}(2\omega_2)$.

(1) The following equation holds as a function of x:

$$\frac{\sigma\big(x-\sum_{i=1}^n\alpha_j\big)\prod_{i<j}\sigma(\alpha_i-\alpha_j)}{\sigma(x)\prod_{i=1}^n\sigma(\alpha_i)^n}\mathrm{e}^{x\sum_{j=1}^n\zeta(\alpha_j)}=\det\big(\Phi^{(i-1)}(x,\alpha_j)\big)_{1\leq i,j\leq n}. \tag{6.17}$$

(2) If $c\in\mathbb{C}$ satisfies $\sigma(c)\neq 0$, we have

$$\frac{\sigma\big(c-\sum_{i=1}^n\alpha_j\big)\prod_{i<j}\sigma(\alpha_i-\alpha_j)}{\sigma(c)\prod_{i=1}^n\sigma(\alpha_i)^{n-1}}=\det\big(\tilde{\phi}_i(\alpha_j)\big)_{1\leq i,j\leq n},$$

where

$$\tilde{\phi}_i(\alpha_j)=\left(\frac{\partial}{\partial v}+\zeta(\alpha_j)\right)^{i-1}\left(\frac{\sigma(v-\alpha_j)}{\sigma(v)}\right)\Big|_{v=c}.$$

Remark 6.4. Formula (6.17) is proved in [13, Lemma 3.1] using the Frobenius–Stickelberger formula (see [27], for example) and some related formulas proved in [18, 19].

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