Symmetry, Integrability and Geometry: Methods and Applications SIGMA **6** (2010), 090, 12 pages

On a Family of 2-Variable Orthogonal Krawtchouk Polynomials

F. Alberto GRÜNBAUM † *and Mizan RAHMAN* ‡

† *Department of Mathematics, University of California, Berkeley, CA 94720, USA*
E-mail: *grunbaum@math.berkeley.edu*
URL: `http://www.math.berkeley.edu/~grunbaum/`

‡ *Department of Mathematics and Statistics, Carleton University, Ottawa, Canada, K1S 5B6*
E-mail: *mrahman@math.carleton.ca*

Received July 25, 2010, in final form December 01, 2010; Published online December 07, 2010
doi:10.3842/SIGMA.2010.090

Abstract. We give a hypergeometric proof involving a family of 2-variable Krawtchouk polynomials that were obtained earlier by Hoare and Rahman [*SIGMA* **4** (2008), 089, 18 pages] as a limit of the $9-j$ symbols of quantum angular momentum theory, and shown to be eigenfunctions of the transition probability kernel corresponding to a "poker dice" type probability model. The proof in this paper derives and makes use of the necessary and sufficient conditions of orthogonality in establishing orthogonality as well as indicating their geometrical significance. We also derive a 5-term recurrence relation satisfied by these polynomials.

Key words: hypergeometric functions; Krawtchouk polynomials in 1 and 2 variables; Appell–Kampe–de Feriet functions; integral representations; transition probability kernels; recurrence relations

2010 Mathematics Subject Classification: 33C45

1 Introduction

It was in the SIDE8 meeting in St-Adele near Montreal that one of us (MR) presented a paper reporting the discovery, by his co-author Michael Hoare and himself [2008], of a "new" system of 2-variable Krawtchouk polynomials, orthogonal with respect to a trinomial distribution. The motivation of their paper was to find eigenvalues and eigenfunctions of the transition probability kernel:

$$K_A(j_1,j_2;i_1,i_2) = \sum_{k_1=0}^{\min(i_1,j_1)}\sum_{k_2=0}^{\min(i_2,j_2)} b(k_1,i_1;\alpha_1)b(k_2,i_2;\alpha_2) \times b_2(j_1-k_1,j_2-k_2;N-k_1-k_2;\beta_1,\beta_2), \tag{1.1}$$

where $b(x,N;p) = \binom{N}{x}p^x(1-p)^{N-x}$ is the binomial distribution, while

$$b_2(x,y;N;p,q) = \binom{N}{x,y}p^xq^y(1-p-q)^{N-x-y},$$

is the trinomial, both normalized to 1. In (1.1) the parameters $(\alpha_1,\alpha_2,\beta_1,\beta_2)$ are probabilities of a two-step cumulative Bernoulli process, and hence necessarily in $(0,1)$, while (i_1,i_2) and (j_1,j_2) represent the initial and final states of the process. Borrowing a result from the angular momentum theory of quantum mechanics the authors of [12] were able to show that the 2-dimensional Krawtchouk polynomials

$$\sum_{i}\sum_{j}\sum_{k}\sum_{l}{}_{0\le i+j+k+l\le N} \frac{(-m)_{i+j}(-n)_{k+l}(-x)_{i+k}(-y)_{j+l}}{i!j!k!l!(-N)_{i+j+k+l}} u_1^i v_1^j u_2^k v_2^l \tag{1.2}$$

do indeed satisfy the requirements for them being the eigenfunctions of (1.1), where (x, y) represents the state-variable and (m, n) the spectral parameters. It goes without saying that for (1.2) to be an orthogonal system with respect to a distribution of the form $b_2(x, y; N; \eta_1, \eta_2)$ the parameters u_1, v_1, u_2, v_2 must be related to η_1, η_2, as well as satisfy some additional conditions among themselves. It was found in [12], again with a cue from the physics literature that these conditions are all satisfied provided the u's and v's are parametrized in the following way

$$\begin{aligned} u_1 &= \frac{(p_1+p_2)(p_1+p_3)}{p_1(p_1+p_2+p_3+p_4)}, \qquad & u_2 &= \frac{(p_1+p_2)(p_2+p_4)}{p_2(p_1+p_2+p_3+p_4)}, \\ v_1 &= \frac{(p_1+p_3)(p_4+p_3)}{p_3(p_1+p_2+p_3+p_4)}, \qquad & v_2 &= \frac{(p_2+p_4)(p_3+p_4)}{p_4(p_1+p_2+p_3+p_4)}, \end{aligned} \tag{1.3}$$

and consequently,

$$\eta_1 = \frac{p_1p_2(p_1+p_2+p_3+p_4)}{(p_1+p_2)(p_1+p_3)(p_2+p_4)}, \qquad \eta_2 = \frac{p_3p_4(p_1+p_2+p_3+p_4)}{(p_2+p_4)(p_3+p_4)(p_1+p_3)}. \tag{1.4}$$

For the origin of the work in [12] the reader may consult [3, 9, 10, 11].

Fortunately, in the audience, a very attentive listener, Masatoshi Noumi, was there to point out to (MR) that these polynomials are not only not new, but a special case of the multivariable generalization of the Gaussian hypergeometric function:

$$F_1^{(n)}(-\mathbf{x}, -\mathbf{m}; -N; \mathbf{u}) = \sum \frac{\prod\limits_{i=1}^{n}(-x_i)_{\sum\limits_{j=1}^{n}\alpha_{ij}} \prod\limits_{i=1}^{n}(-m_i)_{\sum\limits_{j=1}^{n}\alpha_{ji}}}{(-N)_{\sum\limits_{i,j}\alpha_{ij}}} \frac{\prod u_{ij}^{\alpha_{ij}}}{\prod \alpha_{ij}!}, \tag{1.5}$$

where the α_{ij}'s are nonnegative integers taking values from 0 to n, such that $\sum\limits_{i,j}\alpha_{ij} \le N$, N being assumed as a nonnegative integer. Here we are following the notation in [17]. In the original definition of Aomoto and Gelfand [2, 6], N need not be an integer, nor the x's and m's. Furthermore, the space on which their functions are introduced was a bit more general, a generalization we do not need for our purposes. To be sure, these authors' primary interest was not to look at (1.5) as a multidimensional extension of the Krawtchouk polynomials:

$$P_n(x) = {}_2F_1\big(-x, -n; -N; \eta^{-1}\big), \tag{1.6}$$

rather some structures that they contain. H. Mizukawa [17] proved that the functions in (1.5) are the zonal spherical functions on a very special class of Gelfand pairs made up of the reflection groups $G(r, 1, n)$ and the symmetric group S_n. For a very nice account of a way to obtain many discrete orthogonal polynomials in terms of certain $(n + 1, m + 1)$ hypergeometric functions see also the work of Mizukawa and Tanaka [19]. As mentioned above, we learned from Professor M. Noumi that these functions give the multivariable Krawtchouk polynomials independently obtained in [12]. In a very recent paper by Mizukawa, see [18], he has established the orthogonality of Krawtchouk polynomials in n variables by using very different techniques from the ones in this paper. See also the additional comment at the end of our paper.

The origin of the work in [3, 9, 10, 11, 12] is the analysis of a very concrete probabilistic model, namely "poker dice". Its corresponding eigenfunctions are seen in [12] to be given in terms of a family of polynomials that, as indicated above, are now identified with the Gelfand–Aomoto polynomials. It is likely that this may be the first probabilistic application of the Gelfand–Aomoto polynomials. They may also be applicable to other models in the physical sciences.

One should point out that the hypergeometric functions involve both parameters as well as variables. Depending on the issue at hand one can consider these functions as depending on one or the other set of "variables". This is already the case in the classical one variable case as indicated by the expression (1.6) above. The fact that these Krawtchouk polynomials (or the higher level Hahn polynomials) could be so useful in analyzing naturally appearing models in statistical mechanics was not anticipated in the classical book by W. Feller [5], where one can read about the Ehrenfest as well as the Bernoulli–Laplace models. For several applications of the Krawtchouk polynomials to several parts of mathematics see [16]. For a very good general guide to the field see [1].

Our objective in this paper is less ambitious in one sense and more in another – namely, that we still restrict ourselves to the $n = 2$ case, but not necessarily on the reflection group, but to the general situations where the parameters u_{ij}'s will be determined by the requirement of orthogonality. Hoare and Rahman [12] have done that problem, but we will approach it from a different angle. We will refrain from parametrizing the u_{ij} right at the outset, instead looking for conditions they must satisfy among them in order that the 2-variable polynomials:

$$\begin{aligned} &F_1^{(2)}(-m_1,-m_2;-x_1,-x_2;-N;u_1,v_1,u_2,v_2) \\ &\quad := \sum \frac{(-m_1)_{i+j}(-m_2)_{k+l}(-x_1)_{i+k}(-x_2)_{j+l}}{i!j!k!l!(-N)_{i+j+k+l}} u_1^i v_1^j u_2^k v_2^l \equiv P_{m_1,m_2}(x_1,x_2) \end{aligned} \tag{1.7}$$

become orthogonal with respect to the trinomial:

$$b_2(x_1,x_2;N;\eta_1,\eta_2) = \binom{N}{x_1,x_2} \eta_1^{x_1}\eta_2^{x_2}(1-\eta_1-\eta_2)^{N-x_1-x_2}.$$

It may be worth mentioning that a prior knowledge of this weight function is not essential since one could easily derive it by using the binomial generating function of the polynomials $P_{m,n}(x,y)$.

We now state the main results in the paper, namely (1.8) and (1.9) below.

In Sections 2, 3 and 4 we will show that the necessary and sufficient conditions of orthogonality are:

$$\begin{aligned} &\text{(a)} \quad \eta_1 u_1 + \eta_2 v_1 = 1, \\ &\text{(b)} \quad \eta_1 u_2 + \eta_2 v_2 = 1, \\ &\text{(c)} \quad \eta_1 u_1 u_2 + \eta_2 v_1 v_2 = 1, \end{aligned} \tag{1.8}$$

with the η's assumed to be given such that $0 < \eta_1, \eta_2 < 1$ and $\eta_1 + \eta_2 < 1$. One can easily verify that these three conditions are all satisfied by (1.3) and (1.4).

However, one of the main reasons for going back to this problem is to find a 5-term recurrence relation for (1.7), which interestingly, is more easily found by using the p's as in (1.3) and (1.4) than using (1.8) instead. In Section 5 we'll show that, if we denote (1.7) by $P_{m_1,m_2}(x_1,x_2)$ then

$$\begin{aligned} (N-m_1-m_2)\bigg\{ &\frac{p_1p_3(p_2+p_4)(p_1+p_2+p_3+p_4)}{(p_1+p_3)(p_1p_4-p_2p_3)}(P_{m_1+1,m_2}(x_1,x_2)-P_{m_1,m_2}(x_1,x_2)) \\ &- \frac{p_2p_4(p_1+p_3)(p_1+p_2+p_3+p_4)}{(p_2+p_4)(p_1p_4-p_2p_3)}(P_{m_1,m_2+1}(x_1,x_2)-P_{m_1,m_2}(x_1,x_2))\bigg\} \end{aligned}$$

$$
\begin{aligned}
&+ m_1\frac{p_1p_4-p_2p_3}{p_1+p_3}(P_{m_1-1,m_2}(x_1,x_2)-P_{m_1,m_2}(x_1,x_2))\\
&- m_2\frac{p_1p_4-p_2p_3}{p_2+p_4}(P_{m_1,m_2-1}(x_1,x_2)-P_{m_1,m_2}(x,y))\\
&= ((p_1+p_2)x_1-(p_3+p_4)x_2)P_{m_1,m_2}(x_1,x_2).
\end{aligned}\tag{1.9}
$$

This recursion relation is valid when the variables x_1, x_2 are nonnegative integers taking values whose sum is at most N.

If we insist on a difference operator in the variables m_1, m_2 with an eigenvalue that is linear in x_1, x_2 and involves only the four nearest neighbours of m_1, m_2 this is essentially the only choice. This was the result of extensive symbolic computations carried out beginning at the time that [8] was written. This statement was proved in general in [14] a paper that kindly acknowledges this work carried out initially in a special case. If one allows the eight nearest neighbours we get another linearly independent difference operator, a fact also found by symbolic computation by us and independently proved in [14].

There is by now a rather large literature dealing with orthogonal polynomials in several variables. A reference that is still useful is [13, Vol. 2]. A comprehensive treatment is found in [4]. For some of the issues that we are interested in the reader can see [15, 7] and the references in these papers.

It may be useful to point out that when the conditions (1.8) are not met the polynomials $P_{m,n}$ still satisfy difference equations in the indices (m,n), of the type given in [15, 7]. When the conditions (1.8) are met these recursions become much simpler in that they involve a smaller number of neighbouring indices. Having a recursion relation involving the smallest number of neighbours of the index (m,n) might be important in certain numerical implementations of these recursions as well as in potential signal processing applications of these polynomials. In those situations, having a minimal number of sampling points could be a useful feature.

In dealing with the same polynomials Iliev and Terwilliger, see the very recent nice paper [14], have found two 7-term recurrence relations. In fact a referee has kindly pointed out that our 5-term relation (1.9) can be derived by using a combination of these.

It is clear that both in [14] and [18] the replacement of the variables $u_{i,j}$ by the consideration of a matrix with entries $1-u_{i,j}$, properly augmented, played a very important role. This matrix is considered in [14] and an interpretation is given in terms of Lie algebras. In [18] the author proves that the orthogonality of the columns of this matrix with respect to a weight built out of the η_i is the appropriate extension of (1.8).

The $F_1^{(2)}$ notation used in the 2-variable case (1.7) and more generally in the n-variable case (1.5) is a reflection of the fact that these are generalizations of the standard Appell–Kampé de Fériet function

$$
F_1(a;b,b';c;x,y)=\sum_i\sum_j\frac{(a)_{i+j}(b)_i(b')_j}{i!j!(c)_{i+j}}x^iy^j.
$$

A very useful integral representation of this F_1 function is the double integral

$$
\frac{\Gamma(c)}{\Gamma(b)\Gamma(b')\Gamma(c-b-b')}\int_0^1\int_0^1\xi_1^{b-1}\xi_2^{b'-1}(1-\xi_1-\xi_2)^{c-b-b'-1}(1-\xi_1x-\xi_2y)^{-a}d\xi_1d\xi_2,
$$

provided

$$
0<\mathrm{Re}(b,b',c-b-b').
$$

This extends to $F_1^{(2)}$ as well, which can be easily verified:

$$
F_1^{(2)}(a_1,a_2;b_1,b_2;c;u_1,v_1,u_2,v_2)=\frac{\Gamma(c)}{\Gamma(a_1)\Gamma(a_2)\Gamma(c-a_1-a_2)}
$$

$$\times \int_0^1 \int_0^1 \xi_1^{a_1-1}\xi_2^{a_2-1}(1-\xi_1-\xi_2)^{c-a_1-a_2-1}(1-u_1\xi_1-u_2\xi_2)^{-b_1}$$
$$\times (1-v_1\xi_1-v_2\xi_2)^{-b_2}d\xi_1 d\xi_2, \tag{1.10}$$

which we shall find very useful in our calculations, even though the parameters in the case of (1.7) do not satisfy the convergence conditions of the integral in (1.10). We will take the point of view that whatever identities we find by using (1.10) with $0 < \mathrm{Re}(a_1, a_2, c - a_1 - a_2)$, are also valid where a_1, a_2, c are, in fact, negative integers.

For the direct hypergeometric proof that we are planning to give in the following pages it will be necessary to make use of the transformation formulas:

$$\begin{aligned} &F_1^{(2)}(a_1, a_2; b_1, b_2; c; u_1, v_1, u_2, v_2) \\ &\quad = (1-v_1)^{-a_1}(1-v_2)^{-a_2}F_1^{(2)}\left(a_1, a_2; b_1, c-b_1-b_2; c; \frac{u_1-v_1}{1-v_1}, \frac{-v_1}{1-v_1}, \frac{u_2-v_2}{1-v_2}, \frac{-v_2}{1-v_2}\right) \\ &\quad = (1-u_1)^{-a_1}(1-u_2)^{-a_2}F_1^{(2)}\left(a_1, a_2; c-b_1-b_2, b_2; c; \frac{-u_1}{1-u_1}, \frac{v_1-u_1}{1-u_1}, \frac{-u_2}{1-u_2}, \frac{v_2-u_2}{1-u_2}\right), \end{aligned} \tag{1.11}$$

which were proved in Hoare and Rahman [12]. But there is a third transformation that we shall find occasions to use, that is valid when (m_1, m_2) and (x_1, x_2) are pairs of nonnegative integers, as is N, satisfying the triangle inequality: $0 \le m_1 + m_2 \le N$, $0 \le x_1 + x_2 \le N$, and that is

$$F_1^{(2)}(-m_1, -m_2; -x_1, -x_2; -N; u_1, v_1, u_2, v_2) = \frac{(x_1+x_2-N)_{m_1+m_2}}{(-N)_{m_1+m_2}} \tag{1.12}$$
$$\times F_1^{(2)}(-m_1, -m_2; -x_1, -x_2; N+1-x_1-x_2-m_1-m_2; 1-u_1, 1-v_1, 1-u_2, 1-v_2),$$

which is just a generalization of the transformation:

$$\begin{aligned} {}_2F_1(-m, -x; -N; u) &= \frac{(x-N)_m}{(-N)_m}{}_2F_1(-m, -x; N+1-x-m; 1-u) \\ &= \frac{(m-N)_x}{(-N)_x}{}_2F_1(-m, -x; N+1-x-m; 1-u). \end{aligned} \tag{1.13}$$

In fact (1.12) and (1.13) easily extend to the multidimensional case $F_1^{(n)}$, provided one is dealing with terminating series. It may be remarked here that for (1.12) and (1.13) to be true, indeed in the general case of $F_1^{(n)}$, the parameter N need not even be an integer.

2 A general expression for orthogonality sum and proof of (1.8)

Let us denote

$$I_{m_1,m_2}^{n_1,n_2} = \sum_{\substack{x_1,x_2 \\ 0\le x_1+x_2\le N}}\sum b_2(x_1, x_2; N; \eta_1, \eta_2)P_{m_1,m_2}(x_1, x_2)P_{n_1,n_2}(x_1, x_2).$$

At $(m_1, m_2) = (0, 0)$, and $(n_1, n_2) \ne (0, 0)$, this simply represents a generating function for these polynomials, namely:

$$I_{0,0}^{n_1,n_2} = (1-\eta_1 u_1 - \eta_2 v_1)^{n_1}(1-\eta_1 u_2-\eta_2 v_2)^{n_2}.$$

So, at the three points $(0, 0)$, $(1, 0)$ and $(0, 1)$ the pairwise orthogonality between the first and the last two simply amounts to the conditions (a) and (b) given in (1.8). To obtain the condition at the points $(1, 0)$ and $(0, 1)$ we need some more computations.

For ease of computation we will imagine, for the time being, that $-m_1$, $-m_2$, $-N$ are complex numbers a_1, a_2, c such that $0 \le \mathrm{Re}(a_1, a_2, c - a_1 - a_2)$. Then using (1.10) we get

$$
\begin{aligned}
I^{n_1,n_2}_{m_1,m_2} = {}& \frac{\Gamma(c)}{\Gamma(a_1)\Gamma(a_2)\Gamma(c-a_1-a_2)} \int_0^1 \int_0^1 d\xi_1 d\xi_2 \xi_1^{a_1-1}\xi_2^{a_2-1}(1-\xi_1-\xi_2)^{c-a_1-a_2-1} \\
& \times \sum_{x_1}\sum_{x_2} b_2(x_1,x_2;N;\eta_1,\eta_2)P_{n_1,n_2}(x_1,x_2)(1-\xi_1u_1-\xi_2u_2)^{x_1}(1-\xi_1v_1-\xi_2v_2)^{x_2}.
\end{aligned}
$$

Clearly

$$
\begin{aligned}
&\sum_{x_1,x_2}\sum b_2(x_1,x_2;N;\eta_1,\eta_2)(-x_1)_{i+k}(-x_2)_{j+l}(1-\xi_1u_1-\xi_2u_2)^{x_1}(1-\xi_1v_1-\xi_2v_2)^{x_2} \\
&\quad = (-N)_{i+j+k+l}\eta_1^{i+k}\eta_2^{j+l}(1-\xi_1u_1-\xi_2u_2)^{i+k}(1-\xi_1v_1-\xi_2v_2)^{j+l} \\
&\qquad \times \{1-\xi_1(\eta_1u_1+\eta_2v_1)-\xi_2(\eta_1u_2+\eta_2v_2)\}^{N-i-j-k-l} \\
&\quad = (-N)_{i+j+k+l}(\eta_1(1-\xi_1u_1-\xi_2u_2))^{i+k}(\eta_2(1-\xi_1v_1-\xi_2v_2))^{j+l},
\end{aligned}
\tag{2.1}
$$

by virtue of (1.8)(a) and (1.8)(b). For general (m_1, m_2) and (n_1, n_2), we use (2.1) and recast back to the original parameters, getting

$$
\begin{aligned}
I^{n_1,n_2}_{m_1,m_2} = {}& \sum \frac{(-n_1)_{i+j}(-n_2)_{k+l}}{i!j!k!l!}(\eta_1u_1)^i(\eta_2v_1)^j(\eta_1u_2)^k(\eta_2v_2)^l \\
& \times \frac{(-i-j-k-l)_{m_1+m_2}}{(-N)_{m_1+m_2}} F_1^{(2)}(-m_1,-m_2;-i-k,-j-l;-i-j-k-l;u_1,v_1,u_2,v_2).
\end{aligned}
\tag{2.2}
$$

Let us take $(m_1, m_2) = (0, 1)$ and $(n_1, n_2) = (1, 0)$, so that

$$
\begin{aligned}
I^{1,0}_{0,1} &= \sum_{i,j} \frac{(-1)_{i+j}}{i!j!}(\eta_1u_1)^i(\eta_2v_1)^j\frac{(-i-j)}{(-N)}F_1(-1;-i,-j;-i-j;u_2,v_2) \\
&= \frac{(1-v_2)}{N}\sum_{i,j}\frac{(-1)_{i+j}}{i!j!}(\eta_1u_1)^i(\eta_2v_1)^j(i+j)_2F_1\left[\begin{matrix}-1,-i\\-i-j\end{matrix};\frac{u_2-v_2}{1-v_2}\right] \\
&= \frac{1-v_2}{N}\sum_{i,j}\frac{(-1)_{i+j}(\eta_1u_1)^i(\eta_2v_1)^j}{i!j!}j_2F_1\left[\begin{matrix}-1,-i\\j\end{matrix};\frac{1-u_2}{1-v_2}\right] \\
&= \frac{1-v_2}{N}\sum_{i,j}\frac{(-1)_{i+j}}{i!j!}(\eta_1u_1)^i(\eta_2v_1)^j\left(j+i\frac{1-u_2}{1-v_2}\right) \\
&= \frac{1-v_2}{N}\left(\eta_2v_1+\eta_1u_1\frac{1-u_2}{1-v_2}\right) = (\eta_1u_1(1-u_2)+\eta_2v_1(1-v_2))/N
\end{aligned}
$$

which must vanish, so (1.8)(c) must be satisfied in addition to (1.8)(a) and (1.8)(b).

It is worth noting that by solving the first two conditions of (1.8) one can show that the third condition amounts to

$$
U_1V_2 = U_2V_1,
\tag{2.3}
$$

where

$$
U_i = 1-u_i^{-1}, \qquad V_i = 1-v_i^{-1}, \qquad i = 1,2.
$$

Condition (2.3) has a simple geometrical interpretation as a cone embedded in four dimensional space. In a subsequent paper we will look at a geometrical interpretation for the corresponding orthogonality conditions in the case of more than two variables.

We would also like to point out that if $\overline{\eta}_1$ and $\overline{\eta}_2$ are parameters of $b_2(m_1,m_2;N;\overline{\eta}_1,\overline{\eta}_2)$ for the dual orthogonality of the P's, then they must satisfy

$$\begin{aligned} &\text{(a)}\quad \overline{\eta}_1u_1+\overline{\eta}_2u_2=1,\\ &\text{(b)}\quad \overline{\eta}_1v_1+\overline{\eta}_2v_2=1,\\ &\text{(c)}\quad \overline{\eta}_1u_1v_1+\overline{\eta}_2u_2v_2=1. \end{aligned}\tag{2.4}$$

3 Reduction of $I^{n_1,n_2}_{m_1,m_2}$

By the first transformation in (1.11) $F_1^{(2)}$ inside the sum in (2.2) becomes a multiple of F_1, which, transformed by (1.12) gives

$$\begin{aligned} I^{n_1,n_2}_{m_1,m_2} &= (1-v_1)^{m_1}(1-v_2)^{m_2}\sum_{i,j,k,l}\frac{(-n_1)_{i+j}(-n_2)_{k+l}}{i!j!k!l!}(\eta_1u_1)^i(\eta_2v_1)^j(\eta_1u_2)^k(\eta_2v_2)^l\\ &\quad\times\frac{(-j-l)_{m_1+m_2}}{(-N)_{m_1+m+2}}F_1\left(-i-k;-m_1,-m_2;j+l+1-m_1-m_2;\frac{1-u_1}{1-v_1},\frac{1-u_2}{1-v_2}\right). \end{aligned}$$

Set $i+k=r$, $j+l=s$, $i=r-k$, $j=s-l$, to get

$$\begin{aligned} I^{n_1,n_2}_{m_1,m_2} &= (1-v_1)^{m_1}(1-v_2)^{m_2}\sum_{r,s}\frac{(-n_1)_{r+s}}{r!s!}(\eta_1u_1)^r(\eta_2v_1)^s\\ &\quad\times F_1\left(-n_2;-r,-s;n_1+1-r-s;\frac{u_2}{u_1},\frac{v_2}{v_1}\right)\\ &\quad\times\frac{(-s)_{m_1+m_2}}{(-N)_{m_1+m_2}}F_1\left(-r;-m_1,-m_2;s+1-m_1-m_2;\frac{1-u_1}{1-v_1},\frac{1-u_2}{1-v_2}\right). \end{aligned}\tag{3.1}$$

Since

$$\begin{aligned} &F_1\left(-n_2;-r,-s;n_1+1-r-s;\frac{u_2}{u_1},\frac{v_2}{v_1}\right)\\ &\qquad=\frac{(-n_1-n_2)_{r+s}}{(-n_1)_{r+s}}F_1\left(-n_2;-r,-s;-n_1-n_2;1-\frac{u_2}{u_1},1-\frac{v_2}{v_1}\right) \end{aligned}$$

by (1.12), (3.1) reduces to

$$\begin{aligned} I^{n_1,n_2}_{m_1,m_2} &= \frac{(1-v_1)^{m_1}(1-v_2)^{m_2}}{(-N)_{m_1+m_2}}\sum_{r,s}\frac{(-n_1-n_2)_{r+s}}{r!s!}(\eta_1u_1)^r(\eta_2v_1)^s\\ &\quad\times F_1\left(-n_2;-r,-s;-n_1-n_2;1-\frac{u_2}{u_1},1-\frac{v_2}{v_1}\right)\\ &\quad\times(-s)_{m_1+m_2}F_1\left(-r;-m_1,-m_2;s+1-m_1-m_2;\frac{1-u_1}{1-v_1},\frac{1-u_2}{1-v_2}\right). \end{aligned}\tag{3.2}$$

To carry out the summations over r and s we employ the integral formula

$$F_1(a;b,b';c;x,y)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1\xi^{a-1}(1-\xi)^{c-a-1}(1-\xi x)^{-b}(1-\xi y)^{-b'}d\xi,$$

see [13, Vol. 1]. In our case $b=-r$, $b'=-s$, $x=1-\frac{u_2}{u_1}$, $y=1-\frac{v_2}{u_1}$, so in (3.2) we need to compute

$$\sum_{r,s}\frac{(-n_1-n_2)_{r+s}}{r!s!}\left(\eta_1u_1\left(1-\xi\left(1-\frac{u_2}{u_1}\right)\right)\right)^r\left(\eta_2v_1\left(1-\xi\left(1-\frac{v_2}{v_1}\right)\right)\right)^s(-r)_{i+j}$$

$$
\begin{aligned}
&\times (-s)_{m_1+m_2-i-j} = (-1)^{m_1+m_2}(-n_1-n_2)_{m_1+m_2} \\
&\times \left(\eta_1 u_1 \left(1-\xi\left(1-\frac{u_2}{u_1}\right)\right)\right)^{i+j} \left(\eta_2 v_1 \left(1-\xi\left(1-\frac{v_2}{v_1}\right)\right)\right)^{m_1+m_2-i-j} \\
&\times \{1-(1-\xi)(\eta_1 u_1+\eta_2 v_1)-\xi(\eta_1 u_1 u_2+\eta_2 v_1 v_2)\}^{n_1+n_2-m_1-m_2},
\end{aligned} \tag{3.3}
$$

with the implicit assumption that $n_1+n_2 \geq m_1+m_2$. However, by (1.8)(a) and (1.8)(c) the expression in $\{\ \}$ vanishes unless $n_1+n_2=m_1+m_2$. Therefore, (3.3) becomes

$$
\begin{aligned}
&(m_1+m_2)!(\eta_2 v_1)^{m_1+m_2}\left(\frac{\eta_1 u_1}{\eta_2 v_1}\right)^{i+j}\left(1-\xi\left(1-\frac{u_2}{u_1}\right)\right)^{i+j}\left(1-\xi\left(1-\frac{v_2}{v_1}\right)\right)^{m_1+m_2-i-j} \\
&\qquad \times \delta_{m_1+m_2,n_1+n_2},
\end{aligned}
$$

and consequently,

$$
\begin{aligned}
I^{n_1,n_2}_{m_1,m_2} &= \delta_{m_1+m_2,n_1+n_2}(1-v_1)^{m_1}(1-v_2)^{m_2}(\eta_2 v_1)^{m_1+m_2}\frac{(m_1+m_2)!}{(-N)_{m_1+m_2}} \\
&\quad \times \sum_{i,j}\frac{(-m_1)_i(-m_2)_j}{i!j!}\left(-\frac{\eta_1 u_1}{\eta_2 v_1}\right)^{i+j}\left(\frac{1-u_1}{1-v_1}\right)^i\left(\frac{1-u_2}{1-v_2}\right)^j \\
&\quad \times F_1\left(-n_2;-i-j,i+j-m_1-m_2;-m_1-m_2;1-\frac{u_2}{u_1},1-\frac{v_2}{v_1}\right) \\
&= \delta_{m_1+m_2,n_1+n_2}(1-v_1)^{m_1}(1-v_2)^{m_2}(\eta_2 v_1)^{m_1+m_2}\left(\frac{u_2}{u_1}\right)^{n_2}\frac{(m_1+m_2)!}{(-N)_{m_1+m_2}} \\
&\quad \times \sum_{i,j}\frac{(-m_1)_i(-m_2)_j}{i!j!}\left(-\frac{\eta_1 u_1}{\eta_2 v_1}\right)^{i+j}\left(\frac{1-u_1}{1-v_1}\right)^i\left(\frac{1-v_2}{1-u_2}\right)^j \\
&\quad \times {}_2F_1\left(-n_2;i+j-m_1-m_2;-m_1-m_2;1-\frac{u_1 v_2}{u_2 v_1}\right),
\end{aligned} \tag{3.4}
$$

by a special case of the last identity (1.11).

4 Final summations in (3.4)

At the last stage we will set $i+j=k$, $j=k-i$, so that the i-sum becomes

$$
\begin{aligned}
&{}_2F_1\left(-m_1,-k;m_2+1-k;\frac{(1-u_1)(1-v_2)}{(1-u_2)(1-v_1)}\right) \\
&\quad = \frac{(-m_1-m_2)_k}{(-m_2)_k}{}_2F_1\left(-m_1,-k;-m_1-m_2;1-\frac{(1-u_1)(1-v_2)}{(1-u_2)(1-v_1)}\right) \\
&\quad = \frac{(-m_1-m_2)_k}{(-m_2)_k}{}_2F_1\left(-m_1,-k;-m_1-m_2;1-\frac{u_1 v_2}{u_2 v_1}\right),
\end{aligned}
$$

by (2.3) and (1.13). Thus

$$
\begin{aligned}
I^{n_1,n_2}_{m_1,m_2} &= \delta_{m_1+m_2,n_1+n_2}(1-v_1)^{m_1}(1-v_2)^{m_2}(\eta_2 v_2)^{m_1+m_2}\frac{(m_1+m_2)!}{(-N)_{m_1+m_2}}\left(\frac{u_2}{u_1}\right)^{n_2} \\
&\times \sum_{k=0}^{m_1+m_2}\frac{(-m_1-m_2)_k}{k!}\sum_{i=0}^{m_1}\frac{(-m_1)_i(-k)_i}{i!(-m_1-m_2)_i}\sum_{j=0}^{n_2}\frac{(-n_2,k-m_1-m_2)_j}{j!(-m_1-m_2)_j}\left(1-\frac{u_1 v_2}{u_2 v_1}\right)^{i+j},
\end{aligned}
$$

since $-\frac{\eta_1 u_1}{\eta_2 v_1}\frac{(1-u_2)}{(1-v_2)} = 1$, because $\eta_1 u_1(1-u_2) + \eta_2 v_1(1-v_2) = 0$. But, now

$$\begin{aligned}
&\sum_{k=0} \frac{(-m_1-m_2)_k}{k!}(-k)_i(k-m_1-m_2)_j \\
&\qquad = (-1)^i(-m_1-m_2)_{i+j} \sum_{k=0}^{m_1+m_2-i-j} \frac{(i+j-m_1-m_2)_k}{k!} \\
&\qquad = (-1)^i(-m_1-m_2)_{m_1+m_2}\delta_{m_1+m_2,i+j} \\
&\qquad = (m_1+m_2)_{m_1+m_2}(-1)^{m_1+m_2-i}\delta_{m_1+m_2,i+j}.
\end{aligned}$$

So

$$\begin{aligned}
I^{n_1,n_2}_{m_1,m_2} &= \delta_{m_1+m_2,n_1+n_2}(1-v_1)^{m_1}(1-u_2)^{m_2}\left(\eta_2 v_1\left(1-\frac{u_1 v_2}{u_2 v_1}\right)\right)^{m_1+m_2}\left(\frac{u_2}{u_1}\right)^{n_2} \\
&\quad \times \frac{((m_1+m_2)!)^2}{(-N)_{m_1+m_2}} \sum_{i=0}^{m_1} \frac{(-m_1)_i(-1)^{m_1+m_2-i}}{i!(-m_1-m_2)_i} \frac{(-n_2)_{m_1+m_2-i}}{(-m_1-m_2)_{m_1+m_2-i}(m_1+m_2-i)!}.
\end{aligned}$$

The summand is 0 unless $n_2 \geq m_2 + m_1 - i \Rightarrow n_2 \geq m_2$, since $m_1 \geq i$. So we set $i = m_1 + m_2 - n_2 + l$, $l \geq 0$, and get, for the i-sum above

$$\begin{aligned}
&\frac{(-n_2)_{n_2}(-m_1)_{m_1+m_2-n_2}}{(-m_1-m_2)_{m_1+m_2}(m_1+m_2)!} \sum_{l=0}^{n_2-m_2} \frac{(m_2-n_2)_l}{l!} \\
&\qquad = \frac{(-m_2)_{m_2}(-m_1)_{m_1}}{(-m_1-m_2)_{m_1+m_2}(m_1+m_2)!}\delta_{m_2,n_2} = \frac{m_1!m_2!}{(m_1+m_2)!^2}\delta_{m_2,n_2} \quad \Rightarrow \quad m_1 = n_1
\end{aligned}$$

since $m_1 + m_2 = n_1 + n_2$. Thus,

$$\begin{aligned}
I^{n_1,n_2}_{m_1,m_2} &= \delta_{m_1,m_2}\delta_{n_1,n_2}(1-v_1)^{m_1}(1-v_2)^{m_2}\left(\frac{u_2}{u_1}\right)^{m_2}\left(-\eta_2 v_1\left(1-\frac{u_1 v_2}{u_2 v_1}\right)\right)^{m_1+m_2} \\
&\quad \times 1\Big/\binom{N}{m_1, m_2}.
\end{aligned}$$

To determine the coefficient in terms of $\overline{\eta}_1$ and $\overline{\eta}_2$, note that

$$\begin{aligned}
1-\frac{u_1 v_2}{u_2 v_1} &= 1 - \frac{(1-u_1)(1-v_2)}{(1-u_2)(1-v_1)} = \frac{u_1 - v_1 + v_2 - u_2 - (u_1 v_2 - u_2 v_1)}{(1-u_2)(1-v_1)} \\
&= -\frac{D(1-\overline{\eta}_1-\overline{\eta}_2)}{(1-u_2)(1-v_1)},
\end{aligned}$$

from solving (2.4), with $D \equiv u_1 v_2 - u_2 v_1$.

Now, from (2.4)(a) and (2.4)(b) we get

$$\overline{\eta_1} = (v_2-u_2)D^{-1}, \qquad \overline{\eta_2} = (u_1-v_1)D^{-1},$$

while (1.8)(a)–(1.8)(c) give

$$\begin{aligned}
\eta_1 &= \begin{vmatrix} 1 & v_1 \\ 1 & v_1 v_2 \end{vmatrix} \Big/ \begin{vmatrix} u_1 & v_1 \\ u_1 u_2 & v_1 v_2 \end{vmatrix} = \frac{(v_2-1)}{u_1(v_2-u_2)} = -\frac{1-v_1}{u_2(v_1-u_1)}, \\
\eta_2 &= \frac{1-u_2}{v_1(v_2-u_2)} = \frac{1-u_1}{v_2(v_1-u_1)}.
\end{aligned}$$

Hence

$$-\frac{D\eta_2 v_1(1-v_1)}{(1-u_2)(1-v_1)} = -\frac{D}{v_2-u_2} = -\frac{1}{\overline{\eta}_1},$$

from (2.4).

Now,

$$\eta_2 v_1(1-v_2) = -\eta_1 u_1(1-u_2), \qquad \eta_2 v_1(1-v_2)\frac{u_2}{u_1} = -\eta_1 u_2(1-u_2),$$

so

$$-\frac{D\eta_2 v_1(1-v_2)}{(1-u_2)(1-v_1)}\frac{u_2}{u_1} = \frac{D\eta_1 u_2}{(1-u_1)} = -\frac{D}{v_1-u_1} = -\frac{1}{\overline{\eta}_2}.$$

Thus the normalization factor is

$$\left(b_2(m_1,m_2;N;\overline{\eta}_1,\overline{\eta}_2)(1-\overline{\eta}_1-\overline{\eta}_2)^{-N}\right)^{-1}.$$

5 Proof of the recurrence relation (1.9)

By the transformation formula (1.11),

$$P_{m_1,m_2}(x_1,x_2) = (1-u_1)^{x_1}(1-v_1)^{x_2}\sum \frac{(m_1+m_2-N)_{i+j}(-m_2)_{k+l}(-x_1)_{i+k}(-x_2)_{j+l}}{i!j!k!l!(-N)_{i+j+k+l}}$$
$$\times \left(\frac{-u_1}{1-u_1}\right)^i \left(\frac{-v_1}{1-v_1}\right)^j \left(\frac{u_2-u_1}{1-u_1}\right)^k \left(\frac{v_2-v_1}{1-v_1}\right)^l.$$

So, a straightforward calculation gives

$$(N-m_1-m_2)(P_{m_1+1,m_2}(x_1,x_2) - P_{m_1,m_2}(x_1,x_2)) = -(1-u_1)^{x_1}(1-v_1)^{x_2}$$
$$\times \left(u_1'\frac{\partial}{\partial u_1'} + v_1'\frac{\partial}{\partial v_1'}\right)((1-u_1)^{-x_1}(1-v_1)^{-x_2}P_{m_1,m_2}(x_1,x_2)),$$

where $u_i' = u_i/(u_i-1)$, $v_i' = v_i/(v_i-1)$, $i=1,2$.

Clearly $u_i'\frac{\partial}{\partial u_i'} = u_i(1-u_i)\frac{\partial}{\partial u_i}$, etc. Hence, with a similar expression for $P_{m_1,m_2+1} - P_{m_1,m_2}$, we can write

$$(N-m_1-m_2)\{A(P_{m_1+1,m_2} - P_{m_1,m_2}) - B(P_{m_1,m_2+1} - P_{m_1,m_2})\}$$
$$= B(1-u_2)^{x_1}(1-v_2)^{x_2}\left\{u_2(1-u_2)\frac{\partial}{\partial u_2} + v_2(1-v_2)\frac{\partial}{\partial v_2}\right\}$$
$$\times ((1-u_2)^{-x_1}(1-v_2)^{-x_2}P_{m_1,m_2}) - A(1-u_1)^{x_1}(1-v_1)^{x_2}$$
$$\times \left\{u_1(1-u_1)\frac{\partial}{\partial u_1} + v_1(1-v_1)\frac{\partial}{\partial v_1}\right\}((1-u_1)^{-x_1}(1-v_1)^{-x_2}P_{m_1,m_2}) \qquad (5.1)$$

for some suitably chosen constants A and B.

A more convenient form of the right-hand side of (5.1) is

$$\{x_1(Bu_2-Au_1) + x_2(Bv_2-Av_1)\}P_{m_1,m_2} + \left\{B\left(\left(u_2(1-u_2)\frac{\partial}{\partial u_2} + v_2(1-v_2)\frac{\partial}{\partial v_2}\right)\right)\right.$$
$$\left. - A\left(\left(u_1(1-u_1)\frac{\partial}{\partial u_1} + v_1(1-v_1)\frac{\partial}{\partial v_1}\right)\right)\right\}P_{m_1,m_2}.$$

By using the values of A and B indicated in (1.9), and those of u's and v's in (1.3) we find that

$$Bu_2 - Au_1 = p_1 + p_2, \qquad Bv_2 - Av_1 = -(p_3 + p_4).$$

Similarly,

$$\begin{aligned} &Cm_1(P_{m_1-1,m_2} - P_{m_1,m_2}) - Dm_2(P_{m_1,m_2-1} - P_{m_1,m_2}) \\ &\quad = \left\{ D\left(u_2\frac{\partial}{\partial u_2} + v_2\frac{\partial}{\partial v_2}\right) - C\left(u_1\frac{\partial}{\partial u_1} + v_1\frac{\partial}{\partial v_1}\right)\right\} P_{m_1,m_2}. \end{aligned}$$

What we really need to prove is that

$$\begin{aligned} &\left\{ -(A(1-u_1)+C)u_1\frac{\partial}{\partial u_1} - (A(1-v_1)+C)v_1\frac{\partial}{\partial v_1} \right. \\ &\quad \left. + (B(1-u_2)+D)u_2\frac{\partial}{\partial u_2} + (B(1-v_2)+D)v_2\frac{\partial}{\partial v_2}\right\} P_{m_1,m_2}(x_1,x_2) = 0. \end{aligned}$$

Straightforward algebra gives

$$\begin{aligned} &A(1-u_1)+C = p_4, \qquad && A(1-u_2)+C = -p_2, \\ &B(1-u_2)+D = -p_3, \qquad && B(1-v_2)+D = p_1, \end{aligned}$$

so it amounts to showing that

$$\left(-p_4u_1\frac{\partial}{\partial u_1} + p_2v_1\frac{\partial}{\partial v_1} - p_3u_2\frac{\partial}{\partial u_2} + p_1v_2\frac{\partial}{\partial v_2}\right) P_{m_1,m_2} = 0. \tag{5.2}$$

Since the u's and v's are expressed in terms of the 4 p's, what we need now is to express the derivatives in (5.2) in terms of those of the p's. Noting that

$$(F_{u_1}, F_{v_1}, F_{u_2}, F_{v_2})' = J^{-1}(F_{p_1}, F_{p_2}, F_{p_3}, F_{p_4})',$$

for any differentiable function F, with the Jacobian J given by the 4×4 matrix

$$J = (u_{1,j}, v_{1,j}, u_{2,j}, v_{2,j}), \qquad (u_{i,j}) = \left(\frac{\partial u_i}{\partial p_1}, \frac{\partial u_i}{\partial p_2}, \frac{\partial u_i}{\partial p_3}, \frac{\partial u_i}{\partial p_4}\right)', \qquad \text{etc.}, \qquad i = 1, 2,$$

we are reduced to the task of proving that

$$-p_4u_1a_{11} + p_2v_1a_{21} - p_3u_2a_{31} + p_1v_2a_{41} = 0, \tag{5.3}$$

and 3 more similar relations, where the $|J|^{-1}a'_{ij}$ are elements of the inverse matrix J^{-1}, which, of course, exists. By a set of long and messy computations we obtain

$$\begin{aligned} &a_{11} = -p_1v_2\Delta^2/p_4p_2^2p_3^2S^3, \qquad && a_{21} = -u_2\Delta^2/p_1p_2p_4^2S^3, \\ &a_{31} = -v_1\Delta^2/p_1p_3p_4^2S^3, \qquad && a_{41} = -u_1\Delta^2/p_2^2p_3^2S^3, \end{aligned} \tag{5.4}$$

with $S = p_1+p_2+p_3+p_4$, $\Delta = p_1p_4 - p_2p_3$. Substitution of (5.4) proves (5.3). The three other relations are similarly proved. This completes the proof of (1.9).

An additional comment. After this paper was completed we became aware of a recent arXiv posting [14], where the authors point out some important work of H. Mizukawa and H. Tanaka [19]. In a future publication we return to the probabilistic origin of the work of M. Hoare and M. Rahman and we discuss the relation between the approach in [19], based on the notion of character algebras, and our own.

Acknowledgements

We thank one referee in particular for a very methodical job that has rendered this into a more accurate paper. In an area where several people have made important contributions he has helped us tell the story properly.

The research of the first author was supported in part by the Applied Math. Sciences subprogram of the Office of Energy Research, USDOE, under Contract DE-AC03-76SF00098, and by AFOSR under contract FA9550-08-1-0169.

References

[1] Andrews G.E., Askey R., Roy R., Special functions, *Encyclopedia of Mathematics and its Applications*, Vol. 71, Cambridge University Press, Cambridge, 1999.

[2] Aomoto K., Kita M., Hypergeometric functions, Springer, Tokyo, 1994 (in Japanese).

[3] Cooper R.D., Hoare M.R., Rahman M., Stochastic processes and special functions: on the probabilistic origin of some positive kernels associated with classical orthogonal polynomials, *J. Math. Anal. Appl.* **61** (1977), 262–291.

[4] Dunkl C., Xu Y., Orthogonal polynomials of several variables, *Encyclopedia of Mathematics and its Applications*, Vol. 81, Cambridge University Press, Cambridge, 2001.

[5] Feller W., An introduction to probability theory and its applications, Vol. 1, 3rd ed., Wiley 1967.

[6] Gelfand I.M., General theory of hypergeometric functions, *Sov. Math. Dokl.* **33** (1986), 573–577.

[7] Geronimo J.S., Iliev P., Bispectrality of multivariable Racah–Wilson poynomials, *Constr. Approx.* **31** (2010), 417–457, arXiv:0705.1469.

[8] Grünbaum F.A., The Rahman polynomials are bispectral, *SIGMA* **3** (2007), 065, 11 pages, arXiv:0705.0468.

[9] Hoare M.R., Rahman M., Distributive processes in discrete systems, *Phys. A* **97** (1979), 1–41.

[10] Hoare M.R., Rahman M., Cumulative Bernoulli trials and Krawtchouk processes, *Stochastic Process. Appl.* **16** (1983), 113–139.

[11] Hoare M.R., Rahman, M., Cumulative hypergeometric processes: a statistical role for the ${}_nF_{n-1}$ functions, *J. Math. Anal. Appl.* **135** (1988), 615–626.

[12] Hoare M.R., Rahman M., A probabilistic origin for a new class of bivariate polynomials, *SIGMA* **4** (2008), 089, 18 pages, arXiv:0812.3879.

[13] Erdelyi A., Magnus W., Oberhettinger F., Tricomi F.G., Higher transcendental functions, Vols. 1, 2, 3, *Bateman Manuscript Project*, McGraw-Hill Book Co., New York, 1953.

[14] Iliev P., Terwilliger P., The Rahman polynomials and the Lie algebra $sl_3(C)$, arXiv:1006.5062.

[15] Iliev P., Xu Y., Discrete orthogonal polynomials and difference equations in several variables, *Adv. Math.* **212** (2007), 1–36, math.CA/0508039.

[16] Virchenko N., Katchanovski I., Haidey V., Andruskiw R., Voronka R. (Editors), Development of mathematical ideas of Mykhailo Kravchuk, Kyiv – New York, 2004.

[17] Mizukawa H., Zonal spherical functions on the complex reflection groups and $(n+1, m+1)$ hypergeometric functions, *Adv. Math.* **184** (2004), 1–17.

[18] Mizukawa H., Orthogonality relations for multivariate Krawtchouck polynomials, arXiv:1009.1203.

[19] Mizukawa H., Tanaka H., $(n+1, m+1)$-hypergeometric functions associated to character algebras, *Proc. Amer. Math. Soc.* **132** (2004), 2613–2618.