A Particular Solution of a Painlevé System in Terms of the Hypergeometric Function ${}_{n+1}F_n$*

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Received June 23, 2010, in final form September 29, 2010; Published online October 07, 2010
doi:10.3842/SIGMA.2010.078

Abstract. In a recent work, we proposed the coupled Painlevé VI system with $A^{(1)}_{2n+1}$-symmetry, which is a higher order generalization of the sixth Painlevé equation (P_{VI}). In this article, we present its particular solution expressed in terms of the hypergeometric function ${}_{n+1}F_n$. We also discuss a degeneration structure of the Painlevé system derived from the confluence of ${}_{n+1}F_n$.

Key words: affine Weyl group; generalized hypergeometric functions; Painlevé equations

2010 Mathematics Subject Classification: 17B80; 33C20; 34M55

1 Introduction

The main object in this article is the coupled Painlevé VI system with $A^{(1)}_{2n+1}$-symmetry given in [1, 4], or equivalently, the Hamiltonian system $\mathcal{H}_{n+1,1}$ given in [6]. It is expressed as a Hamiltonian system on $\mathbb{P}^1(\mathbb{C})$

$$t(t-1)\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad t(t-1)\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \qquad i = 1,\dots,n, \tag{1}$$

with

$$\begin{aligned} H = {}& \sum_{i=1}^{n} H_{\mathrm{VI}}\left[\sum_{j=0}^{n}\alpha_{2j+1} - \alpha_{2i-1} - \eta, \sum_{j=0}^{i-1}\alpha_{2j}, \sum_{j=i}^{n}\alpha_{2j}, \alpha_{2i-1}\eta; q_i, p_i\right] \\ & + \sum_{1\le i<j\le n}(q_i-1)(q_j-t)\{(q_ip_i+\alpha_{2i-1})p_j + p_i(q_jp_j+\alpha_{2j-1})\}, \end{aligned}$$

where H_{VI} is the Hamiltonian for P_{VI} defined as

$$\begin{aligned} H_{\mathrm{VI}}[\kappa_0,\kappa_1,\kappa_t,\kappa;q,p] = {}& q(q-1)(q-t)p^2 - \kappa_0(q-1)(q-t)p \\ & - \kappa_1 q(q-t)p - (\kappa_t-1)q(q-1)p + \kappa q. \end{aligned}$$

Here $\alpha_0,\dots,\alpha_{2n+1}$ and η are fixed parameters satisfying a relation $\sum\limits_{i=0}^{2n+1}\alpha_i = 1$. We assume that the indices of α_i are congruent modulo $2n+2$. Note that the system (1) includes P_{VI} as the case $n=1$. The aim of this article is to present a particular solution of the system (1) expressed in terms of the hypergeometric function ${}_{n+1}F_n$.

*This paper is a contribution to the Special Issue "Relationship of Orthogonal Polynomials and Special Functions with Quantum Groups and Integrable Systems". The full collection is available at http://www.emis.de/journals/SIGMA/OPSF.html

The hypergeometric function ${}_{n+1}F_n$ is defined by the power series

$$ {}_{n+1}F_n \left[\begin{matrix} a_0, \dots, a_n \\ b_1, \dots, b_n \end{matrix} ; t \right] = \sum_{i=0}^{\infty} \frac{(a_0)_i (a_1)_i \cdots (a_n)_i}{(1)_i (b_1)_i \cdots (b_n)_i} t^i, $$

where $(a)_i$ stands for the factorial function

$$ (a)_i = a(a+1) \cdots (a+i-1) = \frac{\Gamma(a+i)}{\Gamma(a)}. $$

Denoting by $\delta = td/dt$, we see that $x = {}_{n+1}F_n$ satisfies an $(n+1)$-th order linear differential equation

$$ [\delta(\delta + b_1 - 1) \cdots (\delta + b_n - 1) - t(\delta + a_0) \cdots (\delta + a_n)]x = 0, \tag{2} $$

which is called *the generalized hypergeometric equation* [3]. The equation (2) is of Fuchsian type with regular singular points at $t = 0, 1, \infty$ and the Riemann scheme

$$ \left[\begin{matrix} t=0 & t=1 & t=\infty \\ 0 & 0 & a_0 \\ 1-b_1 & 1 & a_1 \\ \vdots & \vdots & \vdots \\ 1-b_{n-1} & n-1 & a_{n-1} \\ 1-b_n & -\sum\limits_{i=1}^{n}(1-b_i) - \sum\limits_{i=0}^{n} a_i & a_n \end{matrix} \right]. $$

Note that ${}_{n+1}F_n$ includes the Gauss hypergeometric function as the case $n = 1$.

In this article, we clarify a relationship between the system (1) and the function ${}_{n+1}F_n$. For $n = 1$ among them, the relationship between P_{VI} and the Gauss hypergeometric function is well known. Under the system (1) of the case $n = 1$, we consider a specialization $p = \eta = 0$. Then we obtain a Riccati equation

$$ t(t-1)\frac{dq}{dt} = \alpha_1 q^2 + \{(\alpha_3 + \alpha_0)t - (\alpha_0 + \alpha_1)\} q - \alpha_3 t. $$

Via a transformation of a dependent variable

$$ q = \frac{t(1-t)}{\alpha_1} \frac{d}{dt} \log\{(t-1)^{\alpha_3} x(t)\}, $$

we obtain the Gauss hypergeometric equation

$$ [\delta(\delta + \alpha_2 + \alpha_3 - 1) - t(\delta + \alpha_1 + \alpha_2 + \alpha_3)(\delta + \alpha_3)]x = 0. $$

The result of this article gives a natural extension of the above fact. For general n, we consider a specialization $p_1 = \cdots = p_n = \eta = 0$. Then we obtain the generalized hypergeometric equation by a certain transformation of dependent variables.

We also discuss a degeneration structure of the system (1) derived from the confluence of ${}_{n+1}F_n$. The confluent hypergeometric functions ${}_{n-r+1}F_n$ $(r = 1, \dots, n+1)$ are defined by the power series

$$ {}_{n-r+1}F_n \left[\begin{matrix} a_r, \dots, a_n \\ b_1, \dots, b_n \end{matrix} ; t \right] = \sum_{i=0}^{\infty} \frac{(a_r)_i \cdots (a_n)_i}{(b_1)_i \cdots (b_n)_i} t^i. $$

The process of confluence ${}_{n-r+2}F_n \to {}_{n-r+1}F_n$ is given by a replacement

$$t \to \varepsilon t, \qquad a_{r-1} \to \varepsilon^{-1},$$

and taking a limit $\varepsilon \to 0$. We see that $x = {}_{n-r+1}F_n$ satisfy the confluent hypergeometric differential equations

$$[\delta(\delta + b_1 - 1)\cdots(\delta + b_n - 1) - t(\delta + a_r)\cdots(\delta + a_n)]x = 0. \tag{3}$$

In this article, we propose a class of higher order Painlevé systems which admit particular solutions expressed in terms of ${}_{n-r+1}F_n$.

Remark 1. In this article, we study a higher order generalization of P_{VI}. On the other hand, for a multi-time generalization, it is known that the Garnier system admits a particular solution in terms of the Appell–Lauricella hypergeometric function F_D [2].

This article is organized as follows. In Section 2, we derive a system of linear differential equations from the system (1) by a specialization $p_1 = \cdots = p_n = \eta = 0$. In Section 3, we give its fundamental solutions expressed in terms of the hypergeometric function ${}_{n+1}F_n$ in a neighborhood of the singular point $t = 0$. In Section 4, we discuss a degeneration structure of the system (1) derived from the confluence of ${}_{n+1}F_n$.

2 Linear differential equations

In this section, we derive a system of linear differential equations from the system (1) by a specialization $p_1 = \cdots = p_n = \eta = 0$.

We first consider a *symmetric form* of (1) in order to derive a system of linear differential equations. Let x_i, y_i $(i = 0, \ldots, n)$ be dependent variables such that

$$t(1-t)\frac{d}{dt}\log x_n = \sum_{i=1}^{n}\{(q_i - 1)(q_i - t)p_i + \alpha_{2i-1}q_i\} + t\alpha_{2n+1} - (t+1)\eta,$$

and

$$x_{i-1} = \frac{x_n q_i}{t}, \qquad y_{i-1} = \frac{tp_i}{x_n}, \qquad i = 1, \ldots, n, \qquad y_n = -\frac{1}{x_n}\left(\sum_{j=1}^{n} q_j p_j + \eta\right).$$

Then we obtain a Hamiltonian system of $(2n+2)$-th order

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \qquad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}, \qquad i = 0, \ldots, n, \tag{4}$$

with a Hamiltonian

$$\begin{aligned} H = {}& \frac{1}{t}\sum_{i=0}^{n}\left\{\frac{1}{2}x_i^2 y_i^2 - \alpha_{2i+2}^{2n-2i-1}x_i y_i + \sum_{j=0}^{i-1} x_i(x_i y_i + \alpha_{2i+1})y_j\right\} \\ & + \frac{1}{1-t}\sum_{i=0}^{n}\sum_{j=0}^{n} x_i(x_i y_i + \alpha_{2i+1})y_j, \end{aligned}$$

where

$$\alpha_k^l = \begin{cases} 0, & l \in \mathbb{Z}_{<0}, \\ \sum\limits_{i=k}^{k+l}\alpha_i, & l \in \mathbb{Z}_{\geq 0}. \end{cases}$$

The dependent variables x_i, y_i and the fixed parameter η satisfy a relation

$$\sum_{i=0}^{n} x_i y_i + \eta = 0.$$

Remark 2. The symmetric form (4) is suggested by the Hamiltonian system given in Theorem 3.2 of [4]. Their relationship is given by

$$t = \frac{1}{t_1^{n+1}}, \qquad x_i = \frac{w_{2i+1}}{t_1^{i-n+\rho_1+\kappa_{2n+1}-\kappa_0}}, \qquad y_i = \frac{t_1^{i-n+\rho_1+\kappa_{2n+1}-\kappa_0}\varphi_{2i+1}}{n+1},$$

and

$$\eta = \sum_{j=0}^{n} \frac{\rho_1 + \kappa_{2i} - \kappa_{2i+1}}{n+1}, \qquad \alpha_{2i} = \frac{1 + \kappa_{2i-1} - 2\kappa_{2i} + \kappa_{2i+1}}{n+1},$$
$$\alpha_{2i+1} = \frac{\kappa_{2i} - 2\kappa_{2i+1} + \kappa_{2i+2}}{n+1},$$

for $i = 0, \dots, n$.

Remark 3. The system (4), or equivalently the system (1), admits the affine Weyl group symmetry of type $A_{2n+1}^{(1)}$; see Appendix B.

We can derive easily a system of linear differential equations from the symmetric form by the specialization $y_0 = \cdots = y_n = \eta = 0$, which is equivalent to $p_1 = \cdots = p_n = \eta = 0$. Let $E_{i,j}$ be the matrix unit defined by

$$E_{i,j} = (\delta_{i,k}\delta_{j,l})_{k,l=0}^{n}.$$

For example, in the case $n = 2$, it is explicitly given as

$$E_{1,1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad E_{1,2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad E_{2,1} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad E_{2,2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then we obtain

Proposition 1. *The system* (4) *admits a specialization*

$$y_i = 0, \qquad i = 0, \dots, n, \qquad \eta = 0.$$

Then a vector of the variables $\mathbf{x} = {}^t(x_0, \dots, x_n)$ *satisfies a system of linear differential equations on* $\mathbb{P}^1(\mathbb{C})$

$$\frac{d\mathbf{x}}{dt} = \left(\frac{A_0}{t} + \frac{A_1}{1-t} \right) \mathbf{x}, \tag{5}$$

with

$$A_0 = \sum_{i=0}^{n-1} \big(-\alpha_{2i+2}^{2n-2i-1} \big) E_{i,i} + \sum_{i=0}^{n-1} \sum_{j=i+1}^{n} \alpha_{2j+1} E_{i,j}, \qquad A_1 = \sum_{i=0}^{n} \sum_{j=0}^{n} \alpha_{2j+1} E_{i,j}.$$

Furthermore, the system (5) *is of Fuchsian type with regular singular points at* $t = 0, 1, \infty$. *The data of eigenvalues of its residue matrices is given as*

$$-\alpha_2^{2n-1}, \dots, -\alpha_{2n}^{1}, 0 \quad \textit{at} \quad t = 0,$$
$$0, \dots, 0, -\sum_{i=0}^{n} \alpha_{2i+1} \quad \textit{at} \quad t = 1,$$
$$\alpha_1^{2n}, \dots, \alpha_{2n-1}^{2}, \alpha_{2n+1} \quad \textit{at} \quad t = \infty.$$

Remark 4. The system (4) also admits a specialization

$$x_i = 0, \qquad i = 0, \dots, n-1, \qquad x_n y_n + \eta = 0, \qquad \eta - \alpha_{2n+1} = 0,$$

which is equivalent to $q_1 = \cdots = q_n = \eta - \alpha_{2n+1} = 0$. Then a vector of the variables $\mathbf{y} = {}^t(y_0, \dots, y_n)$ satisfies a system of linear differential equations

$$\frac{d\mathbf{y}}{dt} = \left(\frac{A_0}{t} + \frac{A_1}{1-t} \right) \mathbf{y},$$

with

$$A_0 = \sum_{i=0}^{n-1} \alpha_{2i+2}^{2n-2i-1} E_{i,i} + \sum_{i=1}^{n-1} \sum_{j=0}^{i-1} (-\alpha_{2i+1}) E_{i,j} + \sum_{j=0}^{n} \alpha_{2n+1} E_{n,j},$$
$$A_1 = \sum_{i=0}^{n-1} \sum_{j=0}^{n} (-\alpha_{2i+1}) E_{i,j} + \sum_{j=0}^{n} \alpha_{2n+1} E_{n,j}.$$

We always assume that

$$\alpha_{2i}^{2j-1} \notin \mathbb{Z}, \qquad \sum_{i=0}^{n} \alpha_{2i+1} \notin \mathbb{Z}, \qquad \alpha_{2i-1}^{2j-1} \notin \mathbb{Z}, \qquad i = 1, \dots, n, \quad j = 1, \dots, n-i+1.$$

In the next section, we describe fundamental solutions of the system (5) in a neighborhood of the singular point $t = 0$ explicitly.

3 Hypergeometric function ${}_{n+1}F_n$

In this section, we give fundamental solutions of the system (5) expressed in terms of the hypergeometric function ${}_{n+1}F_n$ in a neighborhood of the singular point $t = 0$.

For each $k = 0, \dots, n$, we consider a gauge transformation

$$\mathbf{x}^k = t^{\alpha_{2k+2}^{2n-2k-1}} \left(\sum_{i=0}^{n-k-1} t^{-1} E_{i,i+k+1} + \sum_{i=n-k}^{n} E_{i,i-n+k} \right) \mathbf{x}.$$

Then the system (5) is transformed into

$$\frac{d\mathbf{x}^k}{dt} = \left(\frac{A_0^k}{t} + \frac{A_1^k}{1-t} \right) \mathbf{x}^k, \tag{6}$$

with

$$A_0^k = \sum_{i=0}^{n-1} (-\alpha_{2k+2i+4}^{2n-2i-1}) E_{i,i} + \sum_{i=0}^{n-1} \sum_{j=i+1}^{n} \alpha_{2j+2k+3} E_{i,j},$$
$$A_1^k = \sum_{i=0}^{n} \sum_{j=0}^{n} \alpha_{2j+2k+3} E_{i,j}.$$

Recall that indices of the fixed parameters α_i are congruent modulo $2n+2$, from which we have $\alpha_{2k+2n+2}^1 = \alpha_{2k}^1$. We also consider a formal power series of $\mathbf{x}^k$ at $t = 0$

$$\mathbf{x}^k = \sum_{i=0}^{\infty} \mathbf{x}_i^k t^i.$$

Then the system (6) implies

$$
\begin{aligned}
&A_0^k \mathbf{x}_0^k = \mathbf{0}, \\
&\left\{A_0^k - (i+1)I\right\}\mathbf{x}_{i+1}^k = \left(A_0^k - A_1^k - iI\right)\mathbf{x}_i^k, \qquad i \in \mathbb{Z}_{\geq 0},
\end{aligned} \tag{7}
$$

where I stands for the identity matrix. The matrices A_0^k and $A_0^k - (i+1)I$ are of rank n and $n+1$, respectively. It follows that the recurrence formula (7) admits one parameter family of solutions.

For each $k = 0, \dots, n$, we can show that a sequence of vectors

$$
\mathbf{x}_i^k = \begin{bmatrix}
\displaystyle\prod_{j=0}^{n-1} \frac{(\alpha_{2k-2j+1}^{2j})_{i+1}}{(\alpha_{2k-2j}^{2j+1})_{i+1}} \cdot \frac{(\alpha_{2k+3}^{2n})_i}{(\alpha_{2k+2}^{2n+1})_i} \\
\displaystyle\prod_{j=0}^{n-2} \frac{(\alpha_{2k-2j+1}^{2j})_{i+1}}{(\alpha_{2k-2j}^{2j+1})_{i+1}} \cdot \frac{(\alpha_{2k+5}^{2n-2})_i(\alpha_{2k+3}^{2n})_i}{(\alpha_{2k+4}^{2n-1})_i(\alpha_{2k+2}^{2n+1})_i} \\
\vdots \\
\displaystyle\frac{(\alpha_{2k+1})_{i+1}}{(\alpha_{2k}^{1})_{i+1}} \prod_{j=0}^{n-1} \frac{(\alpha_{2k+2j+3}^{2n-2j})_i}{(\alpha_{2k+2j+2}^{2n-2j+1})_i} \\
\displaystyle\prod_{j=0}^{n} \frac{(\alpha_{2k+2j+3}^{2n-2j})_i}{(\alpha_{2k+2j+2}^{2n-2j+1})_i}
\end{bmatrix}, \qquad i \in \mathbb{Z}_{\geq 0},
$$

satisfies the recurrence formula (7) by a direct computation. Note that $\alpha_{2k+2}^{2n+1} = 1$. Therefore we arrive at

Theorem 1. *On a domain $|t| < 1$, the system* (5) *admits fundamental solutions*

$$
\mathbf{x} = t^{-\alpha_{2k+2}^{2n-2k-1}} \begin{bmatrix} f^{k,k} \\ \vdots \\ f^{k,0} \\ tf^{k,n} \\ \vdots \\ tf^{k,k+1} \end{bmatrix}, \qquad k = 0, \dots, n,
$$

where

$$
f^{k,l} = \prod_{i=1}^{l} \frac{\alpha_{2k-2i+3}^{2i-2}}{\alpha_{2k-2i+2}^{2i-1}} \cdot {}_{n+1}F_n \left[\begin{matrix} a_0, \dots, a_n \\ b_1, \dots, b_n \end{matrix} ; t \right],
$$

with

$$
\begin{aligned}
&a_0 = \alpha_{2k-2n+1}^{2n}, \\
&a_i = 1 + \alpha_{2k-2i+3}^{2i-2}, \qquad b_i = 1 + \alpha_{2k-2i+2}^{2i-1}, \qquad i = 1, \dots, l, \\
&a_i = \alpha_{2k-2i+3}^{2i-2}, \qquad b_i = \alpha_{2k-2i+2}^{2i-1}, \qquad i = l+1, \dots, n.
\end{aligned}
$$

Corollary 1. *If the vector $\mathbf{x} = {}^t(x_0, \dots, x_n)$ satisfies the system* (5)*, each component x_i satisfies the generalized hypergeometric equation* (2) *with*

$$
\begin{aligned}
&a_0 = \alpha_1^{2n}, \\
&a_j = 1 + \alpha_{2n-2j+3}^{2j-2}, \qquad b_j = 1 + \alpha_{2n-2j+2}^{2j-1}, \qquad j = 1, \dots, n-i, \\
&a_j = \alpha_{2n-2j+3}^{2j-2}, \qquad b_j = \alpha_{2n-2j+2}^{2j-1}, \qquad j = n-i+1, \dots, n.
\end{aligned}
$$

Remark 5. The system (5) have been already studied by Okubo–Takano–Yoshida [3]. They considered the Fuchsian differential equation of Okubo type and obtained its fundamental solutions at singular points $t = 0, 1$.

4 Degeneration structure

In this section, we discuss a degeneration structure of the system (1) derived from the confluence of ${}_{n+1}F_n$.

For each $r = 1, \dots, n+1$, we consider a Hamiltonian system

$$ {}_{n-r+1}\mathcal{H}_n : \quad \frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \qquad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}, \qquad i = 0, \dots, n, $$

with a Hamiltonian

$$ tH = \sum_{i=0}^{n} \frac{1}{2} x_i y_i \big(x_i y_i - 2\alpha_{2i+2}^{2n-2i-1} \big) + \sum_{i=0}^{r-2} x_{i+1} y_i + \sum_{i=r-1}^{n} \left\{ t x_0 + \sum_{j=i+1}^{n} x_j (x_j y_j + \alpha_{2j+1}) \right\} y_i. $$

Here α_i ($i = 0, \dots, 2n+1$) and η are fixed parameters satisfying

$$ \alpha_{2i} = 0, \quad i = 0, \dots, r-1, \qquad \sum_{j=0}^{n} \alpha_{2j+1} + \sum_{j=r}^{n} \alpha_{2j} = 1, $$

and

$$ \sum_{j=0}^{n} x_j y_j + \eta = 0. $$

Note that

$$ \alpha_{2i+2}^{2n-2i-1} = \sum_{j=2i+2}^{2n+1} \alpha_j = \sum_{j=i+1}^{n} \alpha_{2j+1} + \sum_{j=\max(r,i+1)}^{n} \alpha_{2j}. $$

The system ${}_{n-r+1}\mathcal{H}_n$ is obtained from ${}_{n-r+2}\mathcal{H}_n$ by a replacement

$$ \begin{aligned} &t \to \varepsilon t, \qquad \alpha_{2r-2} \to -\varepsilon^{-1}, \qquad \alpha_{2r-1} \to \alpha_{2r-1} + \varepsilon^{-1}, \\ &x_i \to \varepsilon^{-1} x_i, \qquad y_i \to \varepsilon y_i, \qquad i = 0, \dots, r-2. \end{aligned} $$

and taking a limit $\varepsilon \to 0$, where ${}_{n+1}\mathcal{H}_n$ stands for the system (4).

Remark 6. Such degenerate systems also can be rewritten into the Hamiltonian systems in terms of the canonical coordinates. We give their explicit formulas for $n = 1$ and $n = 2$ in Appendix A.

The system ${}_{n-r+1}\mathcal{H}_n$ admits a specialization

$$ y_i = 0, \quad i = 0, \dots, n, \qquad \eta = 0. $$

Then a vector of the variables $\mathbf{x} = {}^t(x_0, \dots, x_n)$ satisfies a system of linear differential equations

$$ {}_{n-r+1}\mathcal{L}_n : \quad \frac{d\mathbf{x}}{dt} = \left(\frac{A_0}{t} + A_1 \right) \mathbf{x}, $$

with

$$ \begin{aligned} A_0 &= \sum_{i=0}^{n-1} \big(-\alpha_{2i+2}^{2n-2i-1} \big) E_{i,i} + \sum_{i=0}^{r-2} E_{i,i+1} + \sum_{i=r-1}^{n-1} \sum_{j=i+1}^{n} \alpha_{2j+1} E_{i,j}, \\ A_1 &= \sum_{i=r-1}^{n} E_{i,0}. \end{aligned} $$

Note that ${}_{n-r+1}\mathcal{L}_n$ is obtained from ${}_{n-r+2}\mathcal{L}_n$ through the above process of confluence.

In a similar manner as Section 3, we arrive at

Theorem 2. *On a domain $|t| < 1$, the system ${}_{n-r+1}\mathcal{L}_n$ admits fundamental solutions*

$$\mathbf{x} = t^{-\alpha_{2k+2}^{2n-2k-1}} \begin{bmatrix} f_r^{k,k} \\ \vdots \\ f_r^{k,0} \\ t f_r^{k,n} \\ \vdots \\ t f_r^{k,k+1} \end{bmatrix}, \qquad k = 0, \dots, n,$$

where

$$f_r^{k,l} = \prod_{\substack{1 \le i \le l \\ \mathrm{mod}[k-i+1,n+1] \ge r}} \alpha_{2k-2i+3}^{2i-2} \prod_{1 \le i \le l} \frac{1}{\alpha_{2k-2i+2}^{2i-1}} \cdot {}_{n-r+1}F_n \left[\begin{matrix} a_r, \dots, a_n \\ b_1, \dots, b_n \end{matrix} ; t \right],$$

and

$$\mathrm{mod}[i, n+1] = i - m(n+1) \qquad \textit{for} \quad m(n+1) \le i < (m+1)(n+1).$$

Here the parameters $a_r, \dots, a_n$ are given by

$$a_i = \alpha_{2r-2i-1}^{2k-2r+2i+2}, \qquad i = r, \dots, n,$$

for $k+1 \le r$ and $l < k+2$;

$$\begin{aligned} a_i &= 1 + \alpha_{2r-2i-1}^{2k-2r+2i+2}, & i &= r, \dots, r-k+l-2, \\ a_i &= \alpha_{2r-2i-1}^{2k-2r+2i+2}, & i &= r-k+l-1, \dots, n, \end{aligned}$$

for $k+1 \le r$ and $k+2 \le l$;

$$\begin{aligned} a_i &= \alpha_{2r-2i-1}^{2k-2r+2i+2}, & i &= r, \dots, n+r-k-1, \\ a_i &= 1 + \alpha_{2r-2i-1}^{2k-2r+2i+2}, & i &= n+r-k, \dots, n+r-k+l-1, \\ a_i &= \alpha_{2r-2i-1}^{2k-2r+2i+2}, & i &= n+r-k+l, \dots, n, \end{aligned}$$

for $r < k+1$ and $l < k-r+1$;

$$\begin{aligned} a_i &= \alpha_{2r-2i-1}^{2k-2r+2i+2}, & i &= r, \dots, n+r-k-1, \\ a_i &= 1 + \alpha_{2r-2i-1}^{2k-2r+2i+2}, & i &= n+r-k, \dots, n, \end{aligned}$$

for $r < k+1$ and $k-r+1 \le l < k+2$;

$$\begin{aligned} a_i &= 1 + \alpha_{2r-2i-1}^{2k-2r+2i+2}, & i &= r, \dots, r-k+l-2, \\ a_i &= \alpha_{2r-2i-1}^{2k-2r+2i+2}, & i &= r-k+l-1, \dots, n+r-k-1, \\ a_i &= 1 + \alpha_{2r-2i-1}^{2k-2r+2i+2}, & i &= n+r-k, \dots, n, \end{aligned}$$

for $r < k+1$ and $k+2 \le l$. The parameters $b_1, \dots, b_n$ are given by

$$\begin{aligned} b_i &= 1 + \alpha_{2k-2i+2}^{2i-1}, & i &= 1, \dots, l, \\ b_i &= \alpha_{2k-2i+2}^{2i-1}, & i &= l+1, \dots, n. \end{aligned}$$

Corollary 2. *If the vector* $\mathbf{x} = {}^t(x_0, \ldots, x_n)$ *satisfies the system* ${}_{n-r+1}\mathcal{L}_n$*, each component* x_i *satisfies the confluent hypergeometric equation* (3) *with*

$$a_j = 1 + \alpha_{2r-2j-1}^{2n-2r+2j+2}, \qquad j = r, \ldots, n,$$

for $i \leq r-1$,

$$a_j = 1 + \alpha_{2r-2j-1}^{2n-2r+2j+2}, \qquad j = r, \ldots, n+r-i-1,$$
$$a_j = \alpha_{2r-2j-1}^{2n-2r+2j+2}, \qquad j = n+r-i, \ldots, n,$$

for $r-1 < i$ *and*

$$b_j = 1 + \alpha_{2n-2j+2}^{2j-1}, \qquad j = 1, \ldots, n-i,$$
$$b_j = \alpha_{2n-2j+2}^{2j-1}, \qquad j = n-i+1, \ldots, n,$$

for any i.

A Canonical Hamiltonian system

The systems ${}_{n-r+1}\mathcal{H}_n$ can be rewritten into the Hamiltonian systems in terms of canonical coordinates. In this section, we give their explicit formulas for $n = 1$ and $n = 2$. Note that ${}_{3-r}\mathcal{H}_2$ appear in the classification of the fourth order Painlevé type differential equations [5].

A.1 Case $n = 1$, $r = 1$

Under the system ${}_1\mathcal{H}_1$, we take canonical coordinates

$$q = \frac{x_0}{x_1}, \qquad p = -\frac{x_1(x_1 y_1 + \alpha_3)}{x_0}.$$

Via a transformation of the independent variable $t \to -t$, we obtain a Hamiltonian system

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q},$$

with a Hamiltonian

$$tH = q(q-1)p(p+t) - qp(\eta + \alpha_2 - \alpha_3) + (\eta - \alpha_3)p + t\alpha_3 q.$$

It is equivalent to the fifth Painlevé equation.

A.2 Case $n = 1$, $r = 2$

Under the system ${}_0\mathcal{H}_1$, we take canonical coordinates

$$q = \frac{x_1}{x_0}, \qquad p = x_0 y_1.$$

Then we obtain a Hamiltonian system

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q},$$

with a Hamiltonian

$$tH = q^2 p(p-1) + (\eta + \alpha_3)qp + tp - \eta q.$$

It is equivalent to the third Painlevé equation.

A.3 Case $n = 2$, $r = 1$

Under the system ${}_2\mathcal{H}_2$, we take canonical coordinates

$$q_1 = \frac{x_0}{x_1}, \qquad p_1 = -\frac{x_1(x_1y_1 + \alpha_3)}{x_0}, \qquad q_2 = \frac{x_0}{x_2}, \qquad p_2 = -\frac{x_2(x_2y_2 + \alpha_5)}{x_0}.$$

Via a transformation of the independent variable $t \to -t$, we obtain a Hamiltonian system

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \qquad i = 1, 2,$$

with a Hamiltonian

$$\begin{aligned} tH = {}& q_1(q_1 - 1)p_1(p_1 + t) - (\eta + \alpha_2 - \alpha_3 - \alpha_5)q_1p_1 + (\eta - \alpha_3 - \alpha_5)p_1 \\ & + \alpha_3tq_1 + (q_1 - 1)p_1q_2p_2 + (q_1 - 1)(q_1p_1 + \alpha_3)p_2 \\ & + q_2(q_2 - 1)p_2(p_2 + t) - (\eta + \alpha_2 + \alpha_4 - \alpha_5)q_2p_2 + (\eta - \alpha_5)p_2 + \alpha_5tq_2. \end{aligned}$$

A.4 Case $n = 2$, $r = 2$

Under the system ${}_1\mathcal{H}_2$, we take canonical coordinates

$$q_1 = -\frac{x_1}{x_0}, \qquad p_1 = 1 - x_0y_1, \qquad q_2 = -\frac{x_2}{x_0}, \qquad p_2 = -x_0y_2.$$

Via a transformation of the independent variable $t \to -t$, we obtain a Hamiltonian system

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \qquad i = 1, 2,$$

with a Hamiltonian

$$\begin{aligned} tH = {}& q_1^2p_1(p_1 - 1) + (\eta + \alpha_3)q_1p_1 + tp_1 - \alpha_3q_1 + q_1p_1q_2p_2 + p_1q_2(q_2p_2 + \alpha_5) \\ & + q_2^2p_2(p_2 - 1) + (\eta + \alpha_3 + \alpha_4 + \alpha_5)q_2p_2 + tp_2 - \alpha_5q_2. \end{aligned}$$

A.5 Case $n = 2$, $r = 3$

Under the system ${}_0\mathcal{H}_2$, we take canonical coordinates

$$q_1 = -\frac{x_1}{x_0}, \qquad p_1 = 1 - x_0y_1, \qquad q_2 = -\frac{x_2}{x_0}, \qquad p_2 = -x_0y_2.$$

Via a transformation of the independent variable $t \to -t$, we obtain a Hamiltonian system

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \qquad i = 1, 2,$$

with a Hamiltonian

$$\begin{aligned} tH = {}& q_1^2p_1(p_1 - 1) + (\eta + \alpha_3)q_1p_1 - \alpha_3q_1 + q_1p_1q_2p_2 + p_1q_2 \\ & + q_2^2p_2^2 + (\eta + \alpha_3 + \alpha_5)q_2p_2 + tp_2 - q_2. \end{aligned}$$

B Affine Weyl group symmetry

The system (4) admits the affine Weyl group symmetry of type $A^{(1)}_{2n+1}$. In this section, we describe its action on the dependent variables and parameters.

Recall that the affine Weyl group of type $A^{(1)}_{2n+1}$ is generated by the transformations r_i $(i = 0, \dots, 2n+1)$ with the fundamental relations

$$\begin{aligned} &r_i^2 = 1, && i = 0, \dots, 2n+1, \\ &(r_i r_j)^{2-a_{i,j}} = 0, && i, j = 0, \dots, 2n+1, \quad i \neq j, \end{aligned}$$

where

$$\begin{aligned} &a_{i,i} = 2, && i = 0, \dots, 2n+1, \\ &a_{i,i+1} = a_{2n+1,0} = a_{i+1,i} = a_{0,2n+1} = -1, && i = 0, \dots, 2n, \\ &a_{i,j} = 0, && \text{otherwise.} \end{aligned}$$

We define the Poisson structure by

$$\{x_i, y_j\} = -\delta_{i,j}, \qquad i, j = 0, \dots, n.$$

Then the Hamiltonian system (4) is invariant under the birational transformations $r_0, \dots, r_{2n+1}$ defined by

$$\begin{aligned} &r_0(x_j) = t^{-\alpha_0} x_j, \qquad r_0(y_j) = t^{\alpha_0} \left(y_j + \frac{\alpha_0}{x_n - t x_0} \{x_n - t x_0, y_j\} \right), \\ &r_{2i+1}(x_j) = x_j + \frac{\alpha_{2i+1}}{y_i} \{y_i, x_j\}, \qquad r_{2i+1}(y_j) = y_j, \qquad i = 0, \dots, n-1, \\ &r_{2i}(x_j) = x_j, \qquad r_{2i}(y_j) = y_j + \frac{\alpha_{2i}}{x_{i-1} - x_i} \{x_{i-1} - x_i, y_j\}, \qquad i = 1, \dots, n, \\ &r_{2n+1}(x_j) = t^{\alpha_{2n+1}} \left(x_j + \frac{\alpha_{2n+1}}{y_n} \{y_n, x_j\} \right), \qquad r_{2n+1}(y_j) = t^{-\alpha_{2n+1}} y_j, \end{aligned}$$

for $j = 0, \dots, n$ and

$$r_i(\alpha_j) = \alpha_j - a_{i,j} \alpha_i, \qquad r_i(\eta) = \eta + (-1)^i \alpha_i, \qquad i, j = 0, \dots, 2n+1.$$

The group of symmetries $\langle r_0, \dots, r_{2n+1} \rangle$ is isomorphic to the affine Weyl group of type $A^{(1)}_{2n+1}$.

Acknowledgements

The author would like to express his gratitude to Mr. Masaomi Miyamoto of Kobe University for fruitful discussion. The author is also grateful to Professors Masatoshi Noumi, Hidetaka Sakai, Teruhisa Tsuda and Yasuhiko Yamada for helpful comments and advices.

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